

FORMAL INTEGRATION OF DERIVED FOLIATIONS

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ABSTRACT. Frobenius' theorem in differential geometry asserts that every involutive subbundle of the tangent bundle of a manifold M integrates to a decomposition of M into smooth leaves.

We prove an infinitesimal analogue of this result for locally coherent qcqs schemes X over coherent rings. More precisely, we integrate partition Lie algebroids on X to formal moduli stacks $X \rightarrow \widehat{S}$, where \widehat{S} is the formal leaf space and the fibres of $X \rightarrow \widehat{S}$ are the formal leaves. We deduce that deformations of X -families of algebro-geometric objects are controlled by partition Lie algebroids on X . Combining our integration equivalence with a result of Fu, we deduce that Toën–Vezzosi's infinitesimal derived foliations (under suitable finiteness hypotheses) are formally integrable.

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1. INTRODUCTION

A smooth algebraic foliation \mathcal{F} on a smooth complex variety X is a subbundle

$$E_{\mathcal{F}} \subseteq T_X$$

which is closed under the commutator bracket of vector fields, i.e. a locally free Lie algebroid whose anchor map is injective. We call \mathcal{F} *algebraically integrable* if there is a decomposition

$$X = \coprod_{\alpha} \mathcal{F}_{\alpha}$$

of X into smooth complex subvarieties \mathcal{F}_{α} – the leaves – whose tangent vectors span $E_{\mathcal{F}} \subset T_X$. Note that algebraic integrability is a highly nontrivial condition. One can approach the problem of integrating a given smooth foliation \mathcal{F} in three steps.

First, we integrate \mathcal{F} *formally*, which means that for every point $x \in M$, we find power series

$$f_1, \dots, f_k \in \mathbb{C}[[z_1, \dots, z_n]]$$

in local coordinates z_1, \dots, z_n such that a local vector field α lies in $E_{\mathcal{F}}$ if and only if $\alpha \cdot df_i = 0$ for all i . The formal leaf passing through x is then encoded by the functor

$$\mathcal{F}_x^\wedge : A \mapsto \{a_1, \dots, a_n \in A \mid \forall i : f_i(a_1, \dots, a_n) = 0\}$$

from local Artinian \mathbb{C} -algebras to sets.

Varying $x \in X$, we obtain a ‘quotient’ map of prestacks

$$p : X \rightarrow \widehat{S}$$

to the formal leaf space such that \mathcal{F}_x^\wedge sends A to the set of (isomorphism classes of) dotted lifts

$$\begin{array}{ccc} \mathrm{Spec}(\mathbb{C}) & \xrightarrow{x} & X \\ \downarrow & \searrow \text{dotted} & \downarrow \\ \mathrm{Spec}(A) & \xrightarrow{\bar{x}} & \widehat{S}. \end{array}$$

Next, we check the local convergence of these power series to obtain complex manifolds \mathcal{F}_x passing through all points of X , thereby verifying Frobenius theorem for complex manifolds.

Finally, we examine whether or not the various leaves \mathcal{F}_x are algebraic subvarieties of X . If this is the case, we obtain a submersion of Deligne–Mumford stacks $p : X \rightarrow S$ such that $X \rightarrow \widehat{S}$ is the formal completion of X along p .

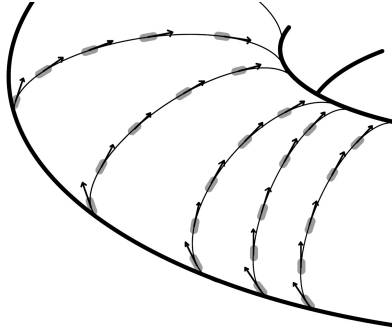


FIGURE 1. A foliation (depicted by tangent vectors) along with its formal leaves (in gray) and leaves (in black).

In this article, we generalise the first step to more general Lie algebroids on more general schemes. More precisely, we replace complex varieties by locally coherent qcqs schemes X over coherent rings R , both possibly derived, and formally integrate *partition Lie algebroids*

$$\rho : E \rightarrow T_{X/R}[1]$$

with ρ not necessarily injective and E not necessarily locally free. The result of our integration procedure is a *formal moduli stack on X* – around each point x in X , we obtain a formal moduli problem \mathcal{F}_x^\wedge encoding the formal leaf passing through x , which can be non-smooth, stacky, and derived.

Formal moduli stacks $X \rightarrow \widehat{S}$ also arise in deformation theory, where they capture infinitesimal derived deformation functors of X -indexed families of algebro-geometric objects

$$\begin{array}{c} M \\ \downarrow \\ X. \end{array}$$

Our integration equivalence then implies that every such deformation functor is governed by an essentially unique partition Lie algebroid on X , generalising various earlier equivalences in characteristic 0 ([Dri, Del, Hin01, DAG X, Pri10, Hen18, GR17a, Nui19b]) and p ([BM, BW]).

Foliations on smooth complex varieties X can also be defined as quotient bundles

$$\Omega_X \twoheadrightarrow \Omega_{\mathcal{F}}$$

inducing maps of commutative differential graded algebras

$$(\Lambda^* \Omega_X, d_{\text{dR}}) \rightarrow (\Lambda^* \Omega_{\mathcal{F}}, \bar{d}_{\text{dR}}).$$

This definition was generalised by Toën–Vezzosi [TV23] to the non-smooth derived setting – they call the resulting structure an *infinitesimal derived foliation*. Fu [Fu24] recently constructed an equivalence between almost perfect infinitesimal derived foliations and dually almost perfect partition Lie algebroids. Combining Fu’s equivalence with our main result, we deduce that almost perfect infinitesimal derived foliations (in the sense of Toën–Vezzosi) are formally integrable.

1.1. Statement of results. A Lie algebroid on a smooth manifold X consists of an anchor map

$$\rho : E \rightarrow T_X$$

of vector bundles together with a bilinear Lie bracket $[-, -]$ on the space $\Gamma(X, E)$ of global sections such that for all global sections $s_1, s_2 \in \Gamma(M, E)$ and all smooth functions f , we have

$$[s_1, f s_2] = \rho(s_1)(f) \cdot s_2 + f[s_1, s_2].$$

Lie algebroids were first introduced by Pradines [Pra67] and have since found numerous applications.

In this article, we will first generalise the theory of Lie algebroids the setting of algebraic geometry over very general bases, and then formally integrate them.

Setup. Let us fix a coherent base ring R , i.e. a commutative ring whose finitely generated ideals are also finitely presented. Let X be a quasi-compact quasi-separated (“qcqs”) derived R -scheme which is locally coherent, meaning that it admits an open cover by affine derived schemes $\text{Spec}(B)$ with $\pi_0(B)$ coherent and $\pi_i(B)$ finitely presented over $\pi_0(B)$.

Every such scheme admits a quasi-coherent cotangent complex $L_{X/R}$ and a tangent complex

$$T_{X/R} = L_{X/R}^{\vee}.$$

This tangent complex belongs to the ∞ -category $\text{QC}_X^{\vee} = \text{Ind}(\text{Coh}_X^{\text{op}})$ of pro-coherent sheaves on X , which is the natural home of duals of quasi-coherent sheaves. Pro-coherent sheaves were first introduced by Deligne [Har66, Appendix]; we refer to Appendix B.2 for further details.

Partition Lie algebroids. In Section 3, we construct the ∞ -category of *partition Lie algebroids*

$$\mathrm{LieAlg}d_{\Delta, X/R}^{\pi}$$

on the scheme X . These homotopical objects exhibit many expected properties:

- (1) (Forgetful functor) There is a monadic forgetful functor $\mathrm{LieAlg}d_{\Delta, X/R}^{\pi} \rightarrow (\mathrm{QC}_X^{\vee})/T_{X/R}[1]$; hence partition Lie algebroids are maps of pro-coherent sheaves

$$\mathfrak{g} \xrightarrow{\rho} T_{X/R}[1]$$

equipped with additional Lie algebraic structure.

- (2) (Global sections) Taking derived global sections gives rise to a functor

$$\mathrm{LieAlg}d_{\Delta, X/R}^{\pi} \rightarrow (\mathrm{LieAlg}_{\Delta, R}^{\pi})/R\Gamma(X, T_{X/R}[1]),$$

where $\mathrm{LieAlg}_{\Delta, R}^{\pi} = \mathrm{LieAlg}_{\Delta, \mathrm{Spec}(R)/R}^{\pi}$ denotes the ∞ -category of R -partition Lie algebras and

$$R\Gamma(X, T_{X/R}[1])$$

is the Kodaira–Spencer partition Lie algebra encoding infinitesimal deformations of the R -scheme X (cf. Section 4.4).

- (3) (Descent) Every étale map

$$Y \rightarrow X$$

induces a functor $\mathrm{LieAlg}d_{\Delta, X/R}^{\pi} \rightarrow \mathrm{LieAlg}d_{\Delta, Y/R}^{\pi}$ lifting the pullback functor on the level of pro-coherent sheaves. Moreover, there is a canonical equivalence of ∞ -categories

$$\mathrm{LieAlg}d_{\Delta, X/R}^{\pi} \xrightarrow{\simeq} \lim_{U \subseteq X \text{ affine open}} \mathrm{LieAlg}d_{\Delta, U/R}^{\pi}.$$

Before proceeding to formal integration, let us highlight various special cases of our construction:

- (a) For k a field of characteristic 0 and $X = \mathrm{Spec}(k)$, the ∞ -category $\mathrm{LieAlg}d_{\Delta, X/k}^{\pi} = \mathrm{LieAlg}_{\Delta, k}^{\pi}$ is modelled by (shifted) differential graded Lie algebras over k . For $X = \mathrm{Spec}(A)$ an affine k -scheme, $\mathrm{LieAlg}d_{\Delta, X/k}^{\pi}$ arises from the category of differential graded Lie algebroids on X with the tame semi-model structure, see [Nui19a, Variant 3.12].
- (b) For k a field of characteristic p and $X = \mathrm{Spec}(k)$, we recover the ∞ -category of partition Lie algebras introduced in [BM]. These can be modelled by cosimplicial-simplicial k -vector spaces with additional operations parametrised by nested chains of partitions, see [BCN, Definition 5.43]. For $X = \mathrm{Spec}(F)$ with F/k a finite purely inseparable field extension, $\mathrm{LieAlg}d_{\Delta, X/k}^{\pi}$ recovers the ∞ -category of F/k -partition Lie algebroids appearing in the purely inseparable Galois correspondence [BW]. For $R = A$ complete local Noetherian with residue field k and $X = \mathrm{Spec}(k)$, the ∞ -category $\mathrm{LieAlg}d_{\Delta, X/A}^{\pi}$ is equivalent to the mixed partition Lie algebras appearing in [BM, Definition 6.22].
- (c) For R a coherent ring and $X = \mathrm{Spec}(R)$, the ∞ -category $\mathrm{LieAlg}d_{\Delta, X/R}^{\pi} = \mathrm{LieAlg}_{\Delta, R}^{\pi}$ appears in [BCN, Section 3]. It admits an explicit model in terms of simplicial-cosimplicial objects, given in [BCN, Theorem 5.42].

Formal moduli stacks. Given X and R as above, we will now introduce *formal moduli stacks*

$$X \rightarrow \widehat{S}$$

under X . Informally, these can be thought of in at least two different ways:

- (1) as decompositions of X into of formal leaves \mathcal{F}_x^{\wedge} parametrised by the points x of \widehat{S} ;
- (2) as infinitesimal deformation functors of X -indexed families of algebro-geometric objects.

To formalise the notion of a formal moduli stack, we will use *animated rings*.

Notation 1.1. We write $\mathrm{CAlg}^{\mathrm{an}} = \mathcal{P}_{\Sigma}(\mathrm{Poly})$ for the ∞ -category of animated rings, which is obtained by freely adding sifted colimits to the category of finitely generated polynomial rings. Note that $\mathrm{CAlg}^{\mathrm{an}}$ is also sometimes denoted by Ani , $\mathrm{DAlg}_{\geq 0}$, CAlg^{Δ} , or SCR in the literature. The last notation emphasises that animated rings can be modelled by simplicial commutative rings.

Formal moduli stacks will belong to the ∞ -category of R -prestacks

$$\mathrm{PrStk}_R$$

i.e. the ∞ -category of (accessible) functors from animated R -algebras to spaces. Every derived R -scheme X gives a prestack $R \mapsto \mathrm{Map}_{\mathrm{Sch}^{\mathrm{der}}}(\mathrm{Spec}(R), X)$, which we denote by the same name.

Definition 1.2 (Formal moduli stacks). A *formal moduli stack* \mathcal{F} under an R -prestack X is a map of R -prestacks

$$X \longrightarrow \widehat{\mathcal{S}}$$

satisfying the following conditions:

- (1) $\widehat{\mathcal{S}}$ has deformation theory, which means that it preserves limits of Postnikov towers and pullbacks along nilpotent extensions of animated R -algebras, see Definition B.6;
- (2) $X \rightarrow \widehat{\mathcal{S}}$ is locally almost of finite presentation, see Definition B.3;
- (3) $X \rightarrow \widehat{\mathcal{S}}$ restricts to an equivalence on all reduced affine schemes.

Write $\mathrm{ModuliStk}_{X/R} \subset (\mathrm{PrStk}_R)_{X/}$ for the full subcategory of formal moduli stacks under X .

Remark 1.3 (Formal leaves of formal moduli stacks). Given an R -point x in X and a formal moduli stack $X \rightarrow \widehat{\mathcal{S}}$, the formal leaf \mathcal{F}_x^{\wedge} passing through x is the formal moduli problem sending an augmented local Artinian animated R -algebra A to the space of dotted lifts

$$\begin{array}{ccc} \mathrm{Spec}(R) & \xrightarrow{x} & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \xrightarrow{\bar{x}} & \widehat{\mathcal{S}}. \end{array}$$

Remark 1.4 (Deformation functors and formal moduli stacks). Let B be a coherent eventually coconnective animated R -algebra. Formal deformation functors of (suitably nice) B -indexed families of algebro-geometric objects over R are then parametrised by functors

$$F: \mathrm{Art}_{R/B} \longrightarrow \mathcal{S}$$

from the ∞ -category

$$\mathrm{Art}_{R/B} \subset \mathrm{CAlg}_{R/B}^{\mathrm{an}}$$

of Artinian extensions of B (cf. Definition 4.1) to the ∞ -category \mathcal{S} of spaces satisfying

- (1) $F(B) \simeq *$;
- (2) F preserves pullbacks along nilpotent extensions.

In Section 5, we show that the ∞ -category of such functors is equivalent to the ∞ -category of formal moduli stacks under $X = \mathrm{Spec}(B)$; this statement is essentially contained in [GR17b, 5.1.2].

Indeed, if $X \rightarrow \widehat{S}$ is a formal moduli stack, then the corresponding functor $\text{Art}_{R/B} \rightarrow \mathcal{S}$ sends an Artinian extension $A \rightarrow B$ over R to the space of dotted lifts in the following diagram:

$$\begin{array}{ccc} X & \longrightarrow & \widehat{S} \\ \downarrow & \searrow \text{dotted} & \downarrow \\ \text{Spec}(A) & \longrightarrow & \text{Spec}(R). \end{array}$$

The main theorem. Given a formal moduli stack $X \rightarrow \widehat{S}$, we can consider the sheaf of ‘derived tangent vectors’ to the formal leaves \mathcal{F}_x^\wedge . To make this idea precise, we use that the ∞ -category QC_X^\vee of pro-coherent sheaves can be identified with the ∞ -category of functors $\text{Coh}_{X, \geq 0} \rightarrow \mathcal{S}$ preserving terminal objects and pullbacks along π_0 -surjections. Here $\text{Coh}_{X, \geq 0}$ denotes the ∞ -category of connective coherent sheaves on X .

The pro-coherent tangent sheaf

$$T_{X/\widehat{S}}$$

then corresponds to the functor sending $I \in \text{Coh}_{X, \geq 0}$ to the space of dotted arrows in the following square:

$$\begin{array}{ccc} X & \longrightarrow & X \\ \downarrow & \searrow \text{dotted} & \downarrow \\ \text{sqz}_X(I) & \longrightarrow & \widehat{S}. \end{array}$$

Here $\text{sqz}_X(I)$ denotes the derived scheme $(X, \mathcal{O}_X \oplus I)$, where $\mathcal{O}_X \oplus I$ is the trivial square-zero extension of the structure sheaf \mathcal{O}_X by I . There is a natural map

$$T_{X/\widehat{S}} \rightarrow T_{X/R},$$

which captures the derived tangents to the formal leaves. We then prove:

Theorem 1.5 (Main theorem). *Let X be a locally coherent qcqs derived R -scheme over a coherent animated ring R . The shifted tangent fibre functor*

$$\begin{aligned} \text{ModuliStk}_{X/R} &\longrightarrow (\text{QC}_X^\vee)_{T_{X/R}[1]} \\ (X \rightarrow Y) &\mapsto (T_{X/Y}[1] \rightarrow T_{X/R}[1]) \end{aligned}$$

lifts to an equivalence of ∞ -categories

$$T_{X/-}[1] : \text{ModuliStk}_{X/R} \xrightarrow{\cong} \text{LieAlg}_{\Delta, X/R}^\pi.$$

Hence every partition Lie algebroid on X integrates uniquely to a formal moduli stack under X .

This equivalence is functorial with respect to almost finitely presented maps.

A key ingredient in the proof of Theorem 1.5 is a Koszul equivalence for animated rings. Before stating it, we introduce some notation.

Definition 1.6. An animated R -algebra A over B is said to be *complete almost finitely augmented* if

- (1) the induced map $\pi_0(A) \rightarrow \pi_0(B)$ is surjective on π_0 with kernel I ;
- (2) the animated ring A is derived I -complete;
- (3) the A -module B is almost perfect as an A -module.

