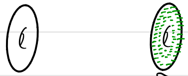
The Bogomolov-Tian-Todorov Theorem

(TeXed notes to follow)

__Warmup:__ Let X be a smooth + proper variety $/\mathbb{C}$.

Consider: $\mathrm{Def}_X^{\heartsuit} : \{\text{local Artin } \mathbb{C}\text{-algebras}\} \longrightarrow \text{Sets}$



$$A \longmapsto \begin{tikzcd} X \arrow[r] \arrow[d] & \tilde{X} \arrow[d, "\substack{\text{smooth}\\+\\\text{proper}}"] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

Kodaira–Spencer: $\mathrm{Def}_X^{\heartsuit}(\mathbb{C}[\varepsilon]/\varepsilon^2) \cong H^1(X, T_X)$

Question: Given $x \in H^1(X, T_X)$, does it extend to a deformation over $\mathbb{C}[\varepsilon]/\varepsilon^3$?

Insight: $H^*(X, T_X)$ is a graded Lie algebra ($\nearrow$ commutator bracket of vector fields)

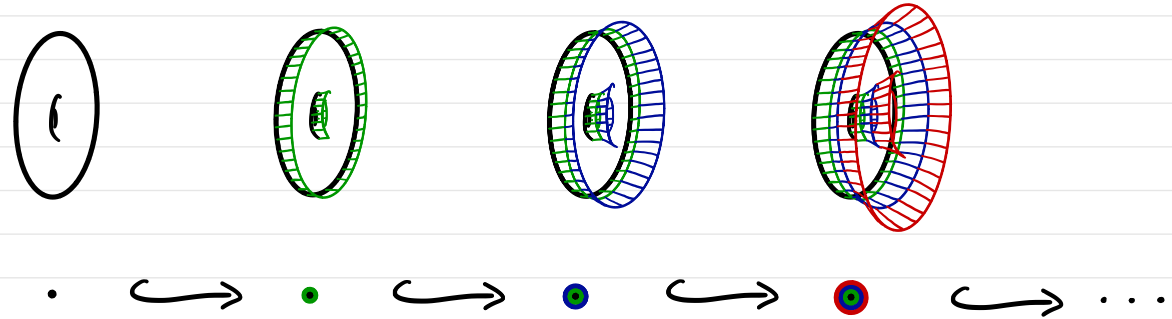
x extends $\Longleftrightarrow [x, x] = 0$.

Given a surjection $\hat{A} \twoheadrightarrow A$ of local Artin $\mathbb{C}$-algelbras with kernel I,

have obstruction classes in $H^2(X, T_X) \otimes I$

__Definition__: X is __unobstructed__ if for all $\hat{A} \twoheadrightarrow A$ as above,

$\mathrm{Def}_X^{\heartsuit}(\hat{A}) \twoheadrightarrow \mathrm{Def}_X^{\heartsuit}(A)$ is surjective.



$$\mathrm{Spec}(\mathbb{C}) \hookrightarrow \mathrm{Spec}(\mathbb{C}[\varepsilon]/\varepsilon^2) \hookrightarrow \mathrm{Spec}(\mathbb{C}[\varepsilon]/\varepsilon^3) \hookrightarrow \mathrm{Spec}(\mathbb{C}[\varepsilon]/\varepsilon^4) \hookrightarrow \cdots$$

A variety X is Calabi-Yau if its canonical bundle is trivial.

Examples: Elliptic curves, abelian / K3 surfaces,

Calabi-Yau threefolds, abelian varieties

"BTT"

<u>Theorem</u> (Bogomolov-Tian-Todorov)

Every Calabi-Yau variety $/\mathbb{C}$ is unobstructed.

<u>Note:</u> This is surprising as the obstruction group $H^2(X, T_X)$ need not vanish. dim: $h^{1(d-2)}$

<u>Ex:</u> The quintic threefold $\{[x_0 : x_1 : x_2 : x_3 : x_4] \mid x_0^5 + \dots + x_4^5 = 0\} \subseteq \mathbb{CP}^4$ has $h^{11} = 1$.

