POINCARÉ–BIRKHOFF–WITT THEOREMS IN HIGHER ALGEBRA

OMAR ANTOLÍN-CAMARENA, LUKAS BRANTNER, AND GIJS HEUTS

Abstract. We extend the classical Poincaré–Birkhoff–Witt theorem to higher algebra by establishing a version that applies to spectral Lie algebras. We deduce this statement from a basic relation between operads in spectra: the commutative operad is the quotient of the associative operad by a right action of the spectral Lie operad. This statement, in turn, is a consequence of a fundamental relation between different $\mathbb{E}_n$-operads, which we articulate and prove. We deduce a variant of the Poincaré–Birkhoff–Witt theorem for relative enveloping algebras of $\mathbb{E}_n$-algebras. Our methods also give a simple construction and description of the higher enveloping $\mathbb{E}_n$-algebras of a spectral Lie algebra.

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1. Introduction

The classical Poincaré–Birkhoff–Witt theorem states that the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ admits a natural filtration whose associated graded is the symmetric algebra on the underlying vector space of $\mathfrak{g}$. This statement may be phrased in the language of operads; it is essentially equivalent to the fact that the quotient of the associative operad by the right action of the Lie operad on it is the commutative operad.

In this paper, we provide an extension of this statement to the world of higher algebra (see Theorem 1.2), replacing the classical associative, Lie, and commutative operads by their lifts to the ∞-category of spectra. Correspondingly, we deduce a Poincaré–Birkhoff–Witt theorem for spectral Lie algebras (Corollary 1.10). In fact, these results are consequences of a more basic relation between different $\mathbb{E}_n$-operads, namely Theorem 1.6, which has no 'classical' analog. In particular we deduce a result in the style of Poincaré–Birkhoff–Witt for $\mathbb{E}_n$-algebras, which we make explicit in Corollary 1.11. Finally, our methods provide a straightforward construction of the higher enveloping algebra of a spectral Lie algebra $\mathfrak{g}$. In Theorem 1.8 we show that the $\mathbb{E}_n$-enveloping algebra of $\mathfrak{g}$ may be calculated as the Chevalley–Eilenberg homology of the n-fold loop object of $\mathfrak{g}$, showing that our construction agrees with the one of Ayala-Francis [AF15] and Knudsen [Knu18] which relies on factorisation homology.

We will now review the results of this paper in more detail. The associative and commutative operads have evident lifts to the world of higher algebra; indeed, they can easily be defined in the ∞-category of spectra (or any symmetric monoidal ∞-category with coproducts). The case of the Lie operad is less straightforward, but Salvatore [Sal98] and Ching [Chi05] have shown that it also admits a lift to the ∞-category of spectra. Indeed, if we define $\mathbb{L}$ to be the Koszul dual $K(\mathbf{Com})$

of the commutative operad, then the operadic suspension $s\mathbb{L}$ (see Section 2) is an operad whose homology is concentrated in degree 0 and reproduces the classical Lie operad in abelian groups. Moreover, this construction recovers the more classical Koszul duality between the commutative and Lie operads in the derived ∞-category Mod_R of any commutative ring R.

To explain our first result, recall that operads in the ∞-category Sp of spectra may be defined as asociative algebra objects in the ∞-category of symmetric sequences in spectra. The monoidal structure is given by the *composition product* (see Section 2 for a brief review). In particular, given a map of operads $\mathcal{O} \to \mathcal{P}$, we may interpret $\mathcal{P}$ as an $\mathcal{O}$-bimodule in symmetric sequences.

Notation 1.1. In what follows, we will denote the associative operad **Ass** by $\mathbb{E}_1$ and the commutative operad **Com** by $\mathbb{E}_\infty$, to make the notation consistent with our later results on $\mathbb{E}_n$-operads. We take all of our operads to be *nonunital* (without emphasising this in the notation), meaning they have no term of 0-ary operations. We will denote by **1** the trivial operad, which has a unit in arity 1 and nothing else. This is also the nonunital $\mathbb{E}_0$-operad.

We can now state our first main theorem:

Theorem 1.2. *There is a commutative square of operads in the ∞-category* Sp *as follows:*

$$\begin{array}{ccc} s\mathbb{L} & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow \\ \mathbb{E}_1 & \longrightarrow & \mathbb{E}_\infty. \end{array}$$

Moreover, the induced map $\mathbb{E}_1 \circ_{s\mathbb{L}} \mathbf{1} \to \mathbb{E}_\infty$ is an equivalence of left $\mathbb{E}_1$-modules.

Remark 1.3. In the statement of the theorem, $\mathbb{E}_1 \circ_{s\mathbb{L}} \mathbf{1}$ denotes the *relative composition product* of $\mathbb{E}_1$ and **1** over $s\mathbb{L}$. For an operad $\mathcal{O}$ with a right module $\mathcal{M}$ and a left module $\mathcal{N}$, such a relative composition product may be computed via the bar construction

$$\mathcal{M} \circ_{\mathcal{O}} \mathcal{N} = \varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} (\mathcal{M} \circ \mathcal{O}^{\circ \bullet} \circ \mathcal{N}).$$

Remark 1.4. The square of Theorem 1.2 is *not* a pushout square of operads (cf. Proposition 4.2).

In fact, Theorem 1.2 will follow as a limiting case of a statement about the relation between different $\mathbb{E}_n$-operads (see Theorem 1.6 below). To be precise, if $\mathbb{E}_n$ denotes the usual (nonunital) operad of little n-cubes in the ∞-category of spaces, then we will be interested in the operad $\Sigma^\infty_+ \mathbb{E}_n$ in spectra. Throughout this paper, we will use the short-hand $\mathbb{E}_n$ for this nonunital, stable version of the $\mathbb{E}_n$-operad. Given integers $m, n \geq 0$, we will generically use the letter ι to denote the usual morphism

$$\mathbb{E}_m \to \mathbb{E}_{m+n}$$

arising from the standard inclusion $\mathbb{R}^m \to \mathbb{R}^{m+n}$ of Euclidean spaces.

Definition 1.5 (Composition squares)**.** A commutative square of operads

$$\begin{array}{ccc} \mathcal{O} & \longrightarrow & \mathcal{P} \\ \downarrow & & \downarrow \\ \mathcal{Q} & \longrightarrow & \mathcal{R} \end{array}$$

is called a *composition square* if the induced map $\mathcal{Q} \circ_{\mathcal{O}} \mathcal{P} \to \mathcal{R}$ is an equivalence (of $\mathcal{Q}$-$\mathcal{P}$-bimodules).

We then establish the following fundamental relation between different $\mathbb{E}_n$-operads, with s denoting operadic suspension (see Section 2):

Theorem 1.6. *Given integers $k, m, n \geq 0$ there is a composition square of operads in spectra*

$$\begin{array}{ccc} \mathbb{E}_{k+m} & \xrightarrow{\iota} & \mathbb{E}_{k+m+n} \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} \\ s^k\mathbb{E}_m & \xrightarrow{\iota} & s^k\mathbb{E}_{m+n}, \end{array}$$

which means that the induced map $s^k\mathbb{E}_m \circ_{\mathbb{E}_{k+m}} \mathbb{E}_{k+m+n} \xrightarrow{\simeq} s^k\mathbb{E}_{m+n}$ *is an equivalence.*

We will formally describe the morphism $\beta\colon \mathbb{E}_{m+k} \to s^k\mathbb{E}_m$ in detail in Section 3. For now, let us outline two ways of thinking about it. Without loss of generality we take $k = 1$ – the general case is obtained by a k-fold composition of such maps. First, β can be obtained as the *Koszul dual* of the morphism $\iota\colon \mathbb{E}_m \to \mathbb{E}_{m+1}$, relying on the fact that the Koszul dual of the operad $\mathbb{E}_m$ is the operad $s^{-m}\mathbb{E}_m$, see Remark 3.2. Secondly, under the 'degree shifting' equivalence $\mathrm{Alg}_{s\mathbb{E}_m}(\mathrm{Sp}) \simeq \mathrm{Alg}_{\mathbb{E}_m}(\mathrm{Sp})$, the morphism β induces a left adjoint functor

$$\beta_!\colon \mathrm{Alg}_{\mathbb{E}_{m+1}}(\mathrm{Sp}) \to \mathrm{Alg}_{\mathbb{E}_m}(\mathrm{Sp}).$$

which may be identified with the usual bar construction. This second perspective is the one used to construct and characterise β in [HL24].