Write $\text{CAlg}_{R/B}^{\text{an, aft, } \wedge} \subset \text{CAlg}_{R/B}^{\text{an}}$ for the full subcategory spanned by those animated R -algebras which are complete almost finitely augmented over B .

Example 1.7. For $R = B$ a complete local Noetherian animated ring, we can identify $\mathrm{CAlg}_{R/B}^{\mathrm{an}, \mathrm{aft}, \wedge}$ with the ∞ -category of augmented animated R -algebras which are complete, local, and Noetherian.

In Section 3, we lift the assignment $A \mapsto (L_{B/A}^\vee[1] \rightarrow L_{B/R}^\vee[1])$ to a functor

$$\mathfrak{D} : \mathrm{CAlg}_{R/B}^{\mathrm{an}} \rightarrow \mathrm{Lie}_{\Delta, X/R}^\pi$$

for $X = \mathrm{Spec}(B)$ and establish the following result:

Theorem 1.8. *The functor \mathfrak{D} restricts to an equivalence between the full subcategory*

$$\mathrm{CAlg}_{R/B}^{\mathrm{an}, \mathrm{aft}, \wedge} \subset \mathrm{CAlg}_{R/B}^{\mathrm{an}}$$

of complete almost finitely augmented objects and the full subcategory

$$\mathrm{Lie}_{\Delta, X/R}^{\pi, \mathrm{dap}, \leq 0} \subset \mathrm{Lie}_{\Delta, X/R}^\pi$$

of partition Lie algebroids $\mathfrak{g} \rightarrow L_{B/R}^\vee[1]$ with \mathfrak{g} dually almost perfect of Tor-amplitude ≤ 0 .

Theorem 1.5 and Theorem 1.8 generalise various earlier results:

- (a) Let k be a field of characteristic 0. For $R = k$ and $X = \mathrm{Spec}(k)$ a point, our results specialise to Lurie–Pridham’s equivalence between formal moduli problems and differential graded Lie algebras [DAG X, Pri10], which in turn extends earlier work work by Deligne [Del], Drinfel’d [Dri], Hinich [Hin01], Manetti [Man09], and others.

Now assume A is a Noetherian and eventually coconnective animated k -algebra. For $R = k$ and $X = \mathrm{Spec}(A)$, our result is contained in [Nui19b], whereas for $R = A$ and $X = \mathrm{Spec}(A)$, it was proven by Hennion [Hen18].

In characteristic 0, the monadicity of the tangent fibre functor is also proven in the work of Gaitsgory–Rozenblyum [GR17b, Chapter 8]. Note, however, that their proof crucially relies on the fact that looped Lie algebras in characteristic 0 are abelian — this fails away from characteristic 0 due to the presence of power operations.

- (b) Let k be a field of characteristic p . For $X = \mathrm{Spec}(k)$ and $R = k$ or, more generally, $R = A$ a complete local Noetherian ring with residue field k , our results are contained in [BM]. For $X = \mathrm{Spec}(L)$ with L a finite purely inseparable field extension over $R = k$, our Koszul statement Theorem 1.8 recovers [BW, Theorem 3.15].

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2. AFFINE KOSZUL DUALITY

We write $\mathrm{CAlg}_R^{\mathrm{der}}$ for the ∞ -category of derived commutative rings and $\mathrm{CAlg}_R^{\mathrm{an}}$ for the ∞ -category of animated R -algebras; a summary of their theory is given in Appendix A. Note that animated rings are the same as connective derived rings:

$$\text{‘animated} = \text{derived} + \text{connective’}.$$

The goal of this section will be to provide some results about derived rings that are derived I -adically complete for ideals I that are of “sufficiently finite type”; most importantly, we prove a comonadicity result (Theorem 2.25) generalising results from [BM].

We will start by recalling a derived analogue of endowing a ring with the I -adic filtration with respect to an ideal I . Instead of working directly with ideals $I \subseteq A$, it is more convenient to work with the corresponding closed immersion $A \rightarrow A/I = B$. We will refer to objects $A \rightarrow A/I = B$ as *arrows* and use the notation $(A \rightarrow B)$.

2.1. Derived filtered and graded rings. To describe the adic filtration associated to an arrow $(A \rightarrow B)$, we first recall how one can obtain the ∞ -category of derived filtered rings as the ∞ -category of $\mathbb{L}\text{Sym}$ -algebras in the ∞ -category of filtered modules.

Notation 2.1. We will consider the following ∞ -categories:

- (a) The ∞ -category $\text{Mod}_{\mathbb{Z}}^{\Delta^1} := \text{Fun}(\Delta^1, \text{Mod}_{\mathbb{Z}})$ of maps of \mathbb{Z} -modules $(X \rightarrow Y)$.
- (b) The ∞ -category $\text{FilMod}_{\mathbb{Z}} = \text{Fun}((\mathbb{Z}_{\geq 0}, \leq)^{\text{op}}, \text{Mod}_{\mathbb{Z}})$ of non-negatively filtered \mathbb{Z} -modules, i.e. diagrams in $\text{Mod}_{\mathbb{Z}}$ of the form

$$\cdots \rightarrow F^1 V \rightarrow F^0 V = V.$$

We will denote objects of $\text{FilMod}_{\mathbb{Z}}$ as $F^* V$, and often denote the underlying object $F^0 V$ of the filtration simply as V .

- (c) The ∞ -category $\text{GrMod}_{\mathbb{Z}} = \text{Fun}((\mathbb{Z}_{\geq 0}, =), \text{Mod}_{\mathbb{Z}})$ of non-negatively graded \mathbb{Z} -modules. We use the notation V^\bullet for objects of $\text{GrMod}_{\mathbb{Z}}$ to emphasise the grading.

We will be interested in the following three functors relating these ∞ -categories:

$$\text{GrMod}_{\mathbb{Z}} \xleftarrow{\text{Gr}} \text{FilMod}_{\mathbb{Z}} \begin{array}{c} \xleftarrow{\text{ker}} \\ \xrightarrow{\text{aug}} \end{array} \text{Mod}_{\mathbb{Z}}^{\Delta^1}.$$

Here $\text{Gr}^\bullet(V)$ is the associated graded of a filtered module $F^* V$, i.e. $\text{Gr}^i(V) = F^i V / F^{i+1} V$. The functor aug sends a filtered module $F^* V$ to the natural augmentation $(V \rightarrow \text{Gr}^0(V))$ to its 0-th graded piece. The functor ker is its *left* adjoint and sends an arrow $(p: V \rightarrow V_0)$ to the filtered module $\cdots \rightarrow 0 \rightarrow 0 \rightarrow \text{fib}(p) \rightarrow V$.

We endow each of these ∞ -categories with a t -structure making it a stably projectively generated ∞ -category in the sense of Definition A.1, as well as a closed symmetric monoidal structure:

- (a) We endow $\text{Mod}_{\mathbb{Z}}^{\Delta^1}$ with the *surjective t -structure*, in which an object $X \rightarrow Y$ is connective if Y and $\text{fib}(X \rightarrow Y)$ (and hence also X) are connective \mathbb{Z} -modules. In other words, $X \rightarrow Y$ is connective if X and Y are connective and $\pi_0(X) \rightarrow \pi_0(Y)$ is surjective. An object $X \rightarrow Y$ is coconnective if X and $\text{fib}(X \rightarrow Y)$ are coconnective \mathbb{Z} -modules. The monoidal structure is the levelwise tensor product

$$(X \rightarrow Y) \otimes (X' \rightarrow Y') = (X \otimes X' \rightarrow Y \otimes Y').$$

- (b) We endow $\text{FilMod}_{\mathbb{Z}}$ with the Day convolution product and the t -structure in which $F^* V$ is (co)connective if each $F^i V$ is (co)connective.
- (c) Likewise, $\text{GrMod}_{\mathbb{Z}}$ is equipped with the Day convolution product and the t -structure in which V^\bullet is (co)connective if each V^\bullet is (co)connective.

The generators of $\text{Mod}_{\mathbb{Z}}^{\Delta^1}$ are the surjections $\mathbb{Z}^{\oplus m+n} \rightarrow \mathbb{Z}^{\oplus n}$ and the generators of $\text{FilMod}_{\mathbb{Z}}$ and $\text{GrMod}_{\mathbb{Z}}$ are the finitely generated free abelian groups, where each generator is of a certain (filtration) weight. Each of these ∞ -categories is then a derived algebraic context in the sense of Definition A.8 (see [Rak20, Section 4.2]) and hence comes with a notion of derived commutative algebra: these are algebras over the monad obtained by right-left extending the (strict) symmetric algebra monad on the categories of surjections of flat abelian groups and of flat filtered/graded abelian groups.

Definition 2.2. We define the ∞ -categories of *animated* (resp. *derived*) *filtered rings*, *animated* (resp. *derived*) *graded rings* and *animated* (resp. *derived*) *surjections* to be the ∞ -categories

$$\begin{aligned} \mathrm{FilCAlg}^{\mathrm{an}} &= \mathrm{CAlg}^{\mathrm{an}}(\mathrm{FilMod}_{\mathbb{Z}, \geq 0}) & \mathrm{FilCAlg}^{\mathrm{der}} &= \mathrm{CAlg}^{\mathrm{der}}(\mathrm{FilMod}_{\mathbb{Z}}) \\ \mathrm{GrCAlg}^{\mathrm{an}} &= \mathrm{CAlg}^{\mathrm{an}}(\mathrm{GrMod}_{\mathbb{Z}, \geq 0}) & \mathrm{GrCAlg}^{\mathrm{der}} &= \mathrm{CAlg}^{\mathrm{der}}(\mathrm{GrMod}_{\mathbb{Z}}) \\ \mathrm{CAlg}^{\mathrm{an}, \Delta^1} &= \mathrm{CAlg}^{\mathrm{an}}(\mathrm{Mod}_{\mathbb{Z}, \geq 0}^{\Delta^1}) & \mathrm{CAlg}^{\mathrm{der}, \Delta^1} &= \mathrm{CAlg}^{\mathrm{der}}(\mathrm{Mod}_{\mathbb{Z}}^{\Delta^1}) \end{aligned}$$

of animated and derived commutative algebras in the sense of Definition A.9.

Remark 2.3. An animated surjection is simply a map of animated rings $A \rightarrow B$ which is surjective on π_0 ; these have been introduced by Z. Mao in [Mao21], where they are called animated pairs.

Remark 2.4. By Remark A.13, every derived filtered or graded ring has an underlying \mathbb{E}_∞ -algebra with respect to Day convolution, and modules are modules over this underlying \mathbb{E}_∞ -algebra.

Taking associated graded objects and augmentations commutes with the $\mathbb{L}\mathrm{Sym}$ -monads, so that we obtain functors

$$\mathrm{GrCAlg}^{\mathrm{der}} \xleftarrow{\mathrm{Gr}} \mathrm{FilCAlg}^{\mathrm{der}} \xrightarrow{\mathrm{aug}} \mathrm{CAlg}^{\mathrm{der}, \Delta^1}.$$

Both functors preserve limits, colimits and connective objects, and the functor aug sends a filtered algebra F^*A to the augmentation arrow $F^0A \rightarrow \mathrm{Gr}^0(A)$.

Definition 2.5. The *adic filtration* functor $\mathrm{adic}: \mathrm{CAlg}^{\mathrm{der}, \Delta^1} \rightarrow \mathrm{FilCAlg}^{\mathrm{der}}$ is the left adjoint to aug .

Since the augmentation functor is left t -exact, taking adic filtrations preserves connective objects. We therefore obtain a diagram of the form

$$\mathrm{GrCAlg}^{\mathrm{an}} \xleftarrow{\mathrm{Gr}} \mathrm{FilCAlg}^{\mathrm{an}} \xrightleftharpoons[\mathrm{aug}]{\mathrm{adic}} \mathrm{CAlg}^{\mathrm{an}, \Delta^1}.$$

where all functors are left adjoints. Note that taking the associated graded does not preserve limits at the connective level.

We give a familiar description of the adic filtration on the level of 0-th homotopy groups.

Proposition 2.6. *Let $(A \rightarrow B)$ be an animated surjection, and $I = \ker(\pi_0 A \rightarrow \pi_0 B)$. Then $\pi_0(\mathrm{adic}(A \rightarrow B))$ is given by the commutative algebra in $\mathrm{Fun}((\mathbb{Z}, \geq), \mathrm{Mod}_R^\heartsuit)$*

$$\dots \rightarrow \mathrm{Sym}_{\pi_0(A)}^3(I) \rightarrow \mathrm{Sym}_{\pi_0(A)}^2(I) \rightarrow I \rightarrow \pi_0(A).$$

In the situation where I is flat as a $\pi_0(A)$ -module, we have that $\mathrm{Sym}_{\pi_0(A)}^n(I) \simeq I^n$. Each map is then an inclusion and $\pi_0(\mathrm{adic}(A \rightarrow B))$ is given by the classical I -adic filtration on $\pi_0(A)$.

Proof. We can assume that both A and B are discrete. The filtered algebra $\pi_0 \mathrm{adic}(A \rightarrow B)$ has the universal property that given any discrete filtered algebra F^*R' , any map $(A \rightarrow B) \rightarrow (R' \rightarrow \mathrm{Gr}^0(R'))$ extends uniquely to a filtered map $\pi_0 \mathrm{adic}(A \rightarrow B) \rightarrow F^*R'$. Note that the category of discrete filtered algebras is equivalent to the category of commutative algebras in $\mathrm{FilMod}_R^\heartsuit = \mathrm{Fun}((\mathbb{Z}, \geq), \mathrm{Mod}_R^\heartsuit)$; in other words, $F^{n+1}R' \rightarrow F^nR'$ need not be injective.

The data of a map of surjections $(A \rightarrow B) \rightarrow (R' \rightarrow \mathrm{Gr}^0(R'))$ is the same as the data of a map $(A, I) \rightarrow (R', F^1R')$ of algebras equipped with an ideal. Let $f_1: I \rightarrow F^1R'$ be induced map on ideals. This extends uniquely to a map of discrete filtered algebra $\mathrm{Sym}_A^* I \rightarrow F^*R'$, given in weight n by

$$\mathrm{Sym}_A^n(f): \mathrm{Sym}_A^n(I) \longrightarrow F^nR; \quad f^{\otimes n}(x_1 \otimes \dots \otimes x_n) = f_1(x_1) \dots f_1(x_n),$$

where the product is taken inside F^*R . \square

Finally, recall that a filtered module is *complete* if $\lim F^n V = 0$ and that a derived filtered commutative ring is complete if it is complete as a filtered module. Equivalently, a filtered module is complete if it is local with respect to all maps inducing equivalences on the associated graded. Since taking symmetric algebras preserves such graded equivalences, it follows that the completion of a derived filtered ring is again a derived filtered ring. In good situations, one can relate the completeness of a filtered derived algebra F^*A to completeness of the underlying algebra F^0A with respect to its augmentation ideal:

Lemma 2.7. *Let A be an animated filtered ring and let I denote the kernel of the augmentation $\pi_0(F^0A) \rightarrow \pi_0(\mathrm{Gr}^0(A))$. Suppose that I is a finitely generated ideal of $\pi_0(F^0A)$. Then the following hold:*

- (1) *If M is a complete filtered A -module, then the underlying F^0A -module F^0M is I -complete [SAG, Definition 7.3.1.1].*
- (2) *Suppose that $A = \mathrm{adic}(F^0A \rightarrow \mathrm{Gr}^0(A))$ is the adic filtration of an animated surjection, and let A^\wedge be its completion. If F^0A is I -complete and $\mathrm{Gr}^0(A) \otimes_{F^0A} F^0A^\wedge \rightarrow \mathrm{Gr}^0(A)$ is an equivalence, then $A \simeq A^\wedge$.*

The proof of Lemma 2.7 makes use of the following version of the “derived Nakayama lemma”:

Lemma 2.8. *Let $A \rightarrow B$ be a map of connective \mathbb{E}_2 -rings inducing a surjection on π_0 and suppose that $I = \ker(\pi_0(A) \rightarrow \pi_0(B))$ is a finitely generated ideal. If M is a connective I -complete A -module such that $B \otimes_A M \simeq 0$, then $M \simeq 0$.*

Proof. Suppose that there exists a smallest integer n such that $\pi_n(M) \neq 0$. Then $\pi_n(M)$ is an I -complete $\pi_0(A)$ -module [SAG, Theorem 7.3.4.1] such that $\pi_n(M) \otimes_{\pi_0(A)} \pi_0(B) \simeq 0$. But [DG02, Proposition 6.5] would then imply that $\pi_n(M) = 0$. \square

Proof of Lemma 2.7. If M is a complete filtered module, then $F^0M \simeq \lim F^0M/F^nM$ is a limit of I -complete F^0A -modules: since each $x \in I$ is of weight ≥ 1 , $x^n: F^0M/F^nM \rightarrow F^0M/F^nM$ is null.