We will discuss a proof by Iacono-Manetti.

<u>Recap</u>: $H^*(X, T_X)$ refines to a differential graded Lie algebra

Dolbeault complex, Lie bracket defined using commutator of vector fields

$$\mathfrak{g}_X := R\Gamma(X, T_X) = \left(\mathcal{A}^{0,0}(T_X) \to \mathcal{A}^{0,1}(T_X) \to \dots\right)$$

The dgla $\mathfrak{g}_X$ also controls the derived deformation functor

$$\mathrm{Def}_X : \mathrm{CAlg}_{\mathbb{C}}^{\mathrm{an},\mathrm{art}} := \left\{ \begin{matrix} \text{simplicial Artin} \\ \text{local } \mathbb{C}\text{-algebras} \end{matrix} \right\} \longrightarrow \mathrm{Spaces}, \quad A \longmapsto \left\{ \begin{tikzcd} X \arrow[r, no head] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \tilde{X} \arrow[d, "\text{smooth + proper}"] \\ \operatorname{Spec} \mathbb{C} \arrow[r] & \operatorname{Spec} A \end{tikzcd} \right\}$$

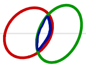
(simplicial = animated; $\tilde{X}$: derived scheme)

Functors like Def_X satisfy the following conditions:

<u>Def</u>^{n}: A formal moduli problem over $\mathbb{C}$ is a functor $D : \mathrm{CAlg}_{\mathbb{C}}^{\mathrm{an},\mathrm{art}} \longrightarrow \mathcal{S}$

such that (1) Normalisation: $D(\mathbb{C}) \simeq *$

(2) Gluing: For all pullbacks $\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A_0 \arrow[d] \\ A_1 \arrow[r] & A_{01} \end{tikzcd}$ with $\pi_0 A_0 \twoheadrightarrow \pi_0 A_{01}$ surjective, $\pi_0 A_1 \twoheadrightarrow \pi_0 A_{01}$



the square $\begin{tikzcd} DA \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & DA_0 \arrow[d] \\ DA_1 \arrow[r] & DA_{01} \end{tikzcd}$ is a pullback in spaces.

Write $\mathrm{Moduli}_{\mathbb{C}} \subseteq \mathrm{Fun}(\mathrm{CAlg}_{\mathbb{C}}^{\mathrm{an},\mathrm{art}}, \mathcal{S})$ for the ∞-category of formal moduli problems.

Every FMP has a tangent fibre $T_D \in Sp$, satisfying $\Omega^{\infty-n} T_D = D(\mathbb{C} \oplus \mathbb{C}[n])$.

The spectrum T_D of a FMP can be promoted to a $/\mathbb{C}$. In fact:

<u>Theorem</u> (Lurie-Pridham)

T_D can be promoted to a differential graded Lie algebra, and the resulting functor $\mathrm{Moduli}_{\mathbb{C}} \longrightarrow \mathrm{dgla}_{\mathbb{C}}$, $D \longmapsto T_D$ is an equivalence.

<u>Example</u>: This correspondence sends Def_X to $\mathfrak{g}_X$.

We now use this correspondence to prove the BTT theorem.

<u>Reduction</u>:

To prove unobstructedness of X, ETS: $\mathfrak{g}_X \simeq$ abelian dgla (vanishing bracket.)

To prove $\mathfrak{g}_X$ abelian, ETS that there is a map of dglas.

$$\mathfrak{g}_X \xrightarrow{2} \mathfrak{h}$$

s.t.
- $\pi_*(2)$ (homology) injective
- $\mathfrak{h}$ abelian

(if so, can find map of abelian dglas $\mathfrak{h} \to \mathfrak{k}$ s.t. $\mathfrak{g}_X \to \mathfrak{h} \to \mathfrak{k}$ is a quasi-isomorphism.)