In addition to Theorem 1.2, we will deduce further limiting cases of Theorem 1.6:

Theorem 1.7. *The following are composition squares of operads, where σ denotes the suspension map (see Section 2):*

$$\begin{array}{ccc} \mathbb{L} & \xrightarrow{\sigma^n} & s^n\mathbb{L} \\ \downarrow & & \downarrow{\scriptstyle\beta} \\ \mathbf{1} & \longrightarrow & \mathbb{E}_n \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{E}_n & \xrightarrow{\iota} & \mathbb{E}_\infty \\ \downarrow & & \downarrow{\scriptstyle\sigma^n} \\ \mathbf{1} & \longrightarrow & s^n\mathbb{E}_\infty. \end{array}$$

The morphism $\beta\colon s^n\mathbb{L} \to \mathbb{E}_n$ featuring in Theorem 1.7 can be obtained as an inverse limit over k of the morphisms $\beta\colon s^{-k}\mathbb{E}_{n+k} \to \mathbb{E}_n$. Alternatively, it can be described as (a shift of) the Koszul dual of the inclusion $\iota\colon \mathbb{E}_n \to \mathbb{E}_\infty$, see Remark 3.2. Identifying the ∞-categories $\mathrm{Alg}_{s^n\mathbb{L}}(\mathrm{Sp})$ and $\mathrm{Alg}_{\mathbb{L}}(\mathrm{Sp})$ via an n-fold degree shift, this morphism induces a left adjoint functor

$$\beta_!\colon \mathrm{Alg}_{\mathbb{L}}(\mathrm{Sp}) \to \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Sp})$$

which we refer to as the $\mathbb{E}_n$*-enveloping algebra* functor and denote by U_n.

A different construction of such a higher enveloping algebra has also been studied by Ayala–Francis [AF15] and Knudsen [Knu18] using the methods of factorization homology. In the special case $n = 1$, the functor U_1 can be thought of as a lift of the classical universal enveloping algebra functor to spectral Lie algebras. The degenerate case $n = 0$ gives the pushforward along the augmentation map $\mathbb{L} \to \mathbf{1}$, which shall be denoted by CE as it is an enhancement of the classical Chevalley–Eilenberg functor.

We will show that the first composition square of Theorem 1.7 easily leads to Theorem 1.8 below. In particular, it shows that our U_n is naturally equivalent to Knudsen's, since the main theorem of [Knu18] provides the same description for his functor.

Theorem 1.8. *Given a spectral Lie algebra $\mathfrak{g} \in \mathrm{Alg}_{\mathbb{L}}(\mathrm{Sp})$ there is a natural equivalence*

$$U_n(\mathfrak{g}) \simeq \mathrm{CE}(\Omega^n\mathfrak{g}).$$

Remark 1.9. The first square of Theorem 1.7 gives an equivalence $\mathbb{E}_n \simeq \mathbf{1} \circ_{\mathbb{L}} s^n\mathbb{L}$. For a given spectrum X, the skeletal filtration on the bar construction $\mathbf{1} \circ_{\mathbb{L}} s^n\mathbb{L}$ therefore gives a spectral sequence converging to $\mathrm{free}_{\mathbb{E}_n}(X)$, which is the one used by Brantner–Hahn–Knudsen [BHK24] to study the (generalised) homology of $\mathbb{E}_n$-algebras.

Theorem 1.2 and Theorem 1.6 imply the following generalisations of the classical PBW theorem via a filtration trick we learned from [GR17]:

Corollary 1.10 (PBW theorem for spectral Lie algebras)**.** *Given a spectral Lie algebra $\mathfrak{g}$, the universal enveloping algebra $U\mathfrak{g}$ admits an exhaustive filtration with associated graded*

$$\mathrm{gr}(U\mathfrak{g}) \simeq \mathrm{free}_{\mathbb{E}_\infty}(\mathrm{forget}(\mathfrak{g})).$$

Corollary 1.11 (PBW theorem for $\mathbb{E}_n$-algebras)**.** *For A an $\mathbb{E}_n$-algebra, the relative enveloping algebra $U_k(A) \in \mathrm{Alg}_{s^k\mathbb{E}_{n-k}}(\mathrm{Sp})$ admits an exhaustive filtration with associated graded*

$$\mathrm{gr}(U_k(A)) \simeq \mathrm{free}_{s^k\mathbb{E}_k^\vee}(\mathrm{forget}(A)).$$

Equivalently, the k-fold bar construction $\mathrm{Bar}^k A$ *admits an exhaustive filtration with associated graded*

$$\mathrm{gr}(\mathrm{Bar}^k A) \simeq \mathrm{free}_{\mathbb{E}_k^\vee}(\Sigma^k \, \mathrm{forget}(A)).$$

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2. Symmetric sequences and operads

In this section, we give a brisk review of the basic facts about symmetric sequences and operads we shall require. We take the point of view that an operad is an algebra for the monoidal structure given by the composition product on the category of symmetric sequences.

To make this precise, write $\mathrm{Fin}^{\simeq}$ for the groupoid of finite sets and bijections.

Definition 2.1. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. The ∞-category of *symmetric sequences* in $\mathcal{C}$ is given by

$$\mathrm{sSeq}(\mathcal{C}) := \mathrm{Fun}(\mathrm{Fin}^{\simeq}, \mathcal{C}).$$

The value of a symmetric sequence A on a set with n elements will be denoted by $A(n)$ and it is an object of $\mathcal{C}$ equipped with an action of the symmetric group Σ_n.

In this paper, it will be sufficient to consider the case where $\mathcal{C}$ is the ∞-category Sp of spectra, although most of what we say will go through in much greater generality. We will need to consider three different monoidal structures on the ∞-category sSeq(Sp).

Day convolution product. The first is given by *Day convolution* (cf. [Gla16] [HA, 2.2.6]) based on the disjoint union monoidal structure on $\mathrm{Fin}^{\simeq}$. Explicitly, it is determined by the formula

$$(A \otimes B)(n) = \bigoplus_{a+b=n} \Sigma^\infty_+ \Sigma_n \otimes_{h(\Sigma_a \times \Sigma_b)} A(a) \otimes B(b).$$

The functor $\iota : \mathrm{Sp} \to \mathrm{sSeq}(\mathrm{Sp})$ given by $\iota(X)(0) = X$ and $\iota(X)(n) = 0$ for $n > 0$ is symmetric monoidal for this Day convolution structure, and we obtain a tensoring of sSeq(Sp) over Sp.