For (2), the map $A \rightarrow A^\wedge$ induces an equivalence on the associated graded, so that it suffices to verify that $F^0A \rightarrow F^0A^\wedge$ is an equivalence. To see this, let us start by observing that since A arises as an adic filtration, Proposition 2.6 implies that

$$\pi_0(F^0A^\wedge) \cong \lim_n \mathrm{coker}(\pi_0(F^nA) \rightarrow \pi_0(F^0A)) \cong \pi_0(F^0A)_I^\wedge$$

agrees with the (classical) I -adic completion of $\pi_0(F^0A)$. Since $\pi_0(F^0A)$ is I -complete, the map $\pi_0(F^0A) \rightarrow \pi_0(F^0A^\wedge)$ is surjective. Consequently, the kernel $I' = \ker(\pi_0(F^0A^\wedge) \rightarrow \pi_0(\mathrm{Gr}^0(A)))$ is the image of I ; in particular, it is finitely generated. It follows from part (1) that F^0A^\wedge is I' -complete and hence I -complete as an A -module. Lemma 2.8 then implies that $F^0A \rightarrow F^0A^\wedge$ is an equivalence if and only if $\mathrm{Gr}^0(A) \rightarrow \mathrm{Gr}^0(A) \otimes_{F^0A} F^0A^\wedge$ is an equivalence. This map is a section of the natural map $\mathrm{Gr}^0(A) \otimes_{F^0A} F^0A^\wedge \rightarrow \mathrm{Gr}^0(A)$. \square

2.2. Cotangent fibre. The usual cotangent complex formalism for derived rings extends in an evident way to the graded/filtered/arrow setting; we refer to Appendix A.1 for a general discussion. We will be mainly interested in the relative cotangent complexes of augmentation maps:

Definition 2.9 (Cotangent fibre). For every A in $\mathrm{GrCALg}^{\mathrm{der}}$ or $\mathrm{FilCALg}^{\mathrm{der}}$, we define the *cotangent fibre* to be the desuspended relative cotangent complex

$$\mathrm{cot}(A) = L_{\mathrm{Gr}^0(A)/A}[-1].$$

Here we view $\mathrm{Gr}^0(A)$ as a derived graded/filtered algebra concentrated in weight 0. Note that $\mathrm{cot}(A)$ is concentrated in weights ≥ 1 , i.e. $\mathrm{Gr}^0(\mathrm{cot}(A)) = 0$.

Likewise, for every derived arrow $(A \rightarrow B)$ we define the *cotangent fibre* to be the desuspended relative cotangent complex

$$\mathrm{cot}(A \rightarrow B) = L_{B/A}[-1].$$

One can also identify this with the relative cotangent complex of the map of arrows $(A \rightarrow B) \rightarrow (B \rightarrow B)$, which is given by the $(B \rightarrow B)$ -module $(\mathrm{cot}(A \rightarrow B), 0)$.

Note that the cotangent fibre is connective for each animated graded/filtered ring and each animated surjection.

Remark 2.10. For any filtered derived ring A and any derived arrow $(B' \rightarrow B)$, there are natural equivalences

$$\begin{aligned} \mathrm{Gr}(\mathrm{cot}(A)) &\simeq \mathrm{cot}(\mathrm{Gr}(A)) \\ \mathrm{aug}(\mathrm{cot}(A)) &\simeq \mathrm{cot}(\mathrm{aug}(A)) \\ F^{*\leq 1} \mathrm{cot}(B' \rightarrow B) &\simeq \mathrm{cot}(\mathrm{adic}(B' \rightarrow B)) \end{aligned}$$

where $F^{*\leq 1} \mathrm{cot}(B' \rightarrow B)$ denotes the B -module $\mathrm{cot}(B' \rightarrow B)$ put in filtration weight 1. The first two equivalences follow from the fact that the functors Gr and aug preserve cotangent complexes (since they commute with taking derived symmetric algebras) and the last equivalence follows by adjunction from the fact that aug preserves square zero extensions.

Lemma 2.11 ([BM, Remark 4.25]). *Let $f: A \rightarrow B$ be a map between derived graded rings. Then the following are equivalent:*

- (1) f is an equivalence.
- (2) $A^0 \rightarrow B^0$ is an equivalence and $\mathrm{cot}(A) \rightarrow \mathrm{cot}(B)$ is an equivalence of graded modules.

Similarly, a map between complete animated filtered rings $f: A \rightarrow B$ is an equivalence if and only if $\mathrm{Gr}^0(A) \rightarrow \mathrm{Gr}^0(B)$ is an equivalence and $\mathrm{cot}(A) \rightarrow \mathrm{cot}(B)$ induces an equivalence on the associated graded.

Proof. The complete filtered case follows from the graded case since equivalences between complete derived animated rings are detected on the associated graded and because the associated graded commutes with taking cotangent complexes. For the graded case, suppose that f satisfies condition (2) and that there exists a lowest weight $n \geq 1$ such that $\mathrm{cofib}(A^n \rightarrow B^n)$ is not zero. Let $A^{\geq n}$ and $B^{\geq n}$ denote the weight $\geq n$ parts of A and B , both of which are graded A -modules, and consider the natural map

$$B' = A \otimes_{\mathbb{L}\mathrm{Sym}_A(A^{\geq n})} \mathbb{L}\mathrm{Sym}_A(B^{\geq n}) \rightarrow B$$

induced by f and the inclusion $B^{\geq n} \rightarrow B$. Using that $\mathbb{L}\mathrm{Sym}_A^{\geq 2}(B^{\geq n})$ and $\mathbb{L}\mathrm{Sym}_A^{\geq 2}(B^{\geq n})$ are concentrated in weights $\geq 2n$ (hence in weights $\geq n+1$), one sees that $B' \rightarrow B$ is an equivalence in weights $\leq n$. It follows that $L_{B/B'}$ is zero in weights $\leq n$; this can be seen for instance by using the bar resolution to resolve B by free B' -algebras, each of which agrees with B' in weight $\leq n$. Consequently, in weight n we find that $L_{B/A}^n \simeq L_{B'/A}^n \simeq \mathrm{cofib}(A^n \rightarrow B^n)$ is non-zero. By part (1) of Lemma A.21, this implies that $B^0 \otimes_B L_{B/A} \simeq \mathrm{cofib}(\mathrm{cot}(A) \rightarrow \mathrm{cot}(B))$ is non-zero, contradicting the assumption that $\mathrm{cot}(A) \simeq \mathrm{cot}(B)$. \square

Lemma 2.12 ([BM, Proposition 4.26]). *For any derived arrow $(A \rightarrow B)$, there is a natural equivalence of derived graded rings*

$$\mathbb{L}\mathrm{Sym}_B(\mathrm{cot}(A \rightarrow B)(1)) \xrightarrow{\sim} \mathrm{Gr}(\mathrm{adic}(A \rightarrow B))$$

where $\mathrm{cot}(A \rightarrow B)(1)$ denotes the cotangent fibre concentrated in weight 1.

Proof. For any graded algebra B^\bullet , the map $\mathbb{L}\mathrm{Sym}_{B^0}(B^1) \rightarrow B^\bullet$ induces an equivalence in degrees ≤ 1 and hence induces a natural equivalence of B^0 -modules $B^1 \rightarrow \mathrm{cot}(B)^1$. Since

$$\mathrm{cot}(\mathrm{Gr}(\mathrm{adic}(A \rightarrow B))) \simeq \mathrm{cot}(A \rightarrow B)(1)$$

is concentrated in weight 1, we obtain an equivalence $\mathrm{cot}(A \rightarrow B) \simeq \mathrm{Gr}(\mathrm{adic}(A \rightarrow B))^1$ of B -modules. This induces a map of derived graded rings as in the lemma, which is an equivalence by Lemma 2.11. \square

2.3. Almost finite type conditions. We will use the cotangent fibre to study finiteness properties of animated (graded or filtered) rings. To this end, let us recall from Definition A.18 that a map of animated (graded or filtered) rings $A \rightarrow B$ is *almost finitely presented* if B is an almost compact object in the ∞ -category of animated (graded or filtered) A -algebras.

Lemma 2.13 ([BM, Proposition 5.15]). *The following are equivalent for an animated graded ring A :*

- (1) *$\mathrm{cot}(A)$ is an almost perfect graded A^0 -module.*
- (2) *the map of graded algebras $A \rightarrow A^0$ is almost finitely presented.*
- (3) *the map of graded algebras $A^0 \rightarrow A$ is almost finitely presented.*

Proof. It is clear that (3) \Rightarrow (2) \Rightarrow (1), so assume that $\mathrm{cot}(A)$ is almost perfect. We can then define a sequence of animated graded algebras $A^0 = A_{(0)} \rightarrow A_{(1)} \rightarrow \dots$ with colimit A , where

$$A_{(n)} = A_{(n-1)} \otimes_{\mathbb{L}\mathrm{Sym}_{A^0}(A_{(n-1)}^n)} \mathbb{L}\mathrm{Sym}_{A^0}(A^n)$$

and $A_{(0)} = A^0$. The natural map $A_{(n)} \rightarrow A$ is an equivalence in weight $\leq n$, so that the colimit is indeed A . Furthermore, an inductive argument shows that $\mathrm{cot}(A_{(n)})$ is precisely the weight $\leq n$ part of $\mathrm{cot}(A)$. Using this, there are equivalences of A^0 -modules concentrated in weight n

$$(A/A_{(n-1)})^n \simeq L_{A/A_{(n-1)}}^n \simeq (A^0 \otimes_A L_{A/A_{(n-1)}})^n \simeq \mathrm{cot}(A)^n.$$

Here the first two equivalences use that $A_{(n-1)} \rightarrow A$ is an equivalence in weight $\leq n-1$. The last equivalence uses the cofibre sequence $\mathrm{cot}(A_{(n-1)}) \rightarrow \mathrm{cot}(A) \rightarrow A^0 \otimes_A L_{A/A_{(n-1)}}$ and the fact that $\mathrm{cot}(A_{(n-1)})$ coincides with the weight $\leq n-1$ part of $\mathrm{cot}(A)$. By assumption, each $\mathrm{cot}(A)^n$ is an almost perfect A^0 -module that becomes increasingly connective as n tends to ∞ . It follows that each $A_{(n-1)} \rightarrow A_{(n)}$ is almost finitely presented, with cofibre becoming increasingly connective as n tends to ∞ . This implies that the colimit A is almost finitely presented as well. \square

Proposition 2.14. *Let $A \rightarrow B$ be a map of animated filtered rings inducing a surjection on π_0 . Then the following are equivalent:*

- (1) *B is an almost finitely presented filtered A -algebra.*
- (2) *$L_{B/A}$ is an almost perfect filtered B -module and the map of filtered algebras $\pi_0(A) \rightarrow \pi_0(B)$ is of finite presentation.*
- (3) *B is almost perfect as a filtered A -module.*

The same assertions apply to animated rings and animated graded rings.

Proof. This is a special case of Proposition A.19. \square

Definition 2.15. We will say that:

- (1) an animated graded algebra A is *almost finitely augmented* if it satisfies the equivalent conditions of Lemma 2.13.
- (2) an animated filtered algebra A is *complete almost finitely augmented* if it is complete and the (filtered) map $A \rightarrow \mathrm{Gr}^0(A)$ satisfies the equivalent conditions of Proposition 2.14.
- (3) an animated surjection $(A \rightarrow B)$ is *complete almost finitely augmented* if $\mathrm{adic}(A \rightarrow B)$ is complete almost finitely augmented.

Lemma 2.16. *Let A be a complete filtered animated ring and let M be a connective filtered A -module. Then M is an almost perfect A -module if and only if it is complete and $\mathrm{Gr}(M)$ is an almost perfect graded $\mathrm{Gr}(A)$ -module.*

Proof. This is a special case of Lemma A.22. □

We now verify that this agrees with the notions introduced in [BM] (in the pointed setting):

Lemma 2.17 ([BM, Proposition 5.28]). *Let A be a complete animated filtered ring. Then A is complete almost finitely augmented if and only if $\mathrm{Gr}(A)$ is an almost finitely augmented graded ring.*

Proof. By Proposition 2.14, A is almost finitely augmented if and only if $\mathrm{Gr}^0(A)$ is almost perfect as a filtered A -module. By Lemma 2.16, this is equivalent to $\mathrm{Gr}^0(A)$ being an almost perfect graded $\mathrm{Gr}(A)$ -module, which in turn is equivalent to $\mathrm{Gr}(A)$ being almost finitely augmented by Proposition 2.14. □

Proposition 2.18. *Let $(A \rightarrow B)$ be an animated surjection and let $I = \ker(\pi_0(A) \rightarrow \pi_0(B))$. Then the following are equivalent:*

- (1) *The map $(A \rightarrow B)$ is complete almost finitely augmented.*
- (2) *The map $(A \rightarrow B)$ is almost finitely presented and A is I -complete.*
- (3) *B is almost perfect as an A -module and A is I -complete.*

Proof. Write $A_{\mathrm{ad}} = \mathrm{adic}(A \rightarrow B)$. Assertions (2) and (3) are equivalent by Proposition 2.14. For (1) \Rightarrow (2), suppose that the adic filtration A_{ad} is complete almost finitely augmented and note that $A \rightarrow B$ is equivalent to $F^0 A_{\mathrm{ad}} \rightarrow \mathrm{Gr}^0(A_{\mathrm{ad}})$. To see that this map is almost finitely presented, note that $F^0: \mathrm{Fil}\mathrm{CAlg}_R^{\mathrm{der}} \rightarrow \mathrm{CAlg}_R^{\mathrm{der}}$ preserves almost finitely presented maps. Indeed, its right adjoint, sending a derived algebra A' to the filtered algebra $\cdots \rightarrow A' \rightarrow A' \rightarrow A'$, preserves filtered colimits and n -coconnective objects.

Since $A_{\mathrm{ad}} \rightarrow \mathrm{Gr}^0(A_{\mathrm{ad}})$ is almost finitely presented, we conclude that $F^0 A_{\mathrm{ad}} \rightarrow \mathrm{Gr}^0(A_{\mathrm{ad}})$ is almost finitely presented as well. Finally, A is I -complete by Lemma 2.7.

Let us now assume that condition (2) holds and show that A_{ad} is complete almost finitely augmented. The functor adic preserves almost finitely presented objects, since its right adjoint preserves filtered colimits and sends n -coconnective objects to $(n+1)$ -coconnective objects. Since the map of animated surjections $(A \rightarrow B) \rightarrow (B \rightarrow B)$ is almost finitely presented, we conclude that $A_{\mathrm{ad}} \rightarrow \mathrm{Gr}^0(A_{\mathrm{ad}})$ is almost finitely presented. It follows that A_{ad}^\wedge is complete, connective (by the Milnor sequence) and $\mathrm{Gr}(A_{\mathrm{ad}}^\wedge) \simeq \mathrm{Gr}(A_{\mathrm{ad}})$ is almost finitely augmented. Lemma 2.17 therefore implies that A_{ad}^\wedge is almost finitely augmented.

It therefore remains to show that $A_{\mathrm{ad}} \rightarrow A_{\mathrm{ad}}^\wedge$ is an equivalence. By Lemma 2.7 and the fact that A_{ad} is I -complete, it suffices to verify that

$$f: \mathrm{Gr}^0(A) \otimes_A F^0 A_{\mathrm{ad}}^\wedge \rightarrow \mathrm{Gr}^0(A)$$

is an equivalence. This map is an isomorphism on π_0 because $\pi_0(F^0 A_{\text{ad}}^\wedge) \cong \pi_0(A)_I^\wedge$ and $\pi_0(\text{Gr}^0(A)) \cong \pi_0(A)/I$. It then suffices to verify that its cotangent complex vanishes. Unravelling the definitions, the cotangent complex of f is the underlying module of the filtered module $\text{cofib}(\text{cot}(A_{\text{ad}}) \rightarrow \text{cot}(A_{\text{ad}}^\wedge))$, which is indeed null: its associated graded is null and $\text{cot}(A_{\text{ad}})$ and $\text{cot}(A_{\text{ad}}^\wedge)$ are both complete, by Remark 2.10 and Proposition 2.18 (and since almost perfect $\text{Gr}^0(A)$ -modules are complete). \square

Corollary 2.19 ([BM, Proposition 5.23]). *Let A be an animated filtered ring that is complete almost finitely augmented. Then the induced pair $F^0 A \rightarrow \text{Gr}^0 A$ is complete almost finitely presented.*

Proof. Since the functor aug preserves free algebras, it sends almost finitely presented maps of filtered animated rings to almost finitely presented maps of pairs. In particular, $F^0 A \rightarrow \text{Gr}^0 A$ is almost finitely presented. Lemma 2.7 implies that A is complete with respect to the kernel of $\pi_0(F^0 A) \rightarrow \pi_0 \text{Gr}^0(A)$, and we conclude by Proposition 2.18. \square

Lemma 2.20. *Let B be an animated (filtered, graded) ring, let M be an almost perfect connective (filtered, graded) B -module and let $f: B_\alpha = B \oplus_\alpha M \rightarrow B$ be a square zero extension of B by M classified by a map $\alpha: L_B[-1] \rightarrow M$. Then the following assertions hold:*

- (1) *Let \mathcal{K} be the thick subcategory of (filtered, graded) B_α -modules generated by the B_α -modules underlying an almost perfect B -module. Then $\mathcal{K} = \text{APerf}_{B_\alpha}$.*
- (2) *f is almost of finite presentation.*

Proof. Assuming (1), we see that B is almost perfect as a B_α -module, so that (2) follows from Proposition 2.14. To see that $\text{APerf}_{B_\alpha} \subseteq \mathcal{K}$, notice that every almost perfect B_α -module N fits into a cofibre sequence where the outer terms arise from almost perfect B -modules

$$M \otimes_B (B \otimes_{B_\alpha} N) \simeq M \otimes_{B_\alpha} N \longrightarrow N \longrightarrow B \otimes_{B_\alpha} N.$$

To see that $\mathcal{K} \subseteq \text{APerf}_{B_\alpha}$, observe that any object $N \in \mathcal{K}$ is eventually connective and that $\pi_k(N)$ is a finitely generated module over $\pi_0(B_\alpha)$ if N is k -connective. Indeed, these properties hold for almost perfect B -modules and are stable under (co)fibrations and retracts. Consequently, every k -connective $N \in \mathcal{K}$ fits into a cofibre sequence $B_\alpha[k]^{oplus n} \rightarrow N \rightarrow N'$ where N' is $(k+1)$ -connective. Notice that B_α is contained in \mathcal{K} , since it fits into a cofibre sequence $M \rightarrow B_\alpha \rightarrow B$ where both M and B are almost perfect B -modules. It follows that N' defines a $(k+1)$ -connective object in \mathcal{K} .

