Lecture 8. Koszul duality for formal moduli problems

In the last lectures, we discussed three examples of objects over $\mathbb{C}$ whose deformation functors are controlled by differential graded Lie algebras: chain complexes, vector bundles, and varieties.

It is natural to ask:

Question. Given an algebro-geometric object Y over $\mathbb{C}$, is there always a differential graded Lie algebra that controls its infinitesimal deformation functor D_Y? Is it unique?

Unfortunately, non-equivalent differential graded Lie algebras can control the same deformation functor, and it is not always possible to pick a preferred one, not even up to quasi-isomorphism.

This is illustrated by the following example (cf. [CFK01, CFK02, Toë14]):

Exercise 8.1 (Hard)**.** *Fix a closed immersion of smooth complex varieties $Z_0 \subset Z$ defined by an ideal sheaf $\mathcal{I} \subset \mathcal{O}_Z$.*

(1) Define the deformation functor encoding deformations of Z_0 inside Z (deforming Z trivially). Show that it is controlled by a differential graded Lie algebra with underlying chain complex $\mathrm{Map}_{\mathcal{O}_{Z_0}}(\mathcal{N}^{\vee}_{Z_0,Z}[1], \mathcal{O}_{Z_0})$, where $\mathcal{N}_{Z_0/Z}$ denotes the normal bundle.
(2) Define the deformation functor encoding deformations of the quotient map of quasi-coherent sheaves $\mathcal{O}_Z \to \mathcal{O}_Z/\mathcal{I}_Z = \mathcal{O}_{Z_0}$ (deforming $\mathcal{O}_Z$ trivially). Show that it is controlled by a differential graded Lie algebra with underlying chain complex $\mathrm{Map}_{\mathcal{O}_Z}(\mathcal{I}, \mathcal{O}_{Z_0})$
(3) Show that the two deformation functors are isomorphic, but that the differential graded Lie algebras need not be quasi-isomorphic.

In fact, it seems impossible to functorially attach a differential graded Lie algebra $\mathfrak{g}_D$ to each deformation functor D such that $D_{\mathfrak{g}_D} \cong D$. This has the disturbing consequence that obstruction classes do not vary functorially in the deformation functor.

8.1. **Formal moduli problems.** To rectify this behaviour, Drinfel'd proposed that one should consider *derived* deformation functors. These are based on simplicial commutative, i.e. animated, rings, see Definition 5.11 in Lecture 5. More precisely, we deform over the following class of rings:

Definition 8.2. Given a field k, an augmented animated k-algebra A is said to be *Artinian* if $\pi_0(A)$ is a local Artin ring with residue field k and $\dim_k \pi_*(A) < \infty$.

We write $\mathrm{CAlg}_k^{\mathrm{an,art}} \subset \mathrm{CAlg}_{k//k}$ for the full subcategory spanned by all Artinian objects.

Derived deformation functors generalise Definition 6.17, and are axiomatised as follows:

Definition 8.3. An (equal characteristic) *formal moduli problem over a field k* is a functor

$$D : \mathrm{CAlg}_k^{\mathrm{an,art}} \to \mathcal{S}$$

from $\mathrm{CAlg}_k^{\mathrm{an,art}}$ to the ∞-category $\mathcal{S}$ of spaces satisfying:

(1) Normalisation: the space $D(k)$ is contractible.
(2) Gluing: applying D to a pullback square

$$\begin{array}{ccc} \widetilde{A} & \longrightarrow & A' \\ \downarrow & & \downarrow \\ A & \longrightarrow & A'' \end{array}$$

with $\pi_0(A') \to \pi_0(A'')$ and $\pi_0(A) \to \pi_0(A'')$ surjective gives another pullback square.

We will write $\mathrm{Moduli}_k \subset \mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{an,art}}, \mathcal{S})$ for the ∞-category of formal moduli problems.

Exercise 8.4. *Show that for any formal moduli problem $D \in \mathrm{Moduli}_k$, the composite*

$$D^{\heartsuit} : \mathrm{CR}^{\mathrm{art}} \hookrightarrow \mathrm{CAlg}_k^{\mathrm{an,art}} \xrightarrow{D} \mathcal{S} \xrightarrow{\pi_0} \mathrm{Set}$$

defines a deformation functor in the sense of Definition 6.17.

Digression. To construct derived refinements of classical deformation functors, we will need a technique known as straightening.

We already briefly touched on unstraightening in Section 2.6: given a functor $F : J \to \mathbf{sSet}$ from an ordinary category J to the category $\mathbf{sSet}$ of simplicial sets, we constructed the relative nerve $N_F(J) \to N(J)$. If F lands in ∞-categories, then $N_F(J) \to N(J)$ is a cocartesian fibration.