Composition product. The second monoidal structure is the *composition product*, which is defined for sSeq(Sp) arguing from universal properties in [Bra, 4.1.2][BCN25, 3.1], following the 1-categorical argument of Carboni described by Kelly [Kel05] and Trimble [Tri]. On objects, the composition product is given by

$$A \circ B = \bigoplus_{n \geq 0} \left(A(n) \otimes B^{\otimes n}\right)_{h\Sigma_n}.$$

One can check that for a spectrum X and symmetric sequence A, the composition $A \circ \iota(X)$ is concentrated in degree 0, that is, it lies in the essential image of ι. Therefore, for a fixed A there is a functor $\mathrm{free}_A \in \mathrm{End}(\mathrm{Sp})$ satisfying $\iota(\mathrm{free}_A(X)) = A \circ \iota(X)$. Explicitly, it is given by

$$\mathrm{free}_A(X) = \bigoplus_{n \geq 0} \left(A(n) \otimes X^{\otimes n}\right)_{h\Sigma_n}.$$

Since this action of A on spectra is given simply by restricting the composition product to the essential image of ι, the functor free: $\mathrm{sSeq}(\mathrm{Sp}) \to \mathrm{End}(\mathrm{Sp})$ is monoidal, where End(Sp) is endowed with the monoidal structure given by composition of functors.

Definition 2.2. *Operads* in spectra are defined to be algebra objects in the monoidal ∞-category $(\mathrm{sSeq}(\mathrm{Sp}), \circ)$, and the ∞-category they form is denoted by $\mathrm{Op}(\mathrm{Sp}) = \mathrm{Alg}(\mathrm{sSeq}(\mathrm{Sp}))$.

The free functor described above therefore induces a functor free: $\mathrm{Op}(\mathrm{Sp}) \to \mathrm{Monad}(\mathrm{Sp})$. One can consider left modules, right modules, and bimodules over an operad $\mathcal{O}$ in $\mathrm{sSeq}(\mathcal{O})$; a left module of the form $\iota(X)$ is equivalently an algebra for the monad $\mathrm{free}_{\mathcal{O}}$ and one says that X is an $\mathcal{O}$-*algebra*. We denote the ∞-category of $\mathcal{O}$-algebras by $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp})$ or simply $\mathrm{Alg}_{\mathcal{O}}$ if no confusion can arise.

Levelwise product. The third and simplest monoidal structure on $\mathrm{sSeq}(\mathrm{Sp})$ we will need is the *levelwise tensor product*, which is simply given by

$$(A \otimes_{\mathrm{lev}} B)(n) = A(n) \otimes B(n).$$

We will use this monoidal structure as an auxiliary tool in our discussion of the operadic suspension and the suspension morphism. The main property we will need is the fact that

$$\otimes_{\mathrm{lev}} : (\mathrm{sSeq}(\mathrm{Sp}) \times \mathrm{sSeq}(\mathrm{Sp}), \circ \times \circ) \to (\mathrm{sSeq}(\mathrm{Sp}), \circ)$$

has a lax monoidal structure [BCN25, Proposition 3.9]. As a consequence, $\otimes_{\mathrm{lev}}$ induces a functor

$$\mathrm{Op}(\mathrm{Sp}) \times \mathrm{Op}(\mathrm{Sp}) \to \mathrm{Op}(\mathrm{Sp})$$

so that given two operads $\mathcal{P}$ and $\mathcal{Q}$, the symmetric sequence $\mathcal{P} \otimes_{\mathrm{lev}} \mathcal{Q}$ is again an operad. On the level of algebras, we obtain a functor

$$\mathrm{Alg}_{\mathcal{P}} \times \mathrm{Alg}_{\mathcal{Q}} \to \mathrm{Alg}_{\mathcal{P} \otimes_{\mathrm{lev}} \mathcal{Q}} \tag{1}$$

sending a pair (A, B) to $A \otimes B$. This functor varies naturally in the pair $(\mathcal{P}, \mathcal{Q})$.

Suspension functor. We now turn our attention to the operation of *operadic suspension.* The suspension of a nonunital operad in spectra $\mathcal{O}$ is an operad $s\mathcal{O}$ such that the corresponding free algebra monads satisfy

$$\mathrm{free}_{s\mathcal{O}} \simeq \Sigma^{-1} \circ \mathrm{free}_{\mathcal{O}} \circ \Sigma,$$

and consequently, $s\mathcal{O}$-algebra structures on X are in one-to-one correspondence $\mathcal{O}$-algebra structures on ΣX. The underlying symmetric sequence of the operadic suspension is given by

$$(s\mathcal{O})(n) = \Sigma^{-1}(\mathbb{S}^1)^{\otimes n} \otimes \mathcal{O}(n),$$

where $\mathbb{S}^1$ is the suspension of the sphere spectrum and the Σ_n action in the left factor permutes the $\mathbb{S}^1$ factors and acts trivially on the suspension coordinate.

To describe the operad structure of $s\mathcal{O}$, consider the endomorphism operad

$$s\mathbb{E}_\infty := \mathrm{End}(\mathbb{S}^{-1})$$

of $\mathbb{S}^{-1}$; our notation reflects the fact that it will be the operadic suspension of the commutative operad. The spectrum of n-ary operations in $s\mathbb{E}_\infty$ is given by $\mathrm{map}(\mathbb{S}^{-n}, \mathbb{S}^{-1}) \simeq \mathbb{S}^{n-1}$, and tracing through the action of Σ_n on $\mathbb{S}^{-n} \simeq (\mathbb{S}^{-1})^{\wedge n}$ we see that the action of Σ_n on $s\mathbb{E}_\infty(n) \simeq \Sigma^{-1}(\mathbb{S}^1)^{\wedge n}$ is the one described above. (Equivalently, $s\mathbb{E}_\infty(n)$ is the representation sphere of ρ, the quotient of the standard n-dimensional permutation representation by its diagonal.) Using the levelwise tensor product, we define:

Definition 2.3. The operadic suspension functor is defined as

$$s := s\mathbb{E}_\infty \otimes_{\mathrm{lev}} (-) : \mathrm{Op}(\mathrm{Sp}) \to \mathrm{Op}(\mathrm{Sp}).$$

On the level of algebras, tensoring with the canonical $s\mathbb{E}_\infty$-algebra $\mathbb{S}^{-1}$ (whose structure map $s\mathbb{E}_\infty \to \mathrm{End}(\mathbb{S}^{-1})$ is the identity) gives a functor

$$\mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}) \to \mathrm{Alg}_{s\mathcal{O}}(\mathrm{Sp}), \quad X \;\mapsto\; \mathbb{S}^{-1} \otimes X \tag{2}$$

via (1). There is also an operadic desuspension operation s^{-1} defined by tensoring with $\mathrm{End}(\mathbb{S}^1)$, which is inverse to s. On the level of algebras, this shows that the morphism in (2) is an equivalence of ∞-categories.

Suspension morphism. Given an operad $\mathcal{O}$ in spectra, we will also need the *suspension morphism*

$$\sigma\colon \mathcal{O} \to s\mathcal{O}.$$

In the special case of $\mathcal{O} = \mathbb{E}_\infty$, the map $\sigma\colon \mathbb{E}_\infty \to \mathrm{End}(\mathbb{S}^{-1})$ is the map endowing $\mathbb{S}^{-1}$ with the $\mathbb{E}_\infty$-ring structure of the reduced spherical cochains of S^1, which can also be described as $\Omega_{\mathbb{E}_\infty}\mathbb{S}^0$. Here $\mathbb{S}^0$, being the monoidal unit of Sp, is equipped with its canonical $\mathbb{E}_\infty$-algebra stucture and $\Omega_{\mathbb{E}_\infty}$ is the loop functor on the ∞-category $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp})$. Notice that by construction this $\mathbb{E}_\infty$-ring structure on $\mathbb{S}^{-1}$ is the image under σ^* of the canonical $s\mathbb{E}_\infty$-algebra structure on $\mathbb{S}^{-1}$ used above.