Given a module $N \in \mathcal{K}$ and $n \geq 0$, we can therefore apply this construction iteratively to obtain a cofibre sequence $K \rightarrow N \rightarrow N/K$ where K is a perfect B_α -module and N/K is n -connective. This implies that N is an almost perfect B_α -module. \square

Corollary 2.21. *Let B be an animated ring, $M \in \text{APerf}(B)_{\geq 0}$ and $B \oplus_\alpha M$ a square zero extension of B by M classified by a map $\alpha: L_B[-1] \rightarrow M$. Then the arrow $(B \oplus_\alpha M \rightarrow B)$ is complete almost finitely augmented.*

Proof. Let I be the kernel of $\pi_0(B \oplus_\alpha M) \rightarrow \pi_0(B)$. This is a finitely generated ideal, with generators arising from the generators of $\pi_0(M)$. By Proposition 2.18, it suffices to prove that $B \oplus_\alpha M \rightarrow B$ is almost of finite presentation and that $B \oplus_\alpha M$ is I -complete. The first condition follows from Lemma 2.20 and the second condition is immediate since $I^2 = 0$. \square

Corollary 2.22. *Let $(A \rightarrow B)$ be an animated arrow which is complete almost finitely presented. For any almost perfect connective A -module M and any square zero extension of A by M , the arrow $(A \oplus_\alpha M \rightarrow B)$ is again complete almost finitely presented.*

Proof. We apply Proposition 2.18. As the composite $A \oplus_\alpha M \rightarrow A \rightarrow B$ of two maps that are almost finitely presented, $A \oplus_\alpha M \rightarrow B$ is almost finitely presented. Let I be the kernel of $\pi_0(A \oplus_\alpha M) \rightarrow \pi_0(B)$. This is generated by finitely many generators x_i of $\pi_0(M)$, which square to zero, and finitely many elements y_i that lift the generators of $\ker(\pi_0(A) \rightarrow \pi_0(B))$. Then $A \oplus_\alpha M$ is evidently (x_i) -complete, and it is (y_i) -complete since it is an extension of A by M , both of which are (y_i) -complete. We conclude that $A \oplus_\alpha M$ is I -complete. \square

2.4. Comonadicity of the cotangent fibre. Throughout this section, we fix a base animated ring R . For each animated surjection $(A \rightarrow B)$ of R -algebras, the cotangent fibre $\cot(A \rightarrow B) = L_{B/A}[-1]$ is a connective B -module that comes equipped with a natural map

$$L_{B/R}[-1] \rightarrow \cot(A \rightarrow B).$$

The goal of this section will be to show that when $(A \rightarrow B)$ is complete almost finitely augmented, its structure can be completely encoded in terms of its cotangent fibre.

Notation 2.23. Let B be an animated R -algebra. We will write

$$\mathrm{CAlg}_{R/B}^{\mathrm{an}, \wedge \mathrm{aft}} \hookrightarrow \mathrm{CAlg}_{R/B}^{\mathrm{an}}$$

for the full subcategory of connective derived R -algebras A endowed with a map $A \rightarrow B$ that exhibits A as complete almost finitely augmented (Definition 2.15).

Definition 2.24. Let B be a derived R -algebra. We define a *derivation* (M, α) on B to be the datum of a B -module M , together with a map $\alpha: L_{B/R}[-1] \rightarrow M$. Write

$$\mathrm{Der}_{B/R} = (\mathrm{Mod}_B)_{L_{B/R}[-1]/}$$

for the ∞ -category of derivations (M, α) on B . If B is connective, let $\mathrm{Der}_{B/R, \geq 0}^{\mathrm{aperf}} \subseteq \mathrm{Der}_{B/R}$ denote the full subcategory spanned by the derivations (M, α) , where M is an almost perfect connective.

Our goal will then be the following:

Theorem 2.25. *Let B be an animated R -algebra. Then the cotangent fibre defines the left adjoint in a comonadic adjunction*

$$(1) \quad \cot: \mathrm{CAlg}_{R/B}^{\mathrm{an}, \wedge \mathrm{aft}} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \mathrm{Der}_{B/R, \geq 0}^{\mathrm{aperf}}: \mathrm{sqz}.$$

Let us first construct the desired adjoint pair. In fact, for later purposes we will construct a version of this adjunction that also encodes naturality in the animated R -algebra B .

Construction 2.26. Let Mod be the ∞ -category of pairs (B, M) consisting of a derived R -algebra and a B -module (cf. Construction A.17). We then have a commuting diagram as follows:

$$\begin{array}{ccc} \mathrm{CAlg}_R^{\mathrm{der}, \Delta^1} & \begin{array}{c} \xrightarrow{L} \\ \xleftarrow{\mathrm{triv}} \end{array} & \mathrm{Mod} \times_{\mathrm{CAlg}_R^{\mathrm{der}}} \mathrm{CAlg}_R^{\mathrm{der}} \\ & \begin{array}{c} \searrow \mathrm{ev}_1 \\ \swarrow \end{array} & \swarrow \\ & & \mathrm{CAlg}_R^{\mathrm{der}} \end{array}$$

Here the two downwards pointing functors send a map of derived rings $A \rightarrow B$, resp. a derived ring with a module (B, M) , to B ; both functors are cocartesian and cartesian fibrations. The pair

(L, triv) is the relative adjunction given by the cotangent complex and trivial square zero extension functors

$$L(A \rightarrow B) = (B, L_{A/R} \otimes_A B) \quad \text{triv}(B, M) = (B \oplus M \rightarrow B).$$

We abuse notation and also write $L: \text{CAlg}_R^{\text{der}} \rightarrow \text{Mod}$ for the functor sending $A \mapsto (A, L_{A/R})$. We then define the ∞ -category Der/R of *derivations* to be the fibre product

$$\begin{array}{ccc} \text{Der}/R & \longrightarrow & \text{Fun}([1], \text{Mod}) \times_{\text{Fun}([1], \text{CAlg}_R^{\text{der}})} \text{CAlg}_R^{\text{der}} \\ \downarrow & & \downarrow \text{ev}_0 \\ \text{CAlg}_R^{\text{der}} & \xrightarrow{L[-1]} & \text{Mod} \times_{\text{CAlg}_R^{\text{der}}} \text{CAlg}_R^{\text{der}}. \end{array}$$

An object of Der/R is given by a tuple (B, M, α) consisting of a derived ring B and a derivation on B , i.e. a B -module M and a map $\alpha: L_{B/R}[-1] \rightarrow M$. By adjunction, one can also identify this with a tuple (B, M, α) of a derived ring, a module and a map of derived pairs $\alpha: (B, B) \rightarrow (B \oplus M[1], B)$ which is the identity on the codomains.

Remark 2.27. The projection $\text{Der}/R \rightarrow \text{CAlg}_R^{\text{der}}; (B, M, \alpha) \mapsto B$ is a cartesian fibration by construction, classifying the functor

$$B \mapsto \text{Der}_{B/R}.$$

Each map $f: B \rightarrow B'$ induces a functor $f_*: \text{Der}_{B'/R} \rightarrow \text{Der}_{B/R}$ by restriction of scalars, sending a derivation (N, β) to $f_*(N, \beta) = (f_*N, \beta')$ with

$$\beta': L_{B/R}[-1] \longrightarrow f_*L_{B'/R}[-1] \xrightarrow{f_*\beta} f_*N$$

This admits a left adjoint, sending a derivation (M, α) to the derivation computed as the pushout of the diagram of B' -modules

$$L_{B'/R}[-1] \longleftarrow f^*L_{B/R}[-1] \xrightarrow{f^*\alpha} f^*M.$$

Consequently, the projection $\text{Der}/R \rightarrow \text{CAlg}_R^{\text{der}}$ is a cocartesian fibration as well.

Remark 2.28. Let $f: B \rightarrow B'$ be a map of derived rings with $L_{B'/B} \simeq 0$. Then the induced adjoint pair from Remark 2.27 has a simpler description: the left adjoint sends (M, α) to the derivation $(f^*M, f^*\alpha)$, with $f^*\alpha: L_{B'/R}[-1] \simeq f^*L_{B/R}[-1] \rightarrow f^*M$. In other words, the adjoint pair is simply given by the functors

$$f^*: \text{Der}_{B/R} \rightleftarrows \text{Der}_{B'/R}: f_*$$

taking base change and direct image at the level of modules.

Definition 2.29. We will write

$$(2) \quad \begin{array}{ccc} \text{CAlg}_R^{\text{der}, \Delta^1} & \begin{array}{c} \xrightarrow{\text{cot}} \\ \xleftarrow{\text{sqz}} \end{array} & \text{Der}/R \\ & \begin{array}{c} \searrow \text{ev}_1 \\ \swarrow \end{array} & \searrow \\ & & \text{CAlg}_R^{\text{der}}. \end{array}$$

for the relative adjunction defined as follows:

- (1) The right adjoint sqz sends a derivation (B, M, α) to the square zero extension $B \oplus_\alpha M \rightarrow B$ classified by α . More precisely, this is given by the fibre product in the ∞ -category of maps of derived R -algebras

$$\begin{array}{ccc} [B \oplus_\alpha M \rightarrow B] & \longrightarrow & [B \rightarrow B] \\ \downarrow & & \downarrow 0 \\ [B \rightarrow B] & \xrightarrow{\alpha} & [B \oplus M[1] \rightarrow B]. \end{array}$$

- (2) The left adjoint cot is given by

$$[A \rightarrow B] \longmapsto (B, L_{B/R}[-1] \rightarrow L_{B/A}[-1]).$$

Proof of Theorem 2.25. The proof is essentially the same as in [BM], although one cannot apply the axiomatic argument from loc. cit. (nor the modification in Section 6) *stricto sensu*. To start, notice that Corollary 2.21 implies that for each animated R -algebra B , the adjoint pair (2) restricts to an adjoint pair $\text{cot}: \text{CAlg}_{R/B}^{\text{an}, \wedge \text{aft}} \rightleftarrows \text{Der}_{B/R, \geq 0}^{\text{aperf}}: \text{sqz}$. We have to verify that this adjoint pair is comonadic. To see that cot is conservative, let $(A \rightarrow B) \rightarrow (A' \rightarrow B)$ be a map in $\text{CAlg}_{R/B}^{\text{an}, \wedge \text{aft}}$ that induces an equivalence on cotangent fibres. The induced map $\text{adic}(A \rightarrow B) \rightarrow \text{adic}(A' \rightarrow B)$ then induces an equivalence on the associated graded by Lemma 2.12. Since the adic filtrations are complete by definition, this implies that $A \rightarrow A'$ is an equivalence.

Next, let A^\bullet be a cosimplicial diagram in $\text{CAlg}_{R/B}^{\text{an}, \wedge \text{aft}}$ such that $\text{cot}(A^\bullet)$ admits a splitting in $\text{Der}_{B/R, \geq 0}^{\text{aperf}}$. Let $\text{Tot}(A^\bullet)$ be the totalisation of A^\bullet , computed in the ∞ -category $\text{CAlg}_{R/B}^{\text{der}}$ of (not necessarily connective) derived R -algebras augmented over B . We need to show that $\text{Tot}(A^\bullet)$ is complete almost finitely presented and that the natural map $\text{cot}(\text{Tot}(A^\bullet)) \rightarrow \text{Tot}(\text{cot}(A^\bullet))$ is an equivalence.

To see this, let $A_{\text{ad}}^\bullet = \text{adic}(A^\bullet \rightarrow B)$ denote the associated diagram of complete almost finitely presented derived filtered algebras. Recall that there are functors

$$\text{GrCAlg}_R^{\text{der}} \xleftarrow{\text{Gr}} \text{FilCAlg}_R^{\text{der}} \xrightarrow{\text{aug}} \text{CAlg}_R^{\text{der}, \Delta^1}$$

preserving both limits (computed in unbounded R -modules) and colimits, and commuting with the functors taking cotangent fibres. In particular, $\text{aug}(\text{Tot}(A_{\text{ad}}^\bullet)) \simeq \text{Tot}(\text{aug}(A_{\text{ad}}^\bullet)) \simeq \text{Tot}(A^\bullet)$ and there is a commuting square

$$\begin{array}{ccc} \text{aug}(\text{cot}(\text{Tot}(A_{\text{ad}}^\bullet))) & \longrightarrow & \text{aug}(\text{Tot}(\text{cot}(A_{\text{ad}}^\bullet))) \\ \sim \downarrow & & \downarrow \sim \\ \text{cot}(\text{Tot}(A^\bullet)) & \longrightarrow & \text{Tot}(\text{cot}(A^\bullet)). \end{array}$$

It therefore suffices to verify that:

- (a) the derived filtered ring $\text{Tot}(A_{\text{ad}}^\bullet)$ is complete almost finitely augmented (hence connective).
- (b) the filtered cotangent fibre preserves this totalisation (in unbounded filtered modules).

For (a), note that $\text{Tot}(A_{\text{ad}}^\bullet)$ is complete as a limit of complete algebras. Its associated graded is

$$\text{Gr}(\text{Tot}(A_{\text{ad}}^\bullet)) \simeq \text{Tot}(\text{Gr}(A_{\text{ad}}^\bullet)) \simeq \text{Tot}[\mathbb{L}\text{Sym}_B(\text{cot}(A^\bullet \rightarrow B))] \simeq \mathbb{L}\text{Sym}_B(\text{Tot}[\text{cot}(A^\bullet \rightarrow B)]).$$

Here the second equivalence is Lemma 2.12 and the last equivalence uses that the augmented cosimplicial diagram $\text{Tot}[\text{cot}(A^\bullet \rightarrow B)] \rightarrow \text{cot}(A^\bullet \rightarrow B)$ admits a splitting, so that $\mathbb{L}\text{Sym}_B$ (or any

other functor) preserves this totalisation. Since $\text{Tot}[\text{cot}(A^\bullet \rightarrow B)]$ is almost perfect by assumption, we conclude that $\text{Gr}(\text{Tot}(A_{\text{ad}}^\bullet))$ is an almost finitely augmented graded algebra (in particular connective). It follows that the complete derived filtered algebra $\text{Tot}(A_{\text{ad}}^\bullet)$ is connective as well, and Lemma 2.17 implies that it is complete almost finitely augmented.

For (b), note that there is a commuting square

$$\begin{array}{ccc} \text{Gr}(\text{cot}(\text{Tot}(A_{\text{ad}}^\bullet))) & \longrightarrow & \text{Gr}(\text{Tot}(\text{cot}(A_{\text{ad}}^\bullet))) \\ \sim \downarrow & & \downarrow \sim \\ \text{cot}(\text{Tot}[\text{Gr}(A_{\text{ad}}^\bullet)]) & \longrightarrow & \text{Tot}[\text{cot}(\text{Gr}(A_{\text{ad}}^\bullet))] \end{array}$$

since taking the associated graded commutes with totalisations (in the unbounded setting) and with cotangent complexes. Since the cosimplicial diagram $\text{Gr}(A_{\text{ad}}^\bullet) \simeq \mathbb{L}\text{Sym}_B(\text{cot}(A^\bullet \rightarrow B))$ has a *split* totalisation (preserved by any functor), the bottom map is an equivalence. Consequently, the map

$$(3) \quad \text{cot}(\text{Tot}(A_{\text{ad}}^\bullet)) \rightarrow \text{Tot}(\text{cot}(A_{\text{ad}}^\bullet))$$

induces an equivalence on the associated graded. The domain is an almost perfect connective filtered module: this follows from Proposition 2.14, since $\text{Tot}(A_{\text{ad}}^\bullet)$ is complete almost finitely presented by part (a). In particular, $\text{cot}(\text{Tot}(A_{\text{ad}}^\bullet))$ is a complete filtered module. The codomain of the map (3) is a limit of complete (in fact, almost perfect connective) filtered B -modules and is therefore complete itself. We conclude that (3) is an equivalence, as desired. \square

In particular, Theorem 2.25 shows that for every animated surjection $(A \rightarrow B)$ which is complete almost finitely augmented, there exists a cosimplicial diagram of square zero extensions of B whose totalisation is equivalent to A : this is simply the comonadic cobar resolution of A . For later purposes, we record the following variant of this construction:

Corollary 2.30. *Let $A \rightarrow B \in \text{CAlg}_{R/B}^{\text{an}, \wedge, \text{aft}}$ and let $M \in \text{APerf}_{B, \geq 0}$. For any map $\alpha: A \rightarrow B \oplus M$ of animated R -algebras augmented over B , there exists a natural coaugmented cosimplicial diagram*

$$A \longrightarrow A^\bullet \longrightarrow B \oplus M$$

in $(\text{CAlg}_{R/B}^{\text{an}, \wedge, \text{aft}})_{/B \oplus M}$ with the following properties:

- (1) For each i , the map $A^i \rightarrow B \oplus M$ is equivalent to $\text{sqz}(M^i, \alpha^i) \rightarrow \text{sqz}(M, 0)$ for a map $(M^i, \alpha^i) \rightarrow (M, 0)$ in $\text{Der}_{B/R, \geq 0}^{\text{aperf}}$.
- (2) The augmented cosimplicial diagram $\text{cot}(A) \rightarrow \text{cot}(A^\bullet)$ in $\text{Der}_{B/R, \geq 0}^{\text{aperf}}$ is split.
- (3) $A \simeq \text{Tot}(A^\bullet)$.