To produce map $\mathfrak{g}_X \xrightarrow{?} \mathfrak{h}$ s.t. $\pi_X(?)$ injective, $\mathfrak{h}$ abelian,

we proceed in several steps.

(1) Every deformation $\tilde{X}$ of X over $A \in \mathrm{CAlg}_{\mathbb{C}}^{\mathrm{an,art}}$ gives a deformation of the map of chain complexes

$$\left(R\Gamma(X, \Omega_X^{\geq n}) \longrightarrow R\Gamma(X, \Omega_X^{\bullet})\right) = \left(dR_X^{\geq n} \longrightarrow dR_X\right)$$

(2) Varying A, obtain a diagram of formal moduli problems

$$\begin{tikzcd} & \mathrm{Def}_X \arrow[dl, bend right] \arrow[d] \arrow[dr] & \\ \mathrm{Fib}(p) \arrow[r] & \mathrm{Def}_{(dR_X^{\geq n} \to dR_X)} \arrow[r, "p"] & \mathrm{Def}_{dR_X} \end{tikzcd},$$

where the green arrow exists as dR_X deforms trivially (this step can fail in char p.)

(Gauss-Manin connection $\rightsquigarrow$ $H^i_{dR}(\tilde{X}/A) \simeq H^i_{dR}(X/k) \underset{\mathbb{C}}{\otimes} A$ for all def's $\tilde{X}$ of X over A.)

(3) Applying Koszul duality equivalence by Lurie-Pridham gives a diagram of dgla's

$$\begin{tikzcd}
& \mathfrak{g}_x \arrow[dl, bend right, "2"'] \arrow[d] \arrow[dr] & \\
\mathfrak{h} \arrow[r] & \mathfrak{g}_1 \arrow[r] & \mathfrak{g}_2
\end{tikzcd},$$

chain complex

Explicit models: $dR_X = R\Gamma(X, \Omega^\bullet_X) = (A^{\bullet,\bullet}_X, d) = \left(\bigoplus_{p,q} \Gamma(X, A^{p,q}_X), \ d = \partial + \bar\partial \right)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\cup$

$dR^{\geq n}_X = R\Gamma(X, \Omega^{\geq n}_X) = (A^{n,\bullet}_X, d)$

chain complex of $\mathbb{C}$-linear endos

Then $(\mathfrak{g}_1 \to \mathfrak{g}_2) \simeq \left(\mathrm{End}^{\times}_{A^{n,\bullet}}(A^{\bullet,\bullet}) \subseteq \mathrm{End}^{\bullet}(A^{\bullet,\bullet}_X) \right)$

preserves

$\mathrm{End}^i(V) = \{ V \to V[i] \}$

$[f, g] = fg - (-1)^{|f||g|} gf$.

$df = [d_V, f] = d_V f - (-1)^{|f|} f d_V$.

(4) Claim: $\mathfrak{g}_1 \to \mathfrak{g}_2$ is injective on π_0

$(\mathfrak{g}_1 \to \mathfrak{g}_2) \simeq \left(\mathrm{End}^{\times}_{A^{n,\bullet}}(A^{\bullet,\bullet}) \subseteq \mathrm{End}^{\bullet}(A^{\bullet,\bullet}_X) \right)$

By Hodge-de-Rham deg, $\pi_\bullet dR^{\geq n}_X \to \pi_\bullet dR_X$ is injective

$\qquad H^{\bullet-n}(X, \Omega^n) \longrightarrow H^\bullet(X, \Omega^\bullet_X)$

Pick a direct sum decomposition of chain cx's $dR_X = dR^{\geq n}_X \oplus dR_X / dR^{\geq n}_X$.

$\qquad = U \oplus V$.