For a general operad $\mathcal{O}$ the definition uses the the previous map σ:

Definition 2.4. Given a nonunital operad $\mathcal{O}$ in spectra, the suspension morphism is the composite

$$\sigma\colon \mathcal{O} \simeq \mathcal{O} \otimes_{\mathrm{lev}} \mathbb{E}_\infty \xrightarrow{\mathrm{id}\otimes_{\mathrm{lev}}\sigma} \mathcal{O} \otimes_{\mathrm{lev}} s\mathbb{E}_\infty \simeq s\mathcal{O}.$$

Since $\mathcal{O}$ is nonunital (i.e. $\mathcal{O}(0) = 0$), the free $\mathcal{O}$-algebra on the zero spectrum 0 has underlying spectrum $\mathrm{free}_{\mathcal{O}}(0) = 0$ and it is a zero object in $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp})$. This implies that the loop functor $\Omega_{\mathcal{O}} : \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}) \to \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp})$ and the forgetful functor forget : $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}) \to \mathrm{Sp}$ satisfy

$$\mathrm{forget} \circ \Omega_{\mathcal{O}} \simeq \Omega \circ \mathrm{forget}\,.$$

Proposition 2.5. *Given a nonunital operad $\mathcal{O}$, the composite*

$$\mathrm{Alg}_{\mathcal{O}} \xrightarrow{\mathbb{S}^{-1}\otimes -} \mathrm{Alg}_{s\mathcal{O}} \xrightarrow{\sigma^*} \mathrm{Alg}_{\mathcal{O}}$$

is equivalent to the loops functor $\Omega_{\mathcal{O}}$. Here $\mathbb{S}^{-1} \otimes -$ is the equivalence in (2) *and σ^* is induced by restricting along the suspension morphism $\sigma\colon \mathcal{O} \to s\mathcal{O}$.*

Proof. The equivalence of operads $\mathcal{O}\otimes_{\mathrm{lev}}\mathbb{E}_\infty \simeq \mathcal{O}$ gives a functor

$$\mathrm{Alg}_{\mathcal{O}} \times \mathrm{Alg}_{\mathbb{E}_\infty} \to \mathrm{Alg}_{\mathcal{O}}$$

via (1). Since $U_{\mathcal{O}}$ creates limits, tensoring with the $\mathbb{E}_\infty$-ring spectrum $\Omega_{\mathbb{E}_\infty}\mathbb{S}^0$ recovers the loops functor $\Omega_{\mathcal{O}}$. The map of pairs of operads

$$(\mathrm{id}, \sigma)\colon (\mathcal{O}, \mathbb{E}_\infty) \to (\mathcal{O}, s\mathbb{E}_\infty)$$

gives rise to a commutative square of ∞-categories

$$\begin{array}{ccc} \mathrm{Alg}_{\mathcal{O}} \times \mathrm{Alg}_{s\mathbb{E}_\infty} & \xrightarrow{\mathrm{id}\times\sigma^*} & \mathrm{Alg}_{\mathcal{O}} \times \mathrm{Alg}_{\mathbb{E}_\infty} \\ \downarrow{\otimes} & & \downarrow{\otimes} \\ \mathrm{Alg}_{s\mathbb{E}_\infty\otimes\mathcal{O}} & \xrightarrow{\sigma^*} & \mathrm{Alg}_{\mathbb{E}_\infty\otimes\mathcal{O}} \end{array}$$

Given an $\mathcal{O}$-algebra X, tracing the pair $(X, \mathbb{S}^{-1})$ through this square gives a natural equivalence

$$\sigma^*(X \otimes \mathbb{S}^{-1}) \simeq X \otimes \Omega_{\mathbb{E}_\infty}\mathbb{S}^0 \simeq \Omega_{\mathcal{O}}X.$$

□

Remark 2.6. Heuts and Land [HL24] construct a suspension morphism $\sigma\colon \mathbb{E}_n \to s\mathbb{E}_n$ in the specific case of the $\mathbb{E}_n$-operad and show that it is characterised essentially uniquely by the fact that it satisfies the conclusion of Proposition 2.5. Since the morphism σ that we constructed above also satisfies that conclusion, it follows that the two constructions of σ agree when they are both defined.

3. Proof of the main result

The aim of this section is to prove Theorem 1.6. In the next section we work out several consequences and special cases, including Theorem 1.2.

To prove Theorem 1.6 it will suffice to establish the following special case: there exists a commutative square

$$\begin{array}{ccc} \mathbb{E}_n & \xrightarrow{\iota} & \mathbb{E}_{n+1} \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} \\ s\mathbb{E}_{n-1} & \xrightarrow{\iota} & s\mathbb{E}_n \end{array}$$

inducing an equivalence

$$s\mathbb{E}_{n-1} \circ_{\mathbb{E}_n} \mathbb{E}_{n+1} \xrightarrow{\simeq} s\mathbb{E}_n$$

of $(s\mathbb{E}_{n-1}, \mathbb{E}_{n+1})$-bimodules. Indeed, the general case of Theorem 1.6 follows by composing copies of this basic square horizontally and/or vertically. The relevant square was constructed by Land and the third author in [HL24, Theorem 3.11]. To summarise, this goes as follows. First one establishes a commutative diagram of left adjoint functors

$$\begin{array}{ccc} \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Sp}) & \xrightarrow{\iota_!} & \mathrm{Alg}_{\mathbb{E}_{n+1}}(\mathrm{Sp}) \\ \downarrow{\scriptstyle\mathrm{Bar}} & & \downarrow{\scriptstyle\mathrm{Bar}} \\ \mathrm{Alg}_{\mathbb{E}_{n-1}}(\mathrm{Sp}) & \xrightarrow{\iota_!} & \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Sp}). \end{array} \tag{3}$$

All of these ∞-categories are monadic over Sp, giving a corresponding square of monads. The functor assigning to an operad in Sp its corresponding monad on Sp is fully faithful on the subcategory of $\mathbb{E}_n$-operads and their (de)suspensions [HL24, Theorem 3.8], so that the given square corresponds essentially uniquely to the desired square of operads. We record the following (which is part of the statement of [HL24, Theorem 3.11]) for later use:

Lemma 3.1. *In the square of operads above we have $\beta \circ \iota \simeq \sigma \simeq \iota \circ \beta$, where $\sigma\colon \mathbb{E}_n \to s\mathbb{E}_n$ denotes the suspension morphism.*

Remark 3.2. Let us write $K\colon \mathrm{Op}(\mathrm{Sp}) \to \mathrm{coOp}(\mathrm{Sp})$ for the Koszul duality functor that takes an operad $\mathcal{O}$ first to its bar construction $\mathrm{Bar}(\mathcal{O})$ (which is a cooperad) and then to the termwise Spanier–Whitehead dual $\mathrm{Bar}(\mathcal{O})^\vee$, which is an operad. It is a theorem of Ching–Salvatore [CS22] that $K\mathbb{E}_n \cong s^{-n}\mathbb{E}_n$, see also Malin's proof [Mal23] or the forthcoming [HL] for an ∞-categorical version. Under this identification, the Koszul dual of the morphism $\iota\colon \mathbb{E}_n \to \mathbb{E}_{n+1}$ is (up to an $n+1$-fold shift) precisely the morphism $\beta\colon \mathbb{E}_{n+1} \to s\mathbb{E}_n$ featuring above. A proof of this fact will appear in [HL].