Proof. Suppose that $U: \mathcal{D} \rightleftarrows \mathcal{C}: R$ is a comonadic adjunction. For every object $c \in \mathcal{C}$, this induces an adjoint pair $U: \mathcal{D}_{/R(c)} \rightleftarrows \mathcal{C}_{/c}: C$ on over-categories, which is comonadic as well. It follows that every object $\alpha: d \rightarrow R(c)$ in $\mathcal{D}_{/R(c)}$ admits a comonadic cobar resolution of the form

$$d \longrightarrow (RU)^\bullet(d) \longrightarrow R(c).$$

Here each $\alpha^n: (RU)^n(d) \rightarrow R(c)$ is the image under R of the map $U(RU)^{n-1}(d) \rightarrow UR(c) \rightarrow c$, where the first map is the image of α^{n-1} under the forgetful functor U . This has the property that the augmented cosimplicial object $U(d) \rightarrow U(RU)^\bullet(d)$ in $\mathcal{C}_{/c}$ is split and (hence) that $d \simeq \text{Tot}((RU)^\bullet(d))$. We now apply this to the situation where $(U, R) = (\text{cot}, \text{sqz})$ is the comonadic adjunction (1), c is the derivation $(M, 0) \in \text{Der}_{B/R, \geq 0}^{\text{aperf}}$ and $d \rightarrow R(c)$ is the map $\alpha: A \rightarrow B \oplus M$. \square

3. PARTITION LIE ALGEBROIDS

Recall that a *Lie algebroid* over a smooth scheme Z over a field k is a quasi-coherent sheaf $\mathfrak{g} \in \mathrm{QC}_Z^\heartsuit$ equipped with an \mathcal{O}_Z -linear anchor map $\rho: \mathfrak{g} \rightarrow T_{Z/R}$ to the tangent sheaf and a k -linear Lie bracket $[-, -]: \mathfrak{g} \otimes_k \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying the Leibniz rule. The goal of this section will be to introduce a derived (and shifted) variant of such Lie algebroids over a qcqs locally coherent derived R -scheme X , which we will refer to as *partition Lie algebroids*. More precisely, we will define a partition Lie algebroid to be given by a map $\rho: \mathfrak{g} \rightarrow T_{X/R}[1]$ in the ∞ -category

$$\mathrm{QC}_X^\vee = \mathrm{Ind}(\mathrm{Coh}_X^{\mathrm{op}})$$

of pro-coherent sheaves on X , together with the structure of an algebra over a certain monad

$$\mathrm{Lie}_{\Delta, X/R}^\pi: (\mathrm{QC}_X^\vee)_{/T_{X/R}[1]} \longrightarrow (\mathrm{QC}_X^\vee)_{/T_{X/R}[1]}.$$

Here $T_{X/R} = L_{X/R}^\vee$ denotes the pro-coherent dual of the cotangent sheaf of X (Example B.27). One can therefore think of $\rho: \mathfrak{g} \rightarrow T_{X/R}[1]$ as the *anchor map* underlying the partition Lie algebroid.

Our construction of the monad $\mathrm{Lie}_{\Delta, X/R}^\pi$ will be based on two ingredients:

- (1) Let $\mathrm{Der}_{X/R, \geq 0}^{\mathrm{aperf}}$ denote the ∞ -category of derivations $\alpha: L_{X/R}[-1] \rightarrow M$ with values in a connective, almost perfect quasi-coherent sheaf on X . We then consider the comonad

$$\mathrm{coLie}_{\Delta, X/R}^\pi: \mathrm{Der}_{X/R, \geq 0}^{\mathrm{aperf}} \longrightarrow \mathrm{Der}_{X/R, \geq 0}^{\mathrm{aperf}}; \quad (M, \alpha) \longrightarrow \mathrm{cot}(\mathcal{O}_X \oplus_\alpha M)$$

sending each derivation $\alpha: L_{X/R}[-1] \rightarrow M$ to the cotangent fibre of the square zero extension $\mathcal{O}_X \oplus_\alpha M$.

- (2) Taking pro-coherent duals determines a fully faithful functor (Lemma A.34)

$$(-)^\vee: \mathrm{Der}_{X/R, \geq 0}^{\mathrm{aperf}, \mathrm{op}} \longleftarrow (\mathrm{QC}_X^\vee)_{/T_{X/R}[1]}.$$

Its essential image consists of maps $F \rightarrow T_{X/R}[1]$ where F is dually almost perfect (Definition A.33) and of non-positive tor-amplitude (Definition A.35).

Our main aim will then be to prove the following:

Theorem 3.1. *Let X be a locally coherent qcqs derived R -scheme. There exists a unique extension*

$$\begin{array}{ccc} \mathrm{Der}_{X/R, \geq 0}^{\mathrm{aperf}, \mathrm{op}} & \xrightarrow{(\mathrm{coLie}_{\Delta, X/R}^\pi)^{\mathrm{op}}} & \mathrm{Der}_{X/R, \geq 0}^{\mathrm{aperf}, \mathrm{op}} \\ (-)^\vee \downarrow & & \downarrow (-)^\vee \\ (\mathrm{QC}_X^\vee)_{/T_{X/R}[1]} & \xrightarrow{\mathrm{Lie}_{\Delta, X/R}^\pi} & (\mathrm{QC}_X^\vee)_{/T_{X/R}[1]} \end{array}$$

to a monad $\mathrm{Lie}_{\Delta, X/R}^\pi$ that preserves sifted colimits.

Definition 3.2. We will refer to the monad $\mathrm{Lie}_{\Delta, X/R}^\pi$ of Theorem 3.1 as the *partition Lie algebroid monad* and define the ∞ -category of *partition Lie algebroids*

$$\mathrm{LieAlg}_{\Delta, X/R}^\pi = \mathrm{Alg}_{\mathrm{Lie}_{\Delta, X/R}^\pi} \left((\mathrm{QC}_X^\vee)_{/T_{X/R}[1]} \right)$$

to be its ∞ -category of algebras.

We will prove the affine case of Theorem 3.1 in Section 3.1. In Section 3.2 we will study the naturality of the ∞ -category $\mathrm{LieAlg}_{\Delta, B/R}^{\pi}$ in the coherent animated R -algebra B . More precisely, we will prove the following in Proposition 3.25:

- (a) For each map $f: B \rightarrow B'$ of that is almost finitely presented, there exists a functor

$$f^{\#}: \mathrm{LieAlg}_{\Delta, B/R}^{\pi} \longrightarrow \mathrm{LieAlg}_{\Delta, B'/R}^{\pi}.$$

- (b) When f is étale, the functor $f^{\#} = f^*$ is given by the usual inverse image at the level of the underlying pro-coherent module, and admits a right adjoint.

We will then use this in Section 3.3 to prove Theorem 3.1 by a descent argument.

3.1. Partition Lie algebroids on affine schemes. Let B be an animated R -algebra and recall the adjoint pair from Definition 2.29

$$\mathrm{cot}: \mathrm{CAlg}_{R/B}^{\mathrm{der}} \begin{array}{c} \xrightarrow{\quad} \\ \longleftarrow \end{array} \mathrm{Der}_{B/R} : \mathrm{sqz}.$$

Here $\mathrm{Der}_{B/R}$ is the ∞ -category of derivations $(M, \alpha) = (M, \alpha: L_{B/R}[-1] \rightarrow M)$ on B (Definition 2.24).

Notation 3.3. Let us write $\mathrm{coLie}_{\Delta, B/R}^{\pi} := \mathrm{cot} \circ \mathrm{sqz}$ for the composite comonad

$$\mathrm{coLie}_{\Delta, B/R}^{\pi} = \mathrm{cot} \circ \mathrm{sqz}: \mathrm{Der}_{B/R} \longrightarrow \mathrm{Der}_{B/R}.$$

Remark 3.4. Given a derivation (M, α) , one can endow $\mathrm{coLie}_{\Delta, B/R}^{\pi}(M, \alpha)$ with a natural decreasing filtration, as follows. Consider $L_{B/R}[-1]$ as a filtered module in weight 0 (i.e. $F^1 = 0$) and consider M as a filtered module in weight 1, i.e., $F^2 M = 0$ and $F^1 M = M$. The derivation α then lifts to a derivation $\alpha: L_{B/R}[-1] \rightarrow F^* M$ and we obtain a filtration

$$F^* \mathrm{coLie}_{\Delta, B/R}^{\pi}(M, \alpha) = \mathrm{cot}(B \oplus_{\alpha} F^* M).$$

Note that $L_{B/R}[-1] \rightarrow F^* M$ is null on the associated graded, so that the associated graded of this filtration is equivalent to $\mathrm{coLie}_{\Delta, B/R}^{\pi}(M, 0)$.

We will use the following notation throughout this paper:

Definition 3.5. Let \mathcal{C} be an ∞ -category such that the inclusion $\tau_{\leq n} \mathcal{C} \subseteq \mathcal{C}$ of its n -truncated objects admits a left adjoint $\tau_{\leq n}$ for each $n \in \mathbb{N}$. We will say that a tower

$$\cdots \rightarrow X_1 \rightarrow X$$

is *almost eventually constant* if for each $n \in \mathbb{N}$, there exists an m such that

$$\tau_{\leq n} X_{i+1} \rightarrow \tau_{\leq n} X_i$$

is an equivalence for all $i \geq m$.

Lemma 3.6. *The comonad $\mathrm{coLie}_{\Delta, B/R}^{\pi}: \mathrm{Der}_{B/R} \rightarrow \mathrm{Der}_{B/R}$ has the following properties:*

- (1) *It preserves derivations (M, α) where M is an almost perfect connective B -module.*
- (2) *It preserves sifted colimits.*
- (3) *For each finite cosimplicial diagram $(M^{\bullet}, \alpha^{\bullet})$ in $\mathrm{Der}_{B/R}$ such that $\mathrm{Tot}(M^{\bullet})$ and all M^t are in $\mathrm{APerf}_{B, \geq 0}$, there is a natural equivalence*

$$\mathrm{coLie}_{\Delta, B/R}^{\pi}(\mathrm{Tot}(M^{\bullet}, \alpha^{\bullet})) \simeq \mathrm{Tot}(\mathrm{coLie}_{\Delta, B/R}^{\pi}(M^{\bullet}, \alpha^{\bullet})).$$

(4) For any tower $\cdots \rightarrow (M_2, \alpha_2) \rightarrow (M_1, \alpha_1)$ in $\text{Der}_{B/R}$ such that $\cdots \rightarrow M_2 \rightarrow M_1$ is eventually constant (i.e. the connectivity of $\text{fib}(M_{n+1} \rightarrow M_n)$ tends to ∞), there is an equivalence

$$\text{coLie}_{\Delta, B/R}^{\pi}(\lim_n(M_n, \alpha_n)) \simeq \lim_n(\text{coLie}_{\Delta, B/R}^{\pi}(M_n, \alpha_n)).$$

Proof. Property (1) follows from Lemma 2.20 and (2) follows from the fact that cot and sqz both preserve sifted colimits. For (3), we use the filtration from Remark 3.4. Since limits in $\text{Der}_{B/R}$ are computed on the underlying B -modules, it then suffices to show that the natural map

$$\mu: \text{cot}(\text{Tot}(B \oplus_{\alpha^{\bullet}} F^* M^{\bullet})) \rightarrow \text{Tot}(\text{cot}(B \oplus_{\alpha^{\bullet}} F^* M^{\bullet}))$$

is an equivalence of filtered modules. Proposition 2.14 and Lemma 2.20 imply that $\text{cot}(B \oplus_{\alpha} F^* M)$ is an almost perfect filtered B -module whenever M is almost perfect. In particular, the domain and codomain of μ are complete, and it remains to verify that the induced map on the associated graded is an equivalence. As the maps α^{\bullet} are null on the associated graded, it will be enough to show that the functor

$$F: \text{Mod}_B \longrightarrow \text{GrMod}_B^{\geq 1}; \quad M \longrightarrow \text{cot}(B \oplus M(1))$$

preserves finite totalisations, where $\text{GrMod}_B^{\geq 1}$ is the ∞ -category of *strictly* positively graded B -modules. Using the bar resolution, the functor F can be obtained via colimits and compositions from the functors $\mathbb{L}\text{Sym}_B^n: \text{GrMod}_B^{\geq 1} \rightarrow \text{GrMod}_B^{\geq 1}$. These are n -excisive and hence preserve finite totalisations [BM, Proposition 3.37]. Since the functors preserving finite totalisations are stable under composition and colimits, the result follows.

For (4), note that if $\text{fib}(M_{n+1} \rightarrow M_n)$ is i -connective, then $B \oplus_{\alpha_{n+1}} M_{n+1} \rightarrow B \oplus_{\alpha_n} M_n$ has i -connective fibres. Consequently, its relative cotangent complex is $(i+1)$ -connective [SAG, Corollary 25.3.6.4]. Taking cotangent fibres, one finds that $\text{coLie}_{\Delta}^{\pi}(M_{n+1}) \rightarrow \text{coLie}_{\Delta}^{\pi}(M_n)$ has i -connective fibres. The tower therefore stabilises after k -truncation for any k , from which the result follows. \square

Let us now suppose that B is a coherent animated R -algebra and consider the dual picture, using linear duality in the ∞ -category $\text{QC}_B^{\vee} = \text{Ind}(\text{Coh}_B^{\text{op}})$ of pro-coherent B -modules. We refer to Appendix B for the theory of pro-coherent sheaves and just recall that there are left adjoint functors (cf. Observation A.30 and Definition A.31)

$$\iota: \text{Mod}_B \longrightarrow \text{QC}_B^{\vee} \quad (-)^{\vee}: \text{Mod}_B \longrightarrow \text{QC}_B^{\vee, \text{op}}.$$

The first functor is fully faithful on eventually connective B -modules and the second is fully faithful on almost perfect B -modules. We will write $\text{dAPerf}_B \subseteq \text{QC}_B^{\vee}$ for the full subcategory of *dually almost perfect* pro-coherent B -modules (Definition A.33). Explicitly, one can identify QC_B^{\vee} with the ∞ -category of exact functors $F: \text{Coh}_B \rightarrow \text{Sp}$. The pro-coherent module $\iota(M)$ then sends $K \mapsto K \otimes_B M$ and the pro-coherent dual M^{\vee} sends $K \mapsto \text{Hom}_B(M, K)$.

Notation 3.7. Given a coherent animated R -algebra $B \in \text{CAlg}_R^{\text{an}}$, we will write $T_{B/R} = L_{B/R}^{\vee}$ for the pro-coherent dual of the cotangent complex and refer to it as the (pro-coherent) *tangent complex*.

By Lemma A.34, pro-coherent duality determines an equivalence

$$(-)^{\vee}: \text{Der}_{B/R}^{\text{aperf}, \text{op}} = ((\text{APerf}_B)_{L_{B/R}[-1]})^{\text{op}} \xrightarrow{\sim} (\text{dAPerf}_B)_{/T_{B/R}[1]}.$$

Using this, we construct the partition Lie algebroid monad in the affine setting:

Proposition 3.8. *Let B be a coherent animated R -algebra. Then there exists a unique extension*

$$\begin{array}{ccc} \mathrm{Der}_{B/R, \geq 0}^{\mathrm{aperf}, \mathrm{op}} & \xrightarrow{(\mathrm{coLie}_{\Delta, B/R}^{\pi})^{\mathrm{op}}} & \mathrm{Der}_{B/R, \geq 0}^{\mathrm{aperf}, \mathrm{op}} \\ (-)^{\vee} \downarrow & & \downarrow (-)^{\vee} \\ (\mathrm{QC}_B^{\vee})_{/T_{B/R}[1]} & \xrightarrow{\mathrm{Lie}_{\Delta, B/R}^{\pi}} & (\mathrm{QC}_B^{\vee})_{/T_{B/R}[1]} \end{array}$$

to a monad $\mathrm{Lie}_{\Delta, B/R}^{\pi}$ that preserves sifted colimits.

Definition 3.9. Following Definition 3.2, we will write $\mathrm{LieAlgd}_{\Delta, B/R}^{\pi}$ for the ∞ -category of partition Lie algebroids over B , that is, algebras over the monad $\mathrm{Lie}_{\Delta, B/R}^{\pi}$.

Before turning to the proof of Proposition 3.8, we record some first consequences of the above definition.

Example 3.10. If $B = R$ is a coherent animated ring, then the monad $\mathrm{Lie}_{\Delta, B/B}^{\pi}$ coincides with the partition Lie algebra monad from [BM, BCN], so that $\mathrm{LieAlgd}_{\Delta, B/B}^{\pi} \simeq \mathrm{LieAlg}_{\Delta, B}^{\pi}$ is equivalent to the ∞ -category of partition Lie algebras over B . In particular, the free partition Lie algebra on a dually almost perfect B -module can be computed explicitly in terms of the partition complex, using the derived functor of Σ_r -invariants (see [BCN, Section 3.6])

$$\mathrm{Lie}_{\Delta, B/B}^{\pi}(M) \simeq \sum_r (\tilde{C}^*(\Sigma|\Pi_r|^{\circ}, B) \otimes_B M^{\otimes r})^{\Sigma_r}.$$

Remark 3.11. For a coherent animated R -algebra B , the free partition Lie algebroid $\mathrm{Lie}_{\Delta, B/R}^{\pi}(M, \alpha)$ can be viewed as a deformation of the free B -linear partition Lie algebra on M . More precisely, $\mathrm{Lie}_{\Delta, B/R}^{\pi}(M, \alpha)$ admits a natural exhaustive increasing filtration whose associated graded is equivalent to the free partition Lie algebra $\mathrm{Lie}_{\Delta, B/R}^{\pi}(M, 0) \simeq \mathrm{Lie}_{\Delta, B/B}^{\pi}(M)$. Indeed, Remark 3.4 and Lemma A.22 provide such a filtration when M is the pro-coherent dual of a connective almost perfect B -module; one then extends by sifted colimits. The resulting filtration is analogous to the filtration on free Lie algebroids constructed in [Kap07].