Note $K = \{ f \in \mathrm{End}(dR_X) \mid \mathrm{im}(f) \subseteq V, \ f|_V = 0 \}$ are complementary

and $\mathrm{End}_U(dR_X)$

(5) Show $h = \mathrm{fib}(p)$ abelian.

$$\begin{tikzcd}
& \Omega g_2 \arrow[rr] \arrow[dl] \arrow[dd] & & 0 \arrow[dl] \arrow[dd] \\
\mathrm{fib}(p) \arrow[rr] \arrow[dd] & & g_1 \arrow[dd] & \\
& 0 \arrow[rr] \arrow[dl] & & g_2 \arrow[dl] \\
0 \arrow[rr] & & g_2 &
\end{tikzcd}$$

$\mathrm{fib}(p)$ admits an explicit description as

$\mathrm{fib}(p) = \{ (x, \; m(t,dt) = m_1 p(t) + m_2 q(t)\,dt) \in g_1 \times g_2[t,dt] \mid m(0) = 0, \; m(1) = f(x) . \}$

Get map of dglas $\mathrm{triv}(g_2)[-1] \longrightarrow \mathrm{fib}(p)$

$m \longmapsto (0, d(tm))$.

Get map $\Omega g_2 \longrightarrow \mathrm{fib}(p)$

Ωg_2 is abelian dgla.

$\pi_\bullet(-)$ surjective

So can find abelian dgla h + map $h \to \Omega g_2$ s.t. $h \to \mathrm{fib}(p)$ is an equivalence.

(6) Show $g_X \longrightarrow h = \mathrm{fib}(p)$ injective on π_*

$$h = \mathrm{fib} = \mathrm{Coker}\Big(\underset{\begin{pmatrix} a & B \\ 0 & c \end{pmatrix}}{\mathrm{End}_{dR_X^{2n}}(dR_X)} \subseteq \underset{\begin{pmatrix} A & B \\ C & D \end{pmatrix}}{\mathrm{End}(dR_X)} \Big)[-1] = \mathrm{Map}(dR_X^{2n}, dR_X / dR_X^{2n})[-1].$$

Now consider

$$\begin{tikzcd}
g_X = R\Gamma(X, T_X) \arrow[d, "q"'] \arrow[dr, bend left] & \\
\mathrm{Map}(dR^{2n}, dR_X^{2n-1} / dR_X^{2n}[-1]) \arrow[r, "r"] & h = \mathrm{fib} = \mathrm{Map}(dR^{2n}, dR_X / dR_X^{2n}[-1])
\end{tikzcd}$$

$\mathrm{Map}(dR^{2n}, dR_X^{2n-1} / dR_X^{2n}[-1]) \cong \mathrm{Map}(R\Gamma(X, \Omega^n), R\Gamma(X, \Omega^{n-1}))$

r is injective on π_* as HdR-deg $\Rightarrow$ $H^*(X, \Omega^{n-1}) \longrightarrow H^\bullet(X, \underbrace{\Omega_X / \Omega_X^n [-1]}_{dR_X / dR_X^{2n}[-1]})$ is an injection.

q is injective on π_* as $H^\bullet(X, T_X) \longrightarrow \mathrm{Map}(H^0(X, \Omega^n), H^*(X, \Omega^{n-1}))$ is inj

Ω_X^n trivial $\Rightarrow$ cup with global section gives iso $H^\bullet(X, T_X) \simeq H^\bullet(X, \Omega_X^{n-1})$

and we can write $H^\bullet(X, T_X) \longrightarrow \pi_\bullet(\mathrm{Map}(R\Gamma(X, \Omega^n), R\Gamma(X, \Omega^{n-1}))) \longrightarrow \mathrm{Map}(H^\bullet(X, \Omega^n), H^\bullet(X, \Omega^{n-1})) \longrightarrow \mathrm{Map}(H^0(X, \Omega^n), H^\bullet(X, \Omega^{n-1}))$

$\square$.