One can show that this construction refines to an equivalence of ∞-categories

$$\mathrm{Fun}(\mathrm{N}(J), \mathrm{Cat}_\infty) \simeq \mathrm{Cat}_{\infty/\mathrm{N}(J)}^{\mathrm{cocart}}$$

between

(1) the ∞-category of functors from $\mathrm{N}(J)$ into the ∞-category Cat_∞ of small ∞-categories and
(2) the (non-full) subcategory $\mathrm{Cat}_{\infty/\mathrm{N}(J)}^{\mathrm{cocart}} \subset \mathrm{Cat}_{\infty/\mathrm{N}(J)}$ consisting of cocartesian fibrations

$$p : \mathcal{C} \to \mathrm{N}(J)$$

and those functors between them which preserve cocartesian morphisms.

In fact, there is a similar equivalence

$$\mathrm{Fun}(\mathcal{C}, \mathrm{Cat}_\infty) \simeq \mathrm{Cat}_{\infty/\mathcal{C}}^{\mathrm{cocart}} \tag{1}$$

for every ∞-category $\mathcal{C}$.

Given a cocartesian fibration $p : X \to \mathcal{C}$, the corresponding functor $F : \mathcal{C} \to \mathrm{Cat}_\infty$ satisfies

$$F(c) \simeq p^{-1}(\{c\}).$$

If $f : c \to c'$ is a morphism, the corresponding functor $F(f) : F(c) \to F(c')$ sends $x \in F(c)$ to the target of a p-cocartesian morphism of f starting at x.

Remark 8.5. Note that if $p : X \to S$ is a left fibration, the corresponding functor lands in the ∞-category $\mathcal{S}$ of spaces.

We are not able to do justice to the technically involved straightening/unstraightening equivalence (1) in these notes, and will treat it as a black box. A comprehensive treatment can be found in Section 3.2 in [Lur09], and we also recommend [HHR21] for a more concise treatment.

We can now construct a derived refinement of the deformation functor in Definition 7.13:

Example 8.6 (Derived infinitesimal deformations of varieties)**.** Let $(Z, \mathcal{O}_Z)$ be a smooth and proper variety over a field k. The (equal characteristic) derived deformation functor Def_X is the formal moduli problem sending $A \in \mathrm{CAlg}_k^{\mathrm{an,art}}$ to the space of pushouts

$$\begin{array}{ccc} A & \longrightarrow & k \\ \downarrow & \ulcorner & \downarrow \\ \mathcal{O}' & \longrightarrow & \mathcal{O}_Z. \end{array}$$

of sheaves of animated rings on Z.

To formally construct the formal moduli problem Def_X, we consider the ∞-category

$$\mathcal{C} := \mathrm{CAlg}^{\mathrm{an,art}} \times_{\mathrm{Shv}(Z,\mathrm{CAlg}^{\mathrm{an}})} \mathrm{Fun}(\Delta^1, \mathrm{Shv}(Z, \mathrm{CAlg}^{\mathrm{an}}))$$

consisting of pairs

$$(A, A \to \mathcal{O}')$$

of animated Artin local k-algebras A and maps of sheaves of animated rings $A \to \mathcal{O}'$. Here we abuse notation and identify A with the corresponding constant sheaf on Z.

Exercise 8.7. *Show that the projection $q : \mathcal{C} \to \mathrm{CAlg}^{\mathrm{an,art}}$ is a cocartesian fibration and describe its cocartesian morphisms.*

We now consider the non-full ∞-category $\mathcal{C}^{\mathrm{cocart}} \subset \mathcal{C}$ containing all objects and only the q-cocartesian morphisms. The cocartesian fibration q then restricts to a left fibration

$$q : \mathcal{C}^{\mathrm{cocart}}_{/(k, k\to\mathcal{O}_X)} \to \mathrm{CAlg}^{\mathrm{an,art}}.$$

Straightening gives a formal moduli problem $\mathrm{Def}_X : \mathrm{CAlg}^{\mathrm{an,art}} \to \mathcal{S}$.

Remark 8.8. Given a classical Artin local k-algebra A, the set $\pi_0(\mathrm{Def}_X(A))$ agrees with the value of the classical deformation functor in Definition 7.13 on A.

To make the correspondence between formal moduli problems and Lie algebras precise, we will extend the Koszul duality for commutative algebras discussed in Lecture 5 to formal moduli problems.

8.2. **The tangent fibre.** Recall from Lecture 5 that given an augmented commutative k-algebra R, its Koszul dual

$$\mathfrak{D}(R) = \cot(R)^\vee \simeq (k \otimes^{\mathbb{L}}_R L_{R/k})^\vee.$$

is defined by first computing its cotangent fibre $\cot(R) = k \otimes^{\mathbb{L}}_R L_{R/k}$, then taking the linear dual, and finally equipping it with additional Lie algebraic structure. It can be thought of as the derived tangent space of the pointed scheme $\mathrm{Spec}(R)$ at the k-point defined by the augmentation.