Proposition 3.12. *Let B be a coherent animated R -algebra. Then the following hold:*

- (1) *There is an equivalence $(\mathrm{LieAlgd}_{\Delta, B/R}^{\pi})_{/0} \simeq \mathrm{LieAlg}_{\Delta, B}^{\pi}$.*
- (2) *Taking the fibre of the anchor map defines a sifted colimit preserving right adjoint functor $\mathrm{LieAlgd}_{\Delta, B/R}^{\pi} \rightarrow \mathrm{LieAlg}_{\Delta, B}^{\pi}$.*

It follows that one can identify $\mathrm{LieAlgd}_{\Delta, B/R}^{\pi}$ with the ∞ -category of algebras over a monad on $\mathrm{LieAlg}_{\Delta, B}^{\pi}$ whose underlying endofunctor sends \mathfrak{g} to $\mathfrak{g} \times T_{B/R}$.

Proof. For (1), note that the forgetful functor $(\mathrm{LieAlgd}_{\Delta, B/R}^{\pi})_{/0} \rightarrow \mathrm{QC}_B^{\vee}$ exhibits $(\mathrm{LieAlgd}_{\Delta, B/R}^{\pi})_{/0}$ as the ∞ -category of algebras over a sifted colimit preserving monad on QC_B^{\vee} . Unravelling the definitions, this monad coincides with $\mathrm{Lie}_{\Delta, B/B}^{\pi}$, so that the result follows from Example 3.10. Part (2) follows from part (1) and the fact that taking pullbacks along the map of partition Lie algebroids $0 \rightarrow T_{B/R}[1]$ preserves limits and sifted colimits. \square

Finally, we record the following reformulation of Theorem 2.25, using pro-coherent duality:

Corollary 3.13. *Let B be a coherent animated R -algebra. Then there is a fully faithful functor*

$$\mathfrak{D}: \mathrm{CAlg}_{R/B}^{\mathrm{an}, \wedge \mathrm{aft}} \xrightarrow{\mathrm{cot}} \mathrm{Coalg}_{\mathrm{coLie}_{\Delta}^{\pi}}(\mathrm{Der}_{B/R, \geq 0}^{\mathrm{aperf}}) \xrightarrow{(-)^{\vee}} (\mathrm{LieAlgd}_{\Delta, B/R}^{\pi})^{\mathrm{op}}$$

whose essential image consists of the partition Lie algebroids whose underlying pro-coherent B -module is dually almost perfect of non-positive tor-amplitude.

Let us now turn to the proof of Proposition 3.8. We will show that the ∞ -category $(\mathrm{QC}_B^\vee)_{/T_{B/R}[1]}$ is generated under sifted colimits by the full subcategory of $M \rightarrow T_{B/R}[1]$ where M is dually almost perfect of nonpositive tor-amplitude (Definition A.35). To make this more precise, recall the process of adding colimits to ∞ -categories from [HTT, Section 5.3.6]:

Definition 3.14. Let \mathcal{K} be a class of small ∞ -categories. An ∞ -category \mathcal{V} is said to be \mathcal{K} -complete if it admits all colimits of diagrams indexed by ∞ -categories in \mathcal{K} .

Suppose that \mathcal{C} is a small ∞ -category equipped with a set of colimiting cocones $\mathcal{R} = \{K_\alpha^\triangleright \rightarrow \mathcal{C}\}$ where each $K_\alpha \in \mathcal{K}$. We will say that a functor $j: \mathcal{C} \rightarrow \mathcal{V}$ exhibits \mathcal{V} as the \mathcal{K} -completion of \mathcal{C} relative to \mathcal{R} if it is the initial \mathcal{K} -complete ∞ -category with a functor from \mathcal{C} that preserves the colimit diagrams in \mathcal{R} .

By [HTT, Proposition 5.3.6.2], the \mathcal{K} -completion of \mathcal{C} relative to \mathcal{R} always exists and the functor $j: \mathcal{C} \rightarrow \mathcal{V}$ is fully faithful.

Proposition 3.15. Let \mathcal{K} be a class of small ∞ -categories, each of which is contractible. Suppose that $j: \mathcal{C} \hookrightarrow \mathcal{V}$ exhibits \mathcal{V} as the \mathcal{K} -completion of \mathcal{C} relative to a set of colimit diagrams $\mathcal{R} = \{K_\alpha^\triangleright \rightarrow \mathcal{C}\}$. For any object $X \in \mathcal{V}$, the induced fully faithful functor

$$j: \mathcal{C}_{/X} \hookrightarrow \mathcal{V}_{/X}$$

exhibits $\mathcal{V}_{/X}$ as the \mathcal{K} -completion of $\mathcal{C}_{/X}$ relative to the set $\mathcal{R}_{/X}$ of colimit diagrams $K_\alpha^\triangleright \rightarrow \mathcal{C}_{/X}$ whose image in \mathcal{C} is contained in \mathcal{R} .

Proof. Let $\mathcal{P}_{\mathcal{R}}(\mathcal{C}) \subseteq \mathcal{P}(\mathcal{C})$ be the ∞ -category of presheaves on \mathcal{C} that send the colimit diagrams in \mathcal{R} to limits. By the proof of [HTT, Proposition 5.3.6.2], we can then identify $\mathcal{V} \subseteq \mathcal{P}_{\mathcal{R}}(\mathcal{C})$ with the smallest full subcategory that contains the representable presheaves and is closed under \mathcal{K} -indexed colimits. Using this, we can identify X with a presheaf $X: \mathcal{C}^{\mathrm{op}} \rightarrow \mathcal{S}$ and $\mathcal{C}_{/X}$ is the full subcategory of representable presheaves over X .

Now recall that the inclusion $\mathcal{C}_{/X} \hookrightarrow \mathcal{P}(\mathcal{C})_{/X}$ induces a natural equivalence $\mathcal{P}(\mathcal{C}_{/X}) \rightarrow \mathcal{P}(\mathcal{C})_{/X}$ [HTT, Corollary 5.1.6.12] (the proof in loc. cit. also applies to non-representable X). This equivalence restricts to an equivalence $\mathcal{P}_{\mathcal{R}}(\mathcal{C})_{/X} \simeq \mathcal{P}_{\mathcal{R}_{/X}}(\mathcal{C}_{/X})$. Since the forgetful functor $\mathcal{P}_{\mathcal{R}_{/X}}(\mathcal{C}_{/X}) \rightarrow \mathcal{P}_{\mathcal{R}}(\mathcal{C})$ detects colimits of \mathcal{K} -indexed diagrams (which are all contractible), it follows that $\mathcal{V}_{/X} \subseteq \mathcal{P}_{\mathcal{R}_{/X}}(\mathcal{C}_{/X})$ is the smallest full subcategory that contains the representable presheaves and is closed under \mathcal{K} -indexed colimits. The result then follows from [HTT, Proposition 5.3.6.2]. \square

Proof of Proposition 3.8. Let $\mathrm{dAPerf}_B^{\leq 0} \subseteq \mathrm{QC}_B^\vee$ denote the full subcategory of dually almost perfect objects of non-positive tor-amplitude (Definition A.35). This ∞ -category is equivalent to $\mathrm{APerf}_{B, \geq 0}^{\mathrm{op}}$ by pro-coherent duality. It now follows from [BCN, Proposition 4.20] that $\mathrm{dAPerf}_B^{\leq 0} \rightarrow \mathrm{QC}_B^\vee$ exhibits QC_B^\vee as the sifted colimit completion of $\mathrm{dAPerf}_B^{\leq 0}$ relative to the set \mathcal{R}_B of colimit diagrams in $\mathrm{dAPerf}_B^{\leq 0}$ that are dual to the following diagrams in $\mathrm{APerf}_{B, \geq 0}$:

- (a) Augmented cosimplicial diagrams $M^\bullet: \Delta_+ \rightarrow \mathrm{APerf}_{B, \geq 0}$ such that $M^{-1} \simeq \mathrm{Tot}(M^\bullet)$ (computed in Mod_B).
- (b) Limits $M_\infty \rightarrow \cdots \rightarrow M_2 \rightarrow M_1$ of towers in $\mathrm{APerf}_{B, \geq 0}$ that are almost eventually constant (see Definition 3.5).

By Proposition 3.15, $(\mathrm{QC}_B^\vee)_{/T_{B/R}[1]}$ is the sifted colimit completion of $(\mathrm{dAPerf}_B^{\leq 0})_{/T_{B/R}[1]}$ relative to $(\mathcal{R}_B)_{/T_{B/R}[1]}$. This implies that restriction along j determines an equivalence between:

- (1) the (monoidal) ∞ -category of sifted colimit preserving endofunctors of $(\mathrm{QC}_B^\vee)_{/T_{B/R}[1]}$ that preserve the full subcategory $(\mathrm{dAPerf}_B^{\leq 0})_{/T_{B/R}[1]}$.
- (2) the (monoidal) ∞ -category of endofunctors of $(\mathrm{dAPerf}_B^{\leq 0})_{/T_{B/R}[1]}$ that preserve all sifted colimits of the form (a) and (b) above.

Using that $(\mathrm{dAPerf}_B^{\leq 0})_{/T_{B/R}[1]} \simeq \mathrm{Der}_{B/R, \geq 0}^{\mathrm{aperf}, \mathrm{op}}$ by pro-coherent duality (Lemma A.34), the result then follows from Lemma 3.6. \square

3.2. Base change. We will now study the functoriality of the ∞ -categories $\mathrm{LieAlgd}_{\Delta, B/R}^\pi$ of partition Lie algebroids with respect to the animated R -algebra B . We will do this by directly construct the cartesian fibration classifying $B \mapsto \mathrm{LieAlgd}_{\Delta, B/R}^\pi$. This is somewhat technical, and we refer to Proposition 3.25 for the final result.

We start by studying the functoriality of the ∞ -category $(\mathrm{QC}_B^\vee)_{/T_{B/R}[1]}$ in the coherent animated R -algebra B . To do this, we will organise these ∞ -categories into a cartesian fibration over the opposite of a certain subcategory of animated R -algebras.

Notation 3.16. Let $\mathrm{CAlg}_R^{\mathrm{an}, \mathrm{coft}} \hookrightarrow \mathrm{CAlg}_R^{\mathrm{an}}$ denote the subcategory whose objects are coherent animated R -algebras and whose morphisms are maps $A \rightarrow B$ that are almost finitely presented. More generally, for any functor $\mathcal{M} \rightarrow \mathrm{CAlg}_R^{\mathrm{an}}$ or $\mathcal{N} \rightarrow \mathrm{CAlg}_R^{\mathrm{an}, \mathrm{op}}$, we denote

$$\mathcal{M}^{\mathrm{coft}} = \mathcal{M} \times_{\mathrm{CAlg}_R^{\mathrm{an}}} \mathrm{CAlg}_R^{\mathrm{an}, \mathrm{coft}} \quad \mathcal{N}^{\mathrm{coft}} = \mathcal{N} \times_{\mathrm{CAlg}_R^{\mathrm{an}, \mathrm{op}}} \mathrm{CAlg}_R^{\mathrm{an}, \mathrm{coft}, \mathrm{op}}.$$

Recall the ∞ -category $\mathrm{Der}_{/R}$ of derivations $(A, M, \alpha) = (A, \alpha: L_{A/R}[-1] \rightarrow M)$ from Construction 2.26, and let $\mathrm{Der}_{/R, \geq 0}^{\mathrm{aperf}} \subseteq \mathrm{Der}_{/R}$ denote the full subcategories of derivations (A, M, α) where M is a connective almost perfect A -module.

Lemma 3.17. *The projection $\pi: \mathrm{Der}_{/R, \geq 0}^{\mathrm{aperf}, \mathrm{coft}} \rightarrow \mathrm{CAlg}_R^{\mathrm{an}, \mathrm{coft}}$ is a cocartesian fibration.*

Proof. Let $f: A \rightarrow B$ be a map of animated R -algebras. Recall from Remark 2.27 that a map $(A, M, \alpha) \rightarrow (B, N, \beta)$ covering f is a cocartesian arrow in $\mathrm{Der}_{/R}$ if the induced square of B -modules

$$\begin{array}{ccc} f^* L_{A/R}[-1] & \longrightarrow & L_{B/R}[-1] \\ f^*(\alpha) \downarrow & & \downarrow \beta \\ f^* M & \longrightarrow & N \end{array}$$

is cocartesian. If f is almost finitely presented, then the cofibres of the horizontal maps are almost perfect connective B -modules. Consequently, N will be an almost perfect B -module if M is an almost perfect connective A -module. We conclude that $\mathrm{Der}_{/R, \geq 0}^{\mathrm{aperf}, \mathrm{coft}} \hookrightarrow \mathrm{Der}_{/R}$ is stable under taking cocartesian lifts of maps in $\mathrm{CAlg}_R^{\mathrm{an}, \mathrm{coft}}$, so that π is indeed a cocartesian fibration. \square

Construction 3.18. Let $\mathrm{Der}_{/R, \geq 0}^{\mathrm{aperf}, \mathrm{coft}, \mathrm{op}} \rightarrow \mathrm{CAlg}_R^{\mathrm{an}, \mathrm{coft}, \mathrm{op}}$ be the opposite of the cocartesian fibration from Lemma 3.17. This classifies a functor $\mathrm{CAlg}_R^{\mathrm{an}, \mathrm{coft}} \rightarrow \mathrm{Cat}_\infty$ sending $A \mapsto \mathrm{Der}_{A/R, \geq 0}^{\mathrm{aperf}, \mathrm{op}}$ and sending each map $f: A \rightarrow B$ to the functor

$$(4) \quad (L_{A/R}[-1] \rightarrow M) \longmapsto (L_{B/R}[-1] \rightarrow L_{B/R}[-1] \oplus_{f^* L_{A/R}[-1]} f^* M).$$

Write \mathcal{R}_A for the collection of colimit diagrams in $\mathrm{Der}_{A/R, \geq 0}^{\mathrm{aperf}, \mathrm{op}}$ opposite to pullback squares of derivations along π_0 -surjections, as well as limits of towers of derivations that are almost eventually constant (cf. Definition 3.5).

By Proposition 3.15 and Lemma A.42, the fully faithful functor

$$(5) \quad (-)^\vee : \mathrm{Der}_{A/R, \geq 0}^{\mathrm{aperf}, \mathrm{op}} \longrightarrow (\mathrm{QC}_A^\vee)_{/T_{A/R}[1]}$$

exhibits $(\mathrm{QC}_A^\vee)_{/T_{A/R}[1]}$ as the completion of $\mathrm{Der}_{A/R, \geq 0}^{\mathrm{aperf}, \mathrm{op}}$ under colimits relative to the set of colimit diagrams \mathcal{R}_A from the proof of Proposition 3.8 (see Definition 3.14).

Each base change functor $\mathrm{Der}_{A/R, \geq 0}^{\mathrm{aperf}, \mathrm{op}} \rightarrow \mathrm{Der}_{B/R, \geq 0}^{\mathrm{aperf}, \mathrm{op}}$ sends colimit diagrams in \mathcal{R}_A to colimit diagrams in \mathcal{R}_B . The universal property of $(\mathrm{QC}_A^\vee)_{/T_{A/R}[1]}$ therefore implies that the maps (5) assemble into a natural transformation between two diagrams of ∞ -categories. Explicitly, this sends $f: A \rightarrow B$ to a commuting square

$$\begin{array}{ccc} \mathrm{Der}_{A/R, \geq 0}^{\mathrm{aperf}, \mathrm{op}} & \xleftarrow{(-)^\vee} & (\mathrm{QC}_A^\vee)_{/T_{A/R}[1]} \\ \downarrow & & \downarrow f^\sharp \\ \mathrm{Der}_{B/R, \geq 0}^{\mathrm{aperf}, \mathrm{op}} & \xleftarrow{(-)^\vee} & (\mathrm{QC}_A^\vee)_{/T_{B/R}[1]} \end{array}$$

where the right vertical functor f^\sharp is the unique colimit preserving extension of the left vertical functor, given by

$$f^\sharp(F \rightarrow T_{A/R}[1]) = f^*F \times_{(f^*L_{A/R})^\vee[1]} T_{A/R}[1].$$

Note that this indeed preserves colimits and is dual to (4).

Definition 3.19. We will write $\mathrm{QC}_{/T[1]}^\vee \rightarrow \mathrm{CAlg}_R^{\mathrm{an}, \mathrm{coft}, \mathrm{op}}$ for the cartesian fibration classified by the functor $A \mapsto (\mathrm{QC}_A^\vee)_{/T_{A/R}[1]}$ from Construction 3.18, and

$$\begin{array}{ccc} \mathrm{Der}_{/R, \geq 0}^{\mathrm{aperf}, \mathrm{coft}, \mathrm{op}} & \xleftarrow{(-)^\vee} & \mathrm{QC}_{/T[1]}^\vee \\ & \searrow & \swarrow \\ & \mathrm{CAlg}_R^{\mathrm{an}, \mathrm{coft}, \mathrm{op}} & \end{array}$$

for the unstraightening of the natural transformation from Construction 3.18.