Before proving Theorem 1.6 it will be convenient to establish a certain recognition criterion for composition squares. Let

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{f} & \mathcal{P} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ \mathcal{Q} & \xrightarrow{f'} & \mathcal{R} \end{array} \tag{4}$$

be a commutative square of operads in spectra. It induces a corresponding square of left adjoint functors between algebra categories

$$\begin{array}{ccc} \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}) & \xrightarrow{f_!} & \mathrm{Alg}_{\mathcal{P}}(\mathrm{Sp}) \\ \downarrow{\scriptstyle g_!} & & \downarrow{\scriptstyle g'_!} \\ \mathrm{Alg}_{\mathcal{Q}}(\mathrm{Sp}) & \xrightarrow{f'_!} & \mathrm{Alg}_{\mathcal{R}}(\mathrm{Sp}). \end{array}$$

Associated to this square is another, namely:

$$\begin{array}{ccc} \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}) & \xleftarrow{f^*} & \mathrm{Alg}_{\mathcal{P}}(\mathrm{Sp}) \\ \downarrow g_! & \Rrightarrow & \downarrow g'_! \\ \mathrm{Alg}_{\mathcal{Q}}(\mathrm{Sp}) & \xleftarrow{(f')^*} & \mathrm{Alg}_{\mathcal{R}}(\mathrm{Sp}). \end{array} \tag{5}$$

This last square is generally only lax commutative, in the sense that there is a natural transformation $g_! \circ f^* \Rightarrow (f')^* g'_!$. This natural transformation is the adjoint of the composite

$$f'_! \circ g_! \circ f^* \cong g'_! \circ f_! \circ f^* \Rightarrow g'_!$$

where the arrow arises from the counit of the adjoint pair $(f_!, f^*)$.

Lemma 3.3. *The square of operads (4) is a composition square if and only if the lax commutative square (5) is commutative, i.e., if the natural transformation $g_! \circ f^* \Rightarrow (f')^* g'_!$ is an equivalence.*

Proof. For a $\mathcal{P}$-algebra X, the natural transformation of the lemma can be identified as the evident map

$$\mathcal{Q} \circ_{\mathcal{O}} X \to \mathcal{R} \circ_{\mathcal{P}} X.$$

On the left-hand side X is implicitly regarded as an $\mathcal{O}$-algebra via f^*. Since both expressions preserve sifted colimits in the variable X, the map is an equivalence if and only if it is so in the special case of free $\mathcal{P}$-algebras $X = \mathcal{P} \circ A$ with $A \in \mathrm{Sp}$. Then it reduces to the natural map

$$(\mathcal{Q} \circ_{\mathcal{O}} \mathcal{P}) \circ A \to \mathcal{R} \circ A.$$

This is a natural equivalence if and only if the underlying map of symmetric sequences $\mathcal{Q} \circ_{\mathcal{O}} \mathcal{P} \to \mathcal{R}$ is an equivalence; indeed, the 'if'-direction is clear, whereas the 'only if' follows by taking the Goodwillie derivatives of the natural transformation above. □

Proof of Theorem 1.6. In the commutative square (3) we can take right adjoints of the horizontal functors to obtain the lax square

$$\begin{array}{ccc} \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Sp}) & \xleftarrow{\iota^*} & \mathrm{Alg}_{\mathbb{E}_{n+1}}(\mathrm{Sp}) \\ \downarrow \mathrm{Bar} & \Rrightarrow & \downarrow \mathrm{Bar} \\ \mathrm{Alg}_{\mathbb{E}_{n-1}}(\mathrm{Sp}) & \xleftarrow{\iota^*} & \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Sp}). \end{array}$$

By Lemma 3.3 it will suffice to show that this square in fact commutes (up to natural equivalence). But this is clear from the fact that ι^* preserves tensor products and sifted colimits, which are the two ingredients to form the bar construction.

□

4. Examples and higher enveloping algebras

We start this section by deducing the composition squares of Theorem 1.2 and Theorem 1.7 and afterwards discuss the implications for higher enveloping algebras.

Proofs of Theorem 1.2 and Theorem 1.7. Starting from the composition square

$$\begin{array}{ccc} \mathbb{E}_{k+m} & \xrightarrow{\iota} & \mathbb{E}_{k+m+n} \\ \downarrow \beta & & \downarrow \beta \\ s^k\mathbb{E}_m & \xrightarrow{\iota} & s^k\mathbb{E}_{m+n}. \end{array}$$

of Theorem 1.6, we can set $m = 0$ and take the colimit as n goes to infinity to obtain a composition square

$$\begin{array}{ccc} \mathbb{E}_k & \xrightarrow{\iota} & \mathbb{E}_\infty \\ \downarrow{\scriptstyle\beta} & & \downarrow \\ \mathbf{1} & \xrightarrow{\iota} & s^k\mathbb{E}_\infty. \end{array}$$

To identify the right-hand vertical map we consider the commutative diagram (using Lemma 3.1):

$$\begin{array}{ccccccccc} \mathbb{E}_n & \xrightarrow{\iota} & \mathbb{E}_{n+1} & \xrightarrow{\iota} & \mathbb{E}_{n+1} & \xrightarrow{\iota} & \cdots & \longrightarrow & \mathbb{E}_\infty \\ \downarrow{\scriptstyle\beta} & \searrow^{\sigma} & \downarrow{\scriptstyle\beta} & \searrow^{\sigma} & \downarrow{\scriptstyle\beta} & \searrow^{\sigma} & & & \downarrow \\ s\mathbb{E}_{n-1} & \longrightarrow & s\mathbb{E}_n & \xrightarrow{\iota} & s\mathbb{E}_{n+1} & \xrightarrow{\iota} & \cdots & \longrightarrow & s\mathbb{E}_\infty. \end{array}$$

We see that the colimit over n of the morphisms $\beta\colon \mathbb{E}_n \to s\mathbb{E}_{n-1}$ can be identified with the colimit of the suspension morphisms $\sigma\colon \mathbb{E}_n \to s\mathbb{E}_n$, which is indeed the morphism $\sigma\colon \mathbb{E}_\infty \to s\mathbb{E}_\infty$ appearing in the statement of Theorem 1.7.