Note that there is an equivalence

$$\mathrm{Map}_{\mathrm{Mod}_k}(\cot(R), V) \simeq \mathrm{Map}_{\mathrm{CAlg}_k^{\mathrm{an,aug}}}(R, \mathrm{sqz}_k(V)),$$

and so $\cot(R)$ 'measures' maps into square-zero extensions $\mathrm{sqz}_k(V) = k \oplus V$.

We can in fact recover the chain complex $\cot(R)^\vee$ from the functor

$$\mathcal{F}_R = \mathrm{Map}_{\mathrm{CAlg}_k^{\mathrm{an}}}(R, -) : \mathrm{CAlg}_k^{\mathrm{an,art}} \to \mathcal{S}$$

corepresented by R. To this end, recall two basic facts from higher category theory:

(1) The ∞-category $\mathrm{Mod}_{k,\geq 0}$ of connective chain complexes over k is equivalent to $\mathcal{P}_\Sigma(\mathrm{Vect}_k^\omega)$, the ∞-category of finite-product-preserving functors from $(\mathrm{Vect}_k^\omega)^{op}$ to spaces.
(2) The ∞-category Mod_k of *all* chain complexes over k arises as the limit

$$\mathrm{Mod}_k \simeq \lim\left(\ldots \to \mathrm{Mod}_{k,\geq 0} \xrightarrow{\Omega} \mathrm{Mod}_{k,\geq 0} \xrightarrow{\Omega} \mathrm{Mod}_{k,\geq 0}\right),$$

where Ω is given by $M \mapsto \tau_{\geq 0}(M[-1])$.

As $\mathcal{F}_R$ is a formal moduli problem, (1) and (2) together imply that the sequence of functors

$$(\ \ldots\ ,\ \ \mathcal{F}_R(k \oplus (-)^\vee[2]),\ \ \mathcal{F}_R(k \oplus (-)^\vee[1]),\ \ \mathcal{F}_R(k \oplus (-)^\vee))$$

defines a chain complex over k.

Exercise 8.9. *Show that the resulting chain complex is equivalent to $\cot(R)^\vee$.*

Definition 8.10. Given a formal moduli problem $D : \mathrm{CAlg}_k^{\mathrm{an,art}} \to \mathcal{S}$, we define the tangent fibre

$$T_D \in \mathrm{Mod}_k$$

by applying the above construction to the functor D instead of $\mathcal{F}_R$.

Unravelling the definitions, we see that for any $V \in \mathrm{Vect}_k^{\omega}$ and any $n \geq 0$, we have an equivalence

$$\mathrm{Map}_{\mathrm{Mod}_k}(V, \tau_{\geq 0} T_D[n])) \simeq \mathcal{F}(k \oplus V^{\vee}[n]).$$

The tangent complex is a powerful invariant; for example, it detects equivalences:

Proposition 8.11 (Conservativity trick)**.** *A natural transformation of formally cohesive functors $\mathcal{F} \to \mathcal{G}$ is an equivalence if and only if the induced map $T_{\mathcal{F}} \to T_{\mathcal{G}}$ is a quasi-isomorphism.*

The tangent fibre is often straightforward to compute via automorphisms of trivial deformations:

Proposition 8.12 (Looping trick)**.** *Given a formal moduli problem D, there is a canonical equivalence of chain complexes over k:*

$$T_D \simeq \Sigma T_{\Omega D}$$

In particular, we have equivalences

$$\pi_n(T_D) \cong \begin{cases} \pi_n(D(k \oplus k[0])) & \text{if } n \geq 1 \\ \pi_1(D(k \oplus k[1-n])) & \text{if } n \leq 1 \end{cases},$$

Here the spaces $D(k \oplus k[i])$ are pointed by the canonical map $* \simeq D(k) \to D(k \oplus k[i])$ which picks out the trivial deformation.

Exercise 8.13 (Tangent fibre of Def_Z)**.** *Use the looping trick to show that the tangent fibre of the formal moduli problem Def_Z in is given by $R\Gamma(Z, T_Z)[1]$.*

8.3. **Formal moduli problens and Lie algebras.** Given a differential graded Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ and a local Artinian simplicial $\mathbb{C}$-algebra A with maximal ideal $\mathfrak{m}_A$, Hinich [Hin01] (generalising work of Goldman-Millson [GM88]) constructed a *space* of Maurer–Cartan elements

$$\mathrm{MC}(\mathfrak{g} \otimes \mathfrak{m}_A).$$

When A is discrete, $\pi_0(\mathrm{MC}(\mathfrak{m}_A \otimes \mathfrak{g}))$ is given by $\{x \in (\mathfrak{g}_Z)_{-1} \otimes \mathfrak{m}_A \,:\, dx + \frac{1}{2}[x,x] = 0\}/_{\substack{\text{gauge}\\ \text{equivalence}}}$.