Our next goal will be to construct a version of the partition Lie algebroid monad in families. To this end, recall from Definition 2.29 that taking cotangent fibres and square zero extensions defines a relative adjunction

$$(6) \quad \begin{array}{ccc} \mathrm{CAlg}_R^{\mathrm{der}, \Delta^1} & \begin{array}{c} \xrightarrow{\mathrm{cot}} \\ \xleftarrow{\mathrm{sqz}} \end{array} & \mathrm{Der}/R \\ & \searrow \mathrm{ev}_1 \quad \swarrow & \\ & \mathrm{CAlg}_R^{\mathrm{der}} & \end{array}$$

We will write $\mathrm{coLie}_\Delta^\pi = \mathrm{cot} \circ \mathrm{sqz}$ for the composite comonad relative to the base $\mathrm{CAlg}_R^{\mathrm{der}}$. On the fibre over each animated R -algebra B , this restricts to the comonad $\mathrm{coLie}_{\Delta, B/R}^\pi$ from Notation 3.3.

Lemma 3.20. *The adjoint pair (6) enjoys the following properties:*

- (1) *Let $f: (A' \rightarrow A) \rightarrow (B' \rightarrow B)$ be a map of animated surjections such that $A \otimes_{A'} B' \simeq B$ and $L_{B/A} \simeq 0$. Then $\mathrm{cot}(A' \rightarrow A) \rightarrow \mathrm{cot}(B' \rightarrow B)$ is a cocartesian arrow in Der/R .*

(2) Let $(A, M, \alpha) \rightarrow (B, N, \beta)$ be a cocartesian arrow in Der/R such that A, M, B, N are all connective and $L_{B/A} \simeq 0$. Then the induced square of animated rings

$$\begin{array}{ccc} A \oplus_{\alpha} M & \longrightarrow & B \oplus_{\beta} N \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

is cocartesian.

Proof. For property (1), note that the left commutative square of animated R -algebras gives rise to the right square of B -modules:

$$\begin{array}{ccc} A' & \longrightarrow & A \\ f' \downarrow & & \downarrow f \\ B' & \longrightarrow & B \end{array} \quad \begin{array}{ccc} f^* L_{A/R} & \longrightarrow & f^* L_{A/A'} \\ \downarrow & & \downarrow \\ L_{B/R} & \longrightarrow & L_{B/B'} \end{array}$$

Since left square is a pushout of derived rings, the right vertical map $f^* L_{A/A'} \rightarrow L_{B/B'}$ is an equivalence. The left vertical map is an equivalence as well, since $f: A \rightarrow B$ was formally étale. It follows that the right square is a pushout square of B -modules, which means precisely that $\text{cot}(A' \rightarrow A) \rightarrow \text{cot}(B' \rightarrow B)$ is a cocartesian arrow in Der/R by Remark 2.28.

Property (2) follows from deformation theory. Indeed, consider the cartesian square of animated rings

$$\begin{array}{ccc} A \oplus_{\alpha} M & \xrightarrow{q_1} & A \\ \downarrow q_2 & \searrow q_0 & \downarrow 0 \\ A & \xrightarrow{\alpha} & A \oplus M[1]. \end{array}$$

and note that we need to show that the natural map $q_2^*(B \oplus_{\beta} N) \rightarrow B$ is an equivalence of connective A -modules. By [SAG, Theorem 16.2.0.2], the above square induces an equivalence

$$(q_1^*, q_2^*): \text{Mod}_{A \oplus_{\alpha} M, \geq 0} \xrightarrow{\sim} \text{Mod}_{A, \geq 0} \times_{\text{Mod}_{A \oplus M[1], \geq 0}} \text{Mod}_{A, \geq 0}$$

with inverse sending $(E_1, E_2, 0^* E_1 \simeq \alpha^* E_2)$ to the fibre product $q_{1*}(E_1) \times_{q_{0*} 0^*(E_1)} q_{2*}(E_2)$. Since $L_{B/A} \simeq 0$, the cocartesian morphism $(A, M, \alpha) \rightarrow (B, N, \beta)$ induces an equivalence $N \simeq B \otimes_A M$ (Remark 2.28). Using this, one sees that the $A \oplus_{\alpha} M$ -module $B \oplus_{\beta} N$ corresponds under the above equivalence to the triple $(B, B, 0^*(B) \simeq \alpha^* B)$, where the equivalence is induced by β . This implies that $q_2^*(B \oplus_{\beta} N) \rightarrow B$ is an equivalence. \square

Lemma 3.21. *The comonad $\text{coLie}_{\Delta}^{\pi}$ restricts to a fibrewise comonad*

$$\begin{array}{ccc} \text{Der}_{/R, \geq 0}^{\text{aperf,coft}} & \xrightarrow{\text{coLie}_{\Delta}^{\pi}} & \text{Der}_{/R, \geq 0}^{\text{aperf,coft}} \\ & \searrow \pi & \swarrow \pi \\ & \text{CAlg}_R^{\text{an,coft}} & \end{array}$$

where π sends a tuple (A, M, α) to the underlying coherent animated R -algebra A . Furthermore, it preserves π -cocartesian maps covering étale maps.

Proof. The comonad restricts by Lemma 3.6 and preserves π -cocartesian morphisms covering étale maps by Lemma 3.20. \square

We then have the following families version of Proposition 3.8:

Proposition 3.22. *There exists a unique extension*

$$\begin{array}{ccc} \mathrm{Der}_{/R, \geq 0}^{\mathrm{aperf}, \mathrm{coft}, \mathrm{op}} & \xrightarrow{(\mathrm{coLie}_{\Delta}^{\pi})^{\mathrm{op}}} & \mathrm{Der}_{/R, \geq 0}^{\mathrm{aperf}, \mathrm{coft}, \mathrm{op}} \\ (-)^{\vee} \downarrow & & \downarrow (-)^{\vee} \\ (\mathrm{QC}^{\vee})_{/T[1]} & \xrightarrow{\mathrm{Lie}_{\Delta}^{\pi}} & (\mathrm{QC}^{\vee})_{/T[1]} \end{array}$$

to a monad $\mathrm{Lie}_{\Delta}^{\pi}$ relative to $\mathrm{CAlg}_R^{\mathrm{an}, \mathrm{coft}, \mathrm{op}}$, that is, a monad in $\mathrm{Cat}_{\infty / \mathrm{CAlg}_R^{\mathrm{an}, \mathrm{coft}, \mathrm{op}}}$, that preserves sifted colimits fibrewise.

Remark 3.23. The relative monad $\mathrm{Lie}_{\Delta}^{\pi}$ restricts to a monad $(\mathrm{QC}_B^{\vee})_{/T_{B/R}[1]} \rightarrow (\mathrm{QC}_B^{\vee})_{/T_{B/R}[1]}$ on the fibre over a coherent animated R -algebra B . This monad preserves sifted colimits and extends the monad $(\mathrm{coLie}_{\Delta, B/R}^{\pi})^{\mathrm{op}}$, so that it is naturally equivalent to the partition Lie algebroid monad $\mathrm{Lie}_{\Delta, B/R}^{\pi}$ of Proposition 3.8. Consequently, Proposition 3.22 shows that the partition Lie algebroid monads $\mathrm{Lie}_{\Delta, B/R}^{\pi}$ depend oplax naturally on the coherent animated R -algebra B .

As the proof of Proposition 3.22 is somewhat technical, we first describe the consequences of Proposition 3.22 at the level of algebras.

Definition 3.24. We define the ∞ -category $\mathrm{LieAlg}_{\Delta, /R}^{\pi}$ of *partition Lie algebroids* to be the ∞ -category of algebras for the monad $\mathrm{Lie}_{\Delta}^{\pi} : \mathrm{QC}_{/T[1]}^{\vee} \rightarrow \mathrm{QC}_{/T[1]}^{\vee}$.

Since $\mathrm{Lie}_{\Delta}^{\pi}$ covers the identity monad on $\mathrm{CAlg}_R^{\mathrm{an}, \mathrm{coft}, \mathrm{op}}$, we obtain a natural projection

$$\mathrm{LieAlg}_{\Delta, /R}^{\pi} \longrightarrow \mathrm{Alg}_{\mathrm{id}}(\mathrm{CAlg}_R^{\mathrm{an}, \mathrm{coft}, \mathrm{op}}) \simeq \mathrm{CAlg}_R^{\mathrm{an}, \mathrm{coft}, \mathrm{op}}.$$

For each coherent animated R -algebra B , the fibre over B is equivalent to the ∞ -category of partition Lie algebroids over B from Definition 3.9.

Proposition 3.25. *The following assertions hold:*

(1) *The forgetful functor defines a map of cartesian fibrations preserving cartesian arrows*

$$(7) \quad \begin{array}{ccc} \mathrm{LieAlg}_{\Delta, /R}^{\pi} & \xrightarrow{\mathrm{forget}} & \mathrm{QC}_{/T[1]}^{\vee} \\ & \searrow & \swarrow \\ & \mathrm{CAlg}_R^{\mathrm{an}, \mathrm{coft}, \mathrm{op}} & \end{array}$$

In particular, every almost finitely presented map $f : A \rightarrow B$ of animated R -algebras induces a commuting square

$$\begin{array}{ccc} \mathrm{LieAlg}_{\Delta, A/R}^{\pi} & \xrightarrow{f^{\sharp}} & \mathrm{LieAlg}_{\Delta, B/R}^{\pi} \\ \mathrm{forget} \downarrow & & \downarrow \mathrm{forget} \\ \mathrm{QC}^{\vee}(A)_{/T_{A/R}[1]} & \xrightarrow{f^{\sharp}} & (\mathrm{QC}_B^{\vee})_{/T_{B/R}[1]} \end{array}$$

where the bottom functor is given by $f^{\sharp}(\mathfrak{g}) = f^*(\mathfrak{g}) \times_{(f^* L_{A/R})^{\vee}[1]} T_{B/R}[1]$.

- (2) If $f: A \rightarrow B$ is étale, then the square (7) is left adjointable, i.e. f^\sharp intertwines the free partition Lie algebroid functors. Consequently, $f^\sharp: \text{LieAlgd}_{\Delta, A/R}^\pi \rightarrow \text{LieAlgd}_{\Delta, B/R}^\pi$ preserves colimits and is given by $\mathfrak{g} \mapsto f^*\mathfrak{g}$ on the underlying pro-coherent modules.
- (3) The cartesian fibration $\text{LieAlgd}_{\Delta, /R}^\pi \rightarrow \text{CAlg}^{\text{an,coft,op}}$ is classified by a functor $A \mapsto \text{LieAlgd}_{\Delta, A/R}^\pi$ satisfying étale descent.

Remark 3.26. Note that for a general finitely presented map $f: A \rightarrow B$, the functor f^\sharp preserves sifted colimits, but it need not preserve limits and it typically does not preserve all colimits.

Proof. Assertion (1) is a formal property of categories of algebras over a fibrewise monad: if $p: \mathcal{Y} \rightarrow \mathcal{X}$ is a cartesian fibration and $T \in \text{End}_{/X}(\mathcal{Y})$ is a fibrewise monad on \mathcal{Y} , then the forgetful functor $\text{Alg}_T(\mathcal{Y}) \rightarrow \mathcal{Y}$ is a map of cartesian fibrations over \mathcal{X} that preserves cartesian edges. To see this, let \mathcal{O} be the operad for algebras and modules and consider the map of \mathcal{O} -monoidal categories $(\text{End}_{/X}(\mathcal{Y}), \mathcal{Y}) \rightarrow (*, \mathcal{X})$. Taking algebras, we then obtain

$$\begin{array}{ccc} \text{Alg}_{\mathcal{O}}(\text{End}_{/X}(\mathcal{Y}), \mathcal{Y}) & \longrightarrow & \text{Alg}_{\mathbb{E}_1}(\text{End}_{/X}(\mathcal{Y})) \times \mathcal{Y} \\ & \searrow & \swarrow \\ & \mathcal{X} & \end{array}$$

It follows from [HA, Corollary 3.2.2.3] that this is a map of cartesian fibrations over \mathcal{X} preserving cartesian arrows. Taking the fibre over a fixed monad $T \in \text{Alg}_{\mathbb{E}_1}(\text{End}_{/X}(\mathcal{Y}))$, one obtains that $\text{Alg}_T(\mathcal{Y}) \rightarrow \mathcal{X}$ is a cartesian fibration as well (with cartesian arrows detected in \mathcal{Y}).

Assertion (2) follows from Lemma 3.6 and (3) follows from (2) and étale descent for $A \mapsto \text{QC}^\vee(A)$. \square

We now turn to the proof of Proposition 3.22. We will need the following technical category-theoretic result about completing families of ∞ -categories under colimits:

Definition 3.27. Let $p: \mathcal{C} \rightarrow \mathcal{B}$ be a cartesian fibration over a small ∞ -category and let \mathcal{K} be a class of small ∞ -categories.

- Let us say that a colimit diagram $K^\triangleright \rightarrow \mathcal{C}_b$ in a fibre is *conserved* if it is preserved by the base change functor $f^*: \mathcal{C}_b \rightarrow \mathcal{C}_{b'}$ for each $b' \rightarrow b$ in \mathcal{B} .
- We will say that p is \mathcal{K} -*complete* if each fibre \mathcal{E}_b is \mathcal{K} -complete and if all of these colimits are conserved.
- By a *collection of conserved colimit diagrams* we will mean a collection $\mathcal{R} = \cup_{b \in \mathcal{B}} \mathcal{R}_b$, where each $\mathcal{R}_b = \{f_\alpha: K_\alpha^\triangleright \rightarrow \mathcal{C}_b\}$ is a set of conserved colimit diagrams in the fibre \mathcal{C}_b and each base change functor sends \mathcal{R}_b to $\mathcal{R}_{b'}$.
- Given another cartesian fibration $q: \mathcal{W} \rightarrow \mathcal{B}$, write $\text{Fun}_{\mathcal{R}/\mathcal{B}}(\mathcal{C}, \mathcal{W})$ for the full subcategory of $\text{Fun}_{/\mathcal{B}}(\mathcal{C}, \mathcal{W})$ on those functors $F: \mathcal{C} \rightarrow \mathcal{W}$ over \mathcal{B} such that each map on fibres $F_b: \mathcal{C}_b \rightarrow \mathcal{W}_b$ preserves the colimits in \mathcal{R}_b . If \mathcal{R} is the collection of all colimits indexed by diagrams from \mathcal{K} , we will simply write $\text{Fun}_{\mathcal{K}/\mathcal{B}}(\mathcal{C}, \mathcal{W})$.

Proposition 3.28. Let \mathcal{B} be a small ∞ -category and consider a fully faithful map of cartesian fibrations, preserving cartesian arrows

$$\begin{array}{ccc} \mathcal{C} & \xleftarrow{j} & \mathcal{V} \\ & \searrow p & \swarrow \hat{p} \\ & \mathcal{B} & \end{array}$$

Let \mathcal{K} be a collection of small ∞ -categories and let \mathcal{R} be a set of conserved colimit diagrams, indexed by ∞ -categories in \mathcal{K} . Suppose that the following condition holds:

- (1) $\hat{p}: \mathcal{V} \rightarrow \mathcal{B}$ is \mathcal{K} -complete.
- (2) Each $j_b: \mathcal{C}_b \rightarrow \mathcal{V}_b$ exhibits \mathcal{V}_b as the \mathcal{K} -completion of \mathcal{C}_b relative to \mathcal{R}_b .

Then for every \mathcal{K} -complete cartesian fibration $q: \mathcal{W} \rightarrow \mathcal{B}$, restriction along j induces an equivalence of ∞ -categories

$$\mathrm{Fun}_{\mathcal{K}/\mathcal{B}}(\mathcal{V}, \mathcal{W}) \simeq \mathrm{Fun}_{\mathcal{R}/\mathcal{B}}(\mathcal{C}, \mathcal{W}).$$

Proof. We will need some details about the construction of the \mathcal{K} -completion \mathcal{V}_b of \mathcal{C}_b relative to \mathcal{R}_b from the proof of [HTT, Proposition 5.3.6.2]. Let $\mathcal{P}_{\mathcal{R}_b}(\mathcal{C}_b) \subseteq \mathcal{P}(\mathcal{C}_b)$ be the ∞ -category of presheaves on \mathcal{C}_b sending all colimits in \mathcal{R}_b to limits and recall that we can identify $\mathcal{V}_b \subseteq \mathcal{P}_{\mathcal{R}_b}(\mathcal{C}_b)$ with the smallest subcategory that contains the representable presheaves and is closed under \mathcal{K} -indexed colimits. Let $L_b: \mathcal{P}(\mathcal{C}_b) \rightarrow \mathcal{P}_{\mathcal{R}_b}(\mathcal{C}_b)$ be the left adjoint to the inclusion and let $\mathcal{E}_b \subseteq \mathcal{P}(\mathcal{C}_b)$ be the inverse image $L_b^{-1}(\mathcal{V}_b)$; note that \mathcal{E}_b is closed under \mathcal{K} -indexed colimits in $\mathcal{P}(\mathcal{C}_b)$. The functor $j_b: \mathcal{C}_b \rightarrow \mathcal{V}_b$ then factors as

$$j_b: \mathcal{C}_b \xrightarrow{h_b} \mathcal{E}_b \xrightarrow{L_b} \mathcal{V}_b$$

where the first functor is the Yoneda embedding and the second is a localisation (at the L_b -local equivalences). All of this depends functorially on the pair $(\mathcal{C}_b, \mathcal{R}_b)$. Consequently, under unstraightening it gives rise to a diagram of cartesian fibrations and maps preserving cartesian arrows

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{h} & \mathcal{E} & \xrightarrow{L} & \mathcal{V} \\ & \searrow p & \downarrow & \swarrow \hat{p} & \\ & & \mathcal{B} & & \end{array}$$

where \mathcal{E} and \mathcal{V} are both \mathcal{K} -complete. By [Hin16], the map L is a localisation, inverting the L_b -local equivalences in each fibre \mathcal{E}_b . Since each localisation L_b preserves \mathcal{K} -indexed colimits and admits a fully faithful right adjoint, it follows that restriction along L defines a fully faithful embedding

$$\mathrm{Fun}_{\mathcal{K}/\mathcal{B}}(\mathcal{V}, \mathcal{W}) \hookrightarrow \mathrm{Fun}_{\mathcal{K}/\mathcal{B}}(\mathcal{E}, \mathcal{W}).$$

Write $\mathrm{Fun}'_{\mathcal{K}/\mathcal{B}}(\mathcal{E}, \mathcal{W})$ for the essential image; it consists of those functors $F: \mathcal{E} \rightarrow \mathcal{W}$ over \mathcal{B} such that each $F_b: \mathcal{E}_b \rightarrow \mathcal{W}_b$ preserves \mathcal{K} -indexed colimits and sends the L_b -local equivalences to equivalences in \mathcal{W}_b . We now claim that the following two assertions hold:

- (a) Let $F \in \mathrm{Fun}_{/\mathcal{B}}(\mathcal{E}, \mathcal{W})$ such that its restriction $F|_{\mathcal{C}}$ is contained in $\mathrm{Fun}_{\mathcal{R}/\mathcal{B}}(\mathcal{C}, \mathcal{W})$. Then $F \in \mathrm{Fun}'_{\mathcal{K}/\mathcal{B}}(\mathcal{E}, \mathcal{W})$ is the q -left Kan extension of $F|_{\mathcal{C}}$ along h if and only if F sends L -local equivalences and preserves \mathcal{K} -indexed colimits fibrewise.
- (b) If $F_0 \in \mathrm{Fun}_{\mathcal{R}/\mathcal{B}}(\mathcal{C}, \mathcal{W})$, then there exists a q -left Kan extension $F \in \mathrm{Fun}_{\mathcal{K}/\mathcal{B}}(\mathcal{E}, \mathcal{W})$ of F_0 along h .