Applying the k-fold operadic desuspension to the composition square of Theorem 1.6 yields a composition square

$$\begin{array}{ccc} s^k\mathbb{E}_{k+m} & \xrightarrow{\iota} & s^k\mathbb{E}_{k+m+n} \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} \\ \mathbb{E}_m & \xrightarrow{\iota} & \mathbb{E}_{m+n}. \end{array}$$

Now we take the limit as k goes to ∞. Remark 3.2 allows us to identify the limit over k of the maps $\beta\colon s^k\mathbb{E}_k \to \mathbf{1}$ with the Koszul dual of the colimit over k of the maps $\iota\colon \mathbf{1} \to \mathbb{E}_k$. Thus $\varprojlim_k s^k\mathbb{E}_k$ is the Koszul dual of $\mathbb{E}_\infty$, which is the spectral Lie operad $\mathbb{L}$. Therefore, the limit of the squares above becomes

$$\begin{array}{ccc} s^m\mathbb{L} & \xrightarrow{\sigma^n} & s^{n+m}\mathbb{L} \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} \\ \mathbb{E}_m & \xrightarrow{\iota} & \mathbb{E}_{m+n}. \end{array}$$

The identification of the top horizontal morphism with the n-fold suspension σ^n is entirely analogous to the argument in the first half of this proof. Now taking $m = 0$ gives the second composition square of Theorem 1.7, whereas setting $m = 1$ and taking the colimit as n goes to ∞ gives the composition square of Theorem 1.2. □

Recall that the morphism $\beta\colon s^n\mathbb{L} \to \mathbb{E}_n$ appearing in Theorem 1.7 induces a functor

$$\beta_!\colon \mathrm{Alg}_{s^n\mathbb{L}}(\mathrm{Sp}) \to \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Sp})$$

or, after identifying the ∞-categories $\mathrm{Alg}_{s^n\mathbb{L}}(\mathrm{Sp})$ and $\mathrm{Alg}_{\mathbb{L}}(\mathrm{Sp})$ via degree shifting, a functor

$$U_n\colon \mathrm{Alg}_{\mathbb{L}}(\mathrm{Sp}) \to \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Sp})$$

that we refer to as the $\mathbb{E}_n$-enveloping algebra functor. Theorem 1.8, which describes U_n as the composition $\mathrm{CE} \circ \Omega^n$, is now straightforward to deduce from our results:

Proof of Theorem 1.8. By Lemma 3.3, the first composition square of Theorem 1.7 gives a commutative square

$$\begin{array}{ccc} \mathrm{Alg}_{\mathbb{L}}(\mathrm{Sp}) & \xleftarrow{(\sigma^n)^*} & \mathrm{Alg}_{s^n\mathbb{L}}(\mathrm{Sp}) \\ \downarrow{\scriptstyle\mathrm{CE}} & & \downarrow{\scriptstyle\beta_!} \\ \mathrm{Sp} & \xleftarrow[\text{forget}]{} & \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Sp}). \end{array}$$

By Proposition 2.5, the top arrow can be identified with the n-fold loop functor of Lie algebras through the following commutative diagram, where the vertical arrow is the n-fold shift:

$$\begin{array}{ccc} \mathrm{Alg}_{s^n\mathbb{L}}(\mathrm{Sp}) & \xrightarrow{(\sigma^n)^*} & \mathrm{Alg}_{\mathbb{L}}(\mathrm{Sp}). \\ \downarrow \simeq & \nearrow_{\Omega^n} & \\ \mathrm{Alg}_{\mathbb{L}}(\mathrm{Sp}) & & \end{array}$$

This concludes the proof. □

We establish one further family of composition squares to be used in our discussion of the PBW theorem below. We will write $\mathbb{E}_k^\vee$ for the termwise Spanier–Whitehead dual of the (stable, nonunital) $\mathbb{E}_k$-operad and use the 'self-duality' of the $\mathbb{E}_k$-operad established by Ching–Salvatore [CS22], which gives an equivalence of symmetric sequences

$$\mathbf{1} \circ_{\mathbb{E}_k} \mathbf{1} \cong s^k\mathbb{E}_k^\vee.$$

Proposition 4.1. *There is a composition square as follows:*

$$\begin{array}{ccc} \mathbb{E}_{k+n} & \longrightarrow & \mathbf{1} \\ \downarrow \beta & & \downarrow \\ s^k\mathbb{E}_n & \longrightarrow & s^k\mathbb{E}_k^\vee. \end{array}$$

Proof. Consider the following diagram:

$$\begin{array}{ccccc} \mathbb{E}_k & \xrightarrow{\iota} & \mathbb{E}_{k+n} & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow \beta & & \downarrow \\ \mathbf{1} & \longrightarrow & s^k\mathbb{E}_n & \longrightarrow & s^k\mathbb{E}_n \circ_{\mathbb{E}_{k+n}} \mathbf{1}. \end{array}$$

The left square is a composition square by the case $m = 0$ of Theorem 1.6, the right square is a composition square by construction. Composing the two shows that the outer rectangle is a composition square, giving an equivalence $s^k\mathbb{E}_n \circ_{\mathbb{E}_{k+n}} \mathbf{1} \simeq \mathbf{1} \circ_{\mathbb{E}_k} \mathbf{1}$. The conclusion now follows from the display preceding the statement of the proposition. □

So far we have focused on commutative squares of operads that are 'composition squares'. A natural question is whether our squares, e.g. the one of Theorem 1.2 involving the Lie, associative, and commutative operads, are also pushout squares in the ∞-category of operads. In the 1-category of operads (in abelian groups, say) the corresponding square *is* indeed a pushout, as one may verify by using the fact that the three operads involved have straightforward presentations with generators in arity 2 and relations in arity 3. However, this turns out to be a bit of an accident:

Proposition 4.2. *The square*

$$\begin{array}{ccc} s\mathbb{L} & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow \\ \mathbb{E}_1 & \longrightarrow & \mathbb{E}_\infty \end{array}$$

is not a pushout in the ∞-category of operads in Sp*. As a consequence, it cannot be the case that the square of Theorem 1.6 is a pushout for all k and n when $m = 0$.*

Proof. The operadic bar construction $B\colon \mathrm{Op}(\mathrm{Sp}) \to \mathrm{sSeq}(\mathrm{Sp})$ is a colimit-preserving functor, which turns the square of the proposition into the following:

$$\begin{array}{ccc} s\mathbb{E}_\infty^\vee & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow \\ s\mathbb{E}_1^\vee & \longrightarrow & \mathbb{L}^\vee \end{array}$$

(This happens to be precisely the termwise Spanier–Whitehead dual of the square we started with, up to operadic suspension.) This square is not a pushout of symmetric sequences; indeed, if it were a pushout then for $n \geq 2$ there would be a cofibre sequence

$$\mathbb{E}_\infty^\vee(n) \to \mathbb{E}_1^\vee(n) \to s^{-1}\mathbb{L}^\vee(n).$$

The homology of these spectra is concentrated in degree zero; their integral homology is finitely generated and free of ranks 1, $n!$, and $(n-1)!$ respectively. The alternating sum of these numbers does not equal zero as soon as $n \geq 3$. This proves the first claim of the proposition. If the squares of Theorem 1.6 were pushouts for all k and n when $m = 0$, then that would imply (by pasting countably many such squares) that the square of the proposition is a pushout, which we have just ruled out. □

5. Relation to the Poincaré–Birkhoff–Witt theorem

Given a Lie algebra $\mathfrak{g}$, we write $U\mathfrak{g}$ for its universal enveloping algebra. Concretely, this is the quotient of the tensor algebra $T\mathfrak{g}$ by the two-sided ideal generated by elements of the form

$$x \otimes y - y \otimes x - [x, y], \;\; x, y \in \mathfrak{g}.$$

This ideal is not homogeneous, so the natural grading on $T\mathfrak{g}$ does not descend to $U\mathfrak{g}$. However, $U\mathfrak{g}$ inherits an increasing filtration whose n^{th} piece $F^n(U\mathfrak{g})$ consists of all images of elements $x_1 \otimes \ldots \otimes x_k$ with $k \leq n$ under the quotient map $T\mathfrak{g} \to U\mathfrak{g}$.

The canonical map $\mathfrak{g} \to U\mathfrak{g}$ lands in the first filtered piece $F^1U\mathfrak{g}$, and we therefore obtain a map of modules

$$\mathfrak{g} \to \mathrm{gr}^1(U\mathfrak{g}).$$

As the associative product on $U\mathfrak{g}$ descends to a commutative product on $\mathrm{gr}(U\mathfrak{g}) = \bigoplus_i \mathrm{gr}^i(U\mathfrak{g})$, we can induce up to obtain a homomorphism of commutative algebras $\mathrm{Sym}^*(\mathfrak{g}) \to \mathrm{gr}^*(U\mathfrak{g})$.