In fact, one can show that this space is equivalent to the mapping space

$$\mathrm{Map}_{\mathrm{dgla}_k}(\mathfrak{D}^{\mathrm{dg}}(A), \mathfrak{g}),$$

where $\mathfrak{D}^{\mathrm{dg}}(A)$ is defined as in Definition 6.5. Varying both A and $\mathfrak{g}$, we obtain a functor

$$\Psi : \mathrm{dgla}_{\mathbb{C}} \to \mathrm{Moduli}_{\mathbb{C}},$$

$$\mathfrak{g} \mapsto \mathrm{Map}_{\mathrm{dgla}_k}(\mathfrak{D}^{\mathrm{dg}}(-), \mathfrak{g})$$

from differential graded Lie algebras to formal moduli problems. The following result by Lurie [Lur10b] and Pridham [Pri10], which generalises earlier work of many others including Kontsevich–Soibelman [KS02] and Manetti [Man09b], asserts that differential graded Lie algebras control derived (infinitesimal) deformation functors:

Theorem 8.14 (Lurie, Pridham)**.** *The functor*

$$\Psi : \mathrm{dgla}_{\mathbb{C}} \to \mathrm{Moduli}_{\mathbb{C}}, \;\; \mathfrak{g} \mapsto \mathrm{Map}_{\mathrm{dgla}_k}(\mathfrak{D}^{\mathrm{dg}}(-), \mathfrak{g})$$

defines an equivalence between the ∞-categories of formal moduli problems and differential graded Lie algebras over $\mathbb{C}$.

Given a formal moduli problem $D \in \mathrm{Moduli}_{\mathbb{C}}$, it is easy to describe the underlying spectrum of the associated differential graded Lie algebra: it is given by the shifted tangent fibre $T_D[-1]$, cf. Definition 8.10. Hence the Lurie–Pridham theorem promotes the tangent fibre construction to an equivalence.

Remark 8.15. Given a smooth and propver variety Z over $\mathbb{C}$, the equivalence in Theorem 8.14 sends the formal moduli problem Def_Z from Example 8.6 to the differential graded Lie algebra

$$\mathfrak{g}_Z = \left(\mathcal{A}^{0,0}(T_Z) \to \mathcal{A}^{0,1}(T_Z) \to \mathcal{A}^{0,2}(T_Z) \to \dots\right).$$

The Lie bracket is defined in terms of the wedge product of differential forms and the commutator of vector fields.

Remark 8.16. Theorem 8.14 in fact holds over any field of characteristic zero.

The equivalence in Theorem 8.14 admits a generalisation to fields k of characteristic p based on partition Lie algebras. Recall that in Lecture 5, we constructed a Koszul duality functor

$$\mathfrak{D} : \mathrm{CAlg}_k^{\mathrm{an,aug}} \to \mathrm{Alg}^{\mathrm{op}}_{\mathrm{Lie}_k^{\pi}}$$

from augmented animated k-algebras to partition Lie algebras, lifting the assignment

$$A \mapsto \cot(A)^{\vee} = (k \otimes_A L_{A/k})^{\vee}.$$

In lecture 6, we stated that $\mathfrak{D}$ restricts to a (contravariant) Koszul equivalence between complete local Noetherian objects and coconnective partition Lie algebras $\mathfrak{g}$ with $\dim(\pi_i(\mathfrak{g})) < \infty$ for all i.

In fact, partition Lie algebras satisfy the following gold standard property (cf. [BM19, Theorem 1.11]), which singles them out as the correct analogues of differential graded Lie algebras in characteristic p:

Theorem 8.17 ([BM19]))**.** *Given a field k of arbitrary characteristic, the functor*

$$\mathrm{Alg}_{\mathrm{Lie}_k^{\pi}} \simeq \mathrm{Moduli}_k, \;\; \mathfrak{g} \mapsto \mathrm{Map}_{\mathrm{dgla}_k}(\mathfrak{D}(-), \mathfrak{g})$$

defines an equivalence between the ∞-categories of partition Lie algebras and formal moduli problems over k.

Hence partition Lie algebras classify derived infinitesimal deformation functors, and thereby provide a useful tool in deformation theory.

Remark 8.18. For $D \in \mathrm{Moduli}_k$ a formal moduli problem over k, the underlying spectrum of the associated partition Lie algebra is given by the tangent fibre T_D of D.

Remark 8.19. Theorem 8.17 admits a generalisation to mixed characteristic formal moduli problems, see [BM19, Section 6].

Next week, we will discuss an application of the equivalence between Lie algebras and formal moduli problems to the theory of Calabi–Yau varieties.

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