Theorem 5.1 (Poincaré–Birkhoff-Witt theorem)**.** *This induced morphism*

$$\mathrm{Sym}^*(\mathfrak{g}) \to \mathrm{gr}^*(U\mathfrak{g})$$

is an isomorphism.

We will now generalise this result using some of the relative composition products we computed above. To this end, we will adopt the strategy in the proof of [GR17, 6.5.2.6] to our setting.

Notation 5.2 (Filtered and graded spectra)**.** We write $\mathbb{N}$ for the poset of nonnegative integers, thought of as a category, and let $\mathbb{N}^{\mathrm{disc}}$ be the category with the same objects and only identity morphisms. The ∞-categories of (nonnegatively) filtered and graded spectra are defined as

$$\mathrm{Sp}^{\mathrm{Fil}} := \mathrm{Fun}(\mathbb{N}, \mathrm{Sp}) \quad \text{and} \quad \mathrm{Sp}^{\mathrm{gr}} := \mathrm{Fun}(\mathbb{N}^{\mathrm{disc}}, \mathrm{Sp}),$$

respectively. Day convolution equips these ∞-categories with symmetric monoidal structures given by

$$(X \otimes Y)_n = \operatorname*{colim}_{i+j \leq n} (X_i \otimes Y_j) \quad \text{and} \quad (X \otimes Y)_n = \bigoplus_{i+j=n} (X_i \otimes Y_j),$$

and the functors

$$\mathrm{Sp} \xrightarrow{c} \mathrm{Sp}^{\mathrm{Fil}} \qquad \mathrm{Sp}^{\mathrm{Fil}} \xrightarrow{\mathrm{gr}} \mathrm{Sp}^{\mathrm{gr}} \qquad \mathrm{Sp}^{\mathrm{Fil}} \xrightarrow{\mathrm{colim}} \mathrm{Sp}$$

are symmetric monoidal. Here c sends $X \in \mathrm{Sp}$ to the constant object $(X \xrightarrow{\mathrm{id}} X \xrightarrow{\mathrm{id}} \ldots) \in \mathrm{Sp}^{\mathrm{Fil}}$, the functor gr sends a filtered spectrum $(X_0 \to X_1 \to \ldots)$ to the graded spectrum $(X_0, X_1/X_0, X_2/X_1, \ldots)$, and colim sends $(X_0 \to X_1 \to \ldots)$ to $\mathrm{colim}_i X_i$.

Let us now fix an ∞-operad $\mathcal{O} \in \mathrm{Op}(\mathrm{Sp}) = \mathrm{Alg}(\mathrm{sSeq}(\mathrm{Sp}))$ in spectra. Using the functors c and $\mathrm{gr} \circ c$, we promote $\mathrm{Sp}^{\mathrm{Fil}}$ and $\mathrm{Sp}^{\mathrm{gr}}$ to Sp-tensored ∞-categories, and obtain ∞-categories $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}^{\mathrm{Fil}})$ and $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}^{\mathrm{Gr}})$ of filtered and graded $\mathcal{O}$-algebras, respectively. Since the functors

c, gr, and colim are lax symmetric monoidal and compatible with the Sp-tensored structure, they induce functors

$$\mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}) \xrightarrow{c} \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}^{\mathrm{Fil}}) \qquad \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}^{\mathrm{Fil}}) \xrightarrow{\mathrm{gr}} \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}^{\mathrm{gr}}) \qquad \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}^{\mathrm{Fil}}) \xrightarrow{\mathrm{colim}} \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}).$$

Remark 5.3 (Adding filtration)**.** There is a more interesting functor from $\mathcal{O}$-algebras to filtered $\mathcal{O}$-algebras: given an $\mathcal{O}$-algebra X, we can equip the filtered spectrum

$$(0 \to X \xrightarrow{\mathrm{id}} X \xrightarrow{\mathrm{id}} \dots)$$

with an $\mathcal{O}$-algebra structure. To this end, consider the functor $c_1 : \mathrm{Sp} \to \mathrm{Sp}^{\mathrm{Fil}}$ sending X in Sp to the filtered spectrum $(0 \to X \xrightarrow{\mathrm{id}} X \xrightarrow{\mathrm{id}} \dots)$ starting in degree 1. It admits a left adjoint $\mathrm{Fun}(\mathbb{N}, \mathrm{Sp}) \to \mathrm{Sp}$ sending a filtered spectrum $(X_0 \to X_1 \to X_2 \to \dots)$ to the cofibre $\mathrm{cof}\,(X_0 \to \mathrm{colim}_n X_n)$.

As the functors $(-)_0, \mathrm{colim} : \mathrm{Fun}(\mathbb{N}, \mathrm{Sp}) \to \mathrm{Sp}$ are both symmetric monoidal, the functor $\mathrm{Sp} \to \mathrm{Sp}^{\to} := \mathrm{Fun}(\Delta^1, \mathrm{Sp})$ given by $X \mapsto (X_0 \to \mathrm{colim}_n X_n)$ is also symmetric monoidal where the arrow category is endowed with the pointwise monoidal structure. The functor $\mathrm{cof} : \mathrm{Sp}^{\to} \to \mathrm{Sp}$ is left adjoint to the functor $X \mapsto (0 \to X)$ which is symmetric monoidal. By [HHLN23, Proposition A], we see first that cof is oplax symmetric monoidal, and next that so is the left adjoint to c_1. Again by [HHLN23, Proposition A], the right adjoint c_1 is a lax symmetric monoidal functor. As it is moreover compatible with the Sp-tensored structures on Sp and $\mathrm{Sp}^{\mathrm{Fil}}$, we obtain a commutative diagram

$$\begin{array}{ccccc} \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}) & \xrightarrow{c_1} & \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}^{\mathrm{Fil}}) & \xrightarrow{\mathrm{colim}} & \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Sp} & \xrightarrow{c_1} & \mathrm{Sp}^{\mathrm{Fil}} & \xrightarrow{\mathrm{colim}} & \mathrm{Sp}. \end{array}$$

where the vertical arrows are forgetful functors and the horizontal arrows compose to the identity. In other words, we have equipped every $\mathcal{O}$-algebra with a natural filtration.

Let us now assume that $\mathcal{O}$ is reduced, i.e. that $\mathcal{O}(0) \simeq 0$ and $\mathcal{O}(1) \simeq \mathbb{S}^0$. We obtain a map of operads $\mathcal{O} \to \mathbf{1}$, where $\mathbf{1}$ is the identity operad. Pulling back along this map gives a functor

$$\mathrm{triv} : \mathrm{Sp}^{\mathrm{gr}} \to \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}^{\mathrm{gr}}).$$

Proposition 5.4. *If $X \in \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}^{\mathrm{gr}})$ is concentrated in degree 1, then there is an equivalence of graded $\mathcal{O}$-algebras*

$$\mathrm{triv}(\mathrm{forget}(X)) \simeq X,$$

where $\mathrm{forget} : \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}) \to \mathrm{Sp}$ *is the forgetful functor.*

Proof. Write $q : \mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ for the cocartesian fibration of ∞-operads that exhibits $\mathrm{Sp}^{\mathrm{gr}}$ as (left)-tensored ∞-category over sSeq(Sp). Here $\mathcal{LM}^{\otimes}$ is the ∞-operad defined in [HA, Section 4.2.1]; it has two objects a (for algebra) and m (for module), and we have $q^{-1}(a) = \mathrm{sSeq(Sp)}$ and $q^{-1}(m) = \mathrm{Sp}^{\mathrm{gr}}$, respectively.

Consider the full subcategory $\mathrm{Sp}^{\mathrm{gr}}_{=1} \subset \mathrm{Sp}^{\mathrm{gr}}$ of symmetric sequences concentrated in degrees 1, which is a localisation. Write $\mathcal{D}^{\otimes} \subset \mathcal{C}^{\otimes}$ for the full subcategory spanned by objects $D_1 \oplus \ldots \oplus D_n$ for which all objects over m belong to $\mathrm{Sp}^{\mathrm{gr}}_{=1} \subset \mathrm{Sp}^{\mathrm{gr}}$. Proposition 2.2.1.9.(3) in [HA] shows that the inclusion $\mathcal{D}^{\otimes} \subset \mathcal{C}^{\otimes}$ is a map of ∞-operads, and we obtain an equivalence

$$\mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}^{\mathrm{gr}}_{=1}) \xrightarrow{\simeq} \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}^{\mathrm{gr}})_{=1}.$$

Here $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}^{\mathrm{gr}})_{=1} \subset \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}^{\mathrm{gr}})$ is the full subcategory of $\mathcal{O}$-algebras concentrated in degree 1.

The action of sSeq(Sp) on $\mathrm{Sp}^{\mathrm{gr}}_{=1}$ factors over the truncation map $F : \mathrm{sSeq(Sp)} \to \mathrm{sSeq(Sp)}_{\leq 1}$ to 1-truncated symmetric sequences, which is monoidal. This gives an equivalence

$$\mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}^{\mathrm{gr}}_{=1}) \xrightarrow{\simeq} \mathrm{Alg}_{\mathcal{O}_1}(\mathrm{Sp}^{\mathrm{gr}}_{=1}) \simeq \mathrm{Sp}.$$

To formally verify this, we model tensored ∞-categories as in [DAG II, 2.1.1] (see also [HA, Remark 4.2.2.25]). The map of tensored ∞-categories $(\mathrm{sSeq}(\mathrm{Sp}) \curvearrowright \mathrm{Sp}^{\mathrm{gr}}_{=1}) \to (\mathrm{sSeq}(\mathrm{Sp})_{\leq 1} \curvearrowright \mathrm{Sp}^{\mathrm{gr}}_{=1})$ is then modelled by a square over Δ^{op}:

$$\begin{array}{ccc} \widetilde{\mathcal{M}}^{\circledast} & \longrightarrow & \mathcal{M}^{\circledast} \\ \downarrow & & \downarrow \\ \mathrm{sSeq}(\mathrm{Sp})^{\circledast} & \xrightarrow{F^{\circledast}} & \mathrm{sSeq}(\mathrm{Sp})^{\circledast}_{\leq 1}. \end{array}$$

The desired equivalence follows as this square is a pullback (see [Bra, 5.2] for further details).

The claim now follows as the composite equivalence $\mathrm{Sp} \xrightarrow{\simeq} \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}^{\mathrm{gr}})_{=1}$ sends $Y \in \mathrm{Sp}$ to $\mathrm{triv}(Y)$. □

Corollary 5.5. *Given an $\mathcal{O}$-algebra X, there is a natural equivalence of graded $\mathcal{O}$-algebras*

$$\mathrm{gr}(c_1(X)) \simeq \mathrm{triv}(\mathrm{forget}(X)).$$

Proof. This follows immediately from Proposition 5.4 as $\mathrm{gr}(c_1(X))$ is concentrated in degree 1. □

Let us now fix a morphism of spectral operads

$$\beta : \mathcal{P} \to \mathcal{Q}.$$

The induced functor $\beta^* : \mathrm{Alg}_{\mathcal{Q}}(\mathrm{Sp}) \to \mathrm{Alg}_{\mathcal{P}}(\mathrm{Sp})$ admits a left adjoint

$$\beta_! : \mathrm{Alg}_{\mathcal{P}}(\mathrm{Sp}) \to \mathrm{Alg}_{\mathcal{Q}}(\mathrm{Sp}),$$

which we call the pushforward along β, and we denote the corresponding functors on filtered and graded algebras by the same name. Note that they are compatible with the functors gr and colim.

Proposition 5.6. *Given a $\mathcal{P}-$algebra X, the $\mathcal{Q}$-algebra $\beta_!(X)$ admits an exhaustive filtration $\beta_!(c_1(X))$ with associated graded $\beta_!(\mathrm{triv}(\mathrm{forget}(X)))$.*

Proof. To verify that this is indeed exhaustive, we compute

$$\mathrm{colim}\, \beta_!(c_1(X))) \simeq \beta_!(\mathrm{colim}\, c_1(X))) \simeq \beta_!(X).$$

Using Corollary 5.5, we obtain

$$\mathrm{gr}(\beta_!(c_1(X)))) \simeq \beta_!(\mathrm{gr}(c_1(X)))) \simeq \beta_!(\mathrm{triv}(\mathrm{forget}(X))).$$

□

We will now establish analogues of the PBW theorem. To this end, we define:

Definition 5.7 (Universal envelope)**.** The *universal enveloping algebra functor*

$$U : \mathrm{Alg}_{s\mathbb{L}}(\mathrm{Sp}) \to \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})$$

is given by the pushforward along the morphism of ∞-operads

$$s\mathbb{L} \to \mathbb{E}_1,$$

which is Koszul dual do the inclusion $\mathbb{E}_1 \to \mathbb{E}_\infty$.

We can now prove Corollary 1.10, the PBW theorem for spectral Lie algebras.

Proof. By Proposition 5.6 gives an exhaustive filtration $U(c_1(\mathfrak{g}))$ with associated graded

$$U(\mathrm{triv}(\mathrm{forget}(\mathfrak{g}))).$$

We have equivalences

$$U(\mathrm{triv}(X)) \simeq U(|\mathrm{Bar}_\bullet(s\mathbb{L}, s\mathbb{L}, \mathrm{triv}(X))|) \simeq |\mathrm{Bar}_\bullet(\mathbb{E}_1, s\mathbb{L}, \mathrm{triv}(X))| \simeq (\mathbb{E}_1 \circ_{s\mathbb{L}} \mathbf{1})(X),$$

and so the claim follows from Theorem 1.2. □

Definition 5.8 (Relative envelope)**.** Given $0 \leq k \leq n$, the *relative enveloping algebra functor*

$$U_k : \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Sp}) \to \mathrm{Alg}_{s^k\mathbb{E}_{n-k}}(\mathrm{Sp})$$

is given by the pushforward along the morphism of ∞-operads

$$\mathbb{E}_n \to s^k\mathbb{E}_{n-k}$$

which is Koszul dual to the inclusion $\mathbb{E}_{n-k} \to \mathbb{E}_n$.

We finish with a proof of the PBW theorem for $\mathbb{E}_n$-algebras Corollary 1.11:

Proof. This follows from Proposition 4.1 by the same argument as in the proof of Corollary 1.10. □

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National Autonomous University of Mexico

Email address: `omar@matem.unam.mx`

Oxford University, Université Paris–Saclay (CNRS)

Email address: `lukas.brantner@maths.ox.ac.uk, lukas.brantner@universite-paris-saclay.fr`

Utrecht University

Email address: `g.s.k.s.heuts@uu.nl`