Given (a) and (b), the result follows: restriction along the localisation L yields an equivalence $\mathrm{Fun}_{\mathcal{K}/\mathcal{B}}(\mathcal{V}, \mathcal{W}) \simeq \mathrm{Fun}'_{\mathcal{K}/\mathcal{B}}(\mathcal{E}, \mathcal{W})$ and restriction along h yields an equivalence $\mathrm{Fun}'_{\mathcal{K}/\mathcal{B}}(\mathcal{E}, \mathcal{W}) \simeq \mathrm{Fun}_{\mathcal{R}/\mathcal{B}}(\mathcal{C}, \mathcal{W})$ with inverse given by q -left Kan extension.

It remains to verify claims (a) and (b). In the case where $\mathcal{B} = *$, these assertions are proven in the proof of [HTT, Proposition 5.3.6.2]; we will reduce to this case by a cofinality argument. To this end, let us fix a presheaf $X \in \mathcal{E}_b \subseteq \mathcal{P}(\mathcal{C}_b)$ in the fibre over $b \in \mathcal{B}$. Then the functor

$$(8) \quad (\mathcal{C}_b)_{/X} := \mathcal{C}_b \times_{\mathcal{E}_b} (\mathcal{E}_b)_{/X} \longrightarrow \mathcal{C} \times_{\mathcal{E}} \mathcal{E}_{/X} =: \mathcal{C}_{/X}$$

is cofinal. To see this, we have to show that for each $g: c \rightarrow X$ in $\mathcal{C}_{/X}$ covering a map $f: b' \rightarrow b$, the ∞ -category $((\mathcal{C}_b)_{/X})_{g/}$ is contractible. Unravelling the definitions, the left fibration $((\mathcal{C}_b)_{/X})_{g/} \rightarrow (\mathcal{C}_b)_{/X}$ classifies the functor

$$(\mathcal{C}_b)_{/X} \longrightarrow \mathcal{S}; \quad (c' \rightarrow X) \longrightarrow \mathrm{Map}_{\mathcal{E}_{b'}}(c, f^*c') \times_{\mathrm{Map}_{\mathcal{E}_{b'}}(c, f^*X)} \{g\}.$$

The colimit of this diagram is contractible: indeed, $f^*: \mathcal{E}_b \rightarrow \mathcal{E}_{b'}$ and $\mathrm{Map}_{\mathcal{E}_{b'}}(c, -): \mathcal{E}_{b'} \rightarrow \mathcal{S}$ both preserve colimits (recall that $\mathcal{E}'_b \subseteq \mathcal{P}(\mathcal{C}'_b)$ and that f^* is the colimit-preserving extension of $f^*: \mathcal{C}_b \rightarrow \mathcal{C}'_b$ by definition).

Using [HTT, Corollary 4.3.1.16] and the fact that (8) is cofinal, we find that:

- (a') $F \in \mathrm{Fun}_{/B}(\mathcal{E}, \mathcal{W})$ is the q -left Kan of its restriction to \mathcal{C} if and only if each $F_b: \mathcal{E}_b \rightarrow \mathcal{W}_b$ is the left Kan extension of its restriction to \mathcal{C}_b .
- (b') $F_0 \in \mathrm{Fun}_{/B}(\mathcal{C}, \mathcal{W})$ admits a q -left Kan extension along h if and only if each $F_{0b}: \mathcal{C}_b \rightarrow \mathcal{W}_b$ admits a left Kan extension along h_b .

Using this, it suffices to verify assertions (a) and (b) fibrewise, where they hold by the proof of [HTT, Proposition 5.3.6.2]. \square

Proof of Proposition 3.22. This follows the same strategy as the proof of Proposition 3.8. By Proposition 3.28 and the proof of Proposition 3.8 (applied fibrewise), restriction defines an equivalence of monoidal ∞ -categories between:

- (1) endofunctors of $\mathrm{QC}_{/T[1]}^\vee$ relative to $\mathrm{CAlg}_R^{\mathrm{an}, \mathrm{coft}, \mathrm{op}}$ that preserve sifted colimits fibrewise and that preserve the full subcategory $\mathrm{Der}_{/R, \geq 0}^{\mathrm{aperf}, \mathrm{coft}, \mathrm{op}}$.
- (2) endofunctors of $\mathrm{Der}_{/R, \geq 0}^{\mathrm{aperf}, \mathrm{coft}, \mathrm{op}}$ relative to $\mathrm{CAlg}_R^{\mathrm{an}, \mathrm{coft}, \mathrm{op}}$ that preserve the sifted colimit diagrams (a) and (b) from the proof of Proposition 3.8 fibrewise.

The result now follows from the fact that the monad $(\mathrm{coLie}_\Delta^\pi)^{\mathrm{op}}$ defines an algebra in the ∞ -category (2), by Lemma 3.6. \square

3.3. Partition Lie algebroids on schemes. We will now deduce Theorem 3.1 from Proposition 3.25 by a descent argument. To establish unicity of the partition Lie algebroid monad $\mathrm{Lie}_{\Delta, X/R}^\pi$, we will need the following observation:

Lemma 3.29. *Let X be a locally coherent qcqs scheme, $T \in \mathrm{QC}_X^\vee$ and write $\mathcal{C}_T \subseteq (\mathrm{QC}_X^\vee)_{/T}$ for the full subcategory of $M \rightarrow T$ where M is dually almost perfect of non-positive tor-amplitude. If \mathcal{D} be an ∞ -category with sifted colimits and $F: (\mathrm{QC}_X^\vee)_{/T} \rightarrow \mathcal{D}$ preserves sifted colimits, then F is left Kan extended from its restriction to $\mathcal{C}_T \subseteq (\mathrm{QC}_X^\vee)_{/T}$.*

Proof. The proof of Lemma A.42 shows that the ∞ -category $\mathrm{QC}_X^\vee \simeq \mathrm{Ind}(\mathrm{Coh}_X^{\mathrm{op}})$ is equivalent to the ∞ -category of functors $(\mathrm{APerf}_{X, \geq 0})^{\mathrm{op}} \rightarrow \mathcal{S}$ that are reduced excisive and preserve limits of almost eventually constant towers (cf. Definition 3.5). Under this equivalence, the dually almost perfect objects of non-positive tor-amplitude simply correspond to the corepresentable functors. Consequently, for every $M \in \mathrm{QC}_X^\vee$ the canonical map $\mathrm{colim}_{N \in \mathcal{C}_M} N \rightarrow M$ is an equivalence.

Now let $M \in (\mathrm{QC}_X^\vee)_{/T}$ and notice that there is an equivalence $(\mathcal{C}_T)_{/M} \simeq \mathcal{C}_M$. To see that F is left Kan extended from \mathcal{C}_T , we need to show that

$$\mathrm{colim}_{N \in (\mathcal{C}_T)_{/M}} F(N) \longrightarrow F(\mathrm{colim}_{N \in (\mathcal{C}_T)_{/M}} N) \longrightarrow F(M)$$

are both equivalences. Now note that $(\mathcal{C}_T)_{/M} \simeq \mathcal{C}_M$, so that the last map is an equivalence. Since \mathcal{C}_M is sifted (it has finite coproducts) and F preserves sifted colimits, the first map is an equivalence as well. \square

Proof of Theorem 3.1. It follows directly from Lemma 3.29 that space of sifted colimit preserving extensions is either empty or contractible. It therefore remains to prove the existence of the monad $\text{Lie}_{\Delta, X/R}^\pi$. Writing \mathfrak{U} for the poset of affine opens of X , Proposition 3.25 provides a natural diagram

$$\mathfrak{U}^{\text{op}} \longrightarrow \text{Fun}^{\text{RAAd}}(\Delta^1, \text{Cat}); \quad U \longmapsto \left((\text{QC}_U^\vee)_{T_{U/R}[1]} \xrightarrow{\text{free}} \text{LieAlgd}_{\Delta, \mathcal{O}(U)/R}^\pi \right)$$

to the ∞ -category of left adjoints and right adjointable squares between them [HA, Definition 4.7.4.16]. The limit of this diagram then yields an adjoint pair [HA, Proposition 4.7.4.19]

$$\text{Lie}_\Delta^\pi: \lim_{U \in \mathfrak{U}^{\text{op}}} (\text{QC}_U^\vee)_{/T_{U/R}[1]} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \lim_{U \in \mathfrak{U}^{\text{op}}} \text{LieAlgd}_{\Delta, \mathcal{O}(U)/R}^\pi: \text{forget}$$

where the left (right) adjoint is given by the limit of the diagram of free (forgetful) functor for all affine open subspaces. Consequently, both adjoints preserve sifted colimits. Since pro-coherent sheaves satisfy descent (Corollary A.39), one obtains a sifted colimit-preserving monad

$$\text{Lie}_{\Delta, X/R}^\pi: (\text{QC}_X^\vee)_{/T_{X/R}[1]} \longrightarrow (\text{QC}_X^\vee)_{/T_{X/R}[1]}.$$

By construction, the restriction of this monad to the full subcategory $\text{Der}_{X/R, \geq 0}^{\text{aperf, op}}$ is the limit of the monads

$$\left(\text{cot otriv}(-^\vee) \right)^\vee: \text{Der}_{\mathcal{O}(U)/R, \geq 0}^{\text{aperf, op}} \longrightarrow \text{Der}_{\mathcal{O}(U)/R, \geq 0}^{\text{aperf, op}}$$

for all affine opens $U \subseteq X$. By descent for almost perfect complexes, this monad naturally equivalent to $\left(\text{cot otriv}(-^\vee) \right)^\vee$, as desired. \square

Remark 3.30. The proof of Theorem 3.1 shows that the ∞ -category $\text{LieAlgd}_{\Delta, X/R}^\pi$ from Definition 3.2 coincides with the definition as the limit of the categories $\text{LieAlgd}_{\Delta, U/R}^\pi$ from Definition 3.9 for all affine open subschemes.

4. INFINITESIMAL DEFORMATIONS OF FAMILIES

In this section, we will show that the ∞ -category of partition Lie algebroids over a coherent animated R -algebra B is equivalent to the ∞ -category of deformation functors, or *formal moduli problems* in the terminology of Lurie [SAG, Chapter 12], defined on a certain ∞ -category of Artinian extensions of B . In Section 4.4, we will use this to show that every partition Lie algebroid over B has an underlying R -linear partition Lie algebra.

4.1. Artinian extensions. We begin by introducing the following variant of Artin local rings relative to B :

Definition 4.1 (Artinian extensions). Let B be a coherent animated R -algebra. We will say that a map of animated R -algebras $A \rightarrow B$ is an *Artinian extension* if it satisfies the following three conditions:

- (1) $A \rightarrow B$ is a nilpotent extension, that is, $\pi_0(A) \rightarrow \pi_0(B)$ is surjective with nilpotent kernel.
- (2) $A \rightarrow B$ is almost of finite presentation.
- (3) the fibre of $A \rightarrow B$ is eventually coconnective.

A map of animated R -algebras $A \rightarrow B$ is said to be an *almost Artinian extension* if it satisfies conditions (1) and (2). We will write $\text{Art}_{R/B} \subseteq \text{AArt}_{R/B} \subseteq \text{CAlg}_{R/B}^{\text{an}}$ for the full subcategory on the Artinian extensions of B .

Remark 4.2. Let A be a discrete commutative ring and let $I \subseteq A$ be a finitely presented nilpotent ideal such that A/I is a coherent ring. Let us recall the following facts:

- (1) A is coherent [Gla89, Theorem 4.1.1].
- (2) For every $k \geq 0$, the ideal I^k is finitely generated, and hence finitely presented over A .
- (3) If M is a finitely presented A -module, then $I^k M / I^{k+1} M$ is a finitely presented A -module and hence a finitely presented A/I -module [Gla89, Theorem 2.1.8, Theorem 2.2.1].

Proposition 4.3. Let $A' \rightarrow A$ be a map in $\text{CAlg}_{R/B}^{\text{an}}$ with $A \in \text{Art}_{R/B}$. Then the following assertions are equivalent:

- (1) $A' \in \text{Art}_{R/B}$ and $A' \rightarrow A$ is a nilpotent extension.
- (2) There exists a finite chain of maps $A' = A_n \rightarrow A_{n-1} \rightarrow \cdots \rightarrow A_0 = A$ in $\text{CAlg}_{R/B}^{\text{an}}$ where each $A_i \rightarrow A_{i-1}$ is a square zero extension by a connective coherent B -module, viewed as an A_{i-1} -module via the canonical map $A_{i-1} \rightarrow B$.

Proof. If $A' \rightarrow A$ satisfies condition (2), it is clearly a nilpotent extension. We will prove by induction on i that the map $A_i \rightarrow B$ exhibits B as an almost perfect A_i -module. This is evident for $i = 0$. Assume that $B \in \text{APerf}_{A_{i-1}}$, so that every almost perfect B -module is almost perfect as an A_{i-1} -module. It follows that $A_i \rightarrow A_{i-1}$ is a square zero extension by an almost perfect A_{i-1} -module. Lemma 2.20 now implies that every almost perfect A_{i-1} -module is almost perfect as an A_i -module, so that $B \in \text{APerf}_{A_i}$ as well. Proposition 2.14 now implies that $A_i \rightarrow B$ is almost finitely presented. Since $A_i \rightarrow B$ is clearly a nilpotent extension with eventually coconnective fibre, we conclude that $A_i \rightarrow B$ is an Artinian extension, so that (1) follows.

For the converse, we first suppose that $A' \rightarrow A$ is a nilpotent extension with $A' \in \text{Art}_{R/B}$ such that $A \in \text{APerf}_{A'}$. Write

$$I = \ker(\pi_0(A') \rightarrow \pi_0(B)) \quad \text{and} \quad J = \ker(\pi_0(A') \rightarrow \pi_0(A)).$$

Both I and J are finitely presented nilpotent ideals, since $A' \rightarrow A$ and $A' \rightarrow B$ are almost finitely presented. In particular, $\pi_0(A')$ is a coherent ring by Remark 4.2, and likewise for $\pi_0(A)$.

We now fix $m \geq 0$ such that $I^m = 0$ and $t \geq 1$ such that $\text{cofib}(A' \rightarrow A)$ is t -coconnective; such t exists since $A' \rightarrow B$ and $A \rightarrow B$ both have eventually coconnective fibres. We will construct a sequence of animated A' -algebras $A' = A_{(t+1)} \rightarrow \cdots \rightarrow A_{(3)} \rightarrow A_{(2)} \rightarrow A_{(1)} \rightarrow A_{(0)} = A$ with the following properties:

- (a) For each i , $\text{cofib}(A \rightarrow A_{(i)})$ is a i -connective and t -coconnective almost perfect A' -module.
- (b) Each $A_{(i+1)} \rightarrow A_{(i)}$ decomposes into m square zero extensions, each by the i -fold suspension of a finitely presented discrete $\pi_0(B)$ -module.

Since finitely presented discrete $\pi_0(B)$ -modules are coherent as B -modules (Definition A.41), this provides the desired decomposition.

To begin, we define $A_{(1)} = \pi_0(A') \times_{\pi_0(A)} A$ and note that $A_{(1)} \rightarrow A_{(0)} = A$ is the base change of the sequence of square zero extensions

$$\pi_0(A') \rightarrow \pi_0(A')/I^{m-1}J \rightarrow \cdots \rightarrow \pi_0(A')/IJ \rightarrow \pi_0(A')/J = \pi_0(A)$$

by the finitely presented discrete $\pi_0(B)$ -modules $I^k J/I^{k+1}J$ (by Remark 4.2). Note that $\pi_0(A') \cong \pi_0(A_{(1)})$ and that $\text{cofib}(A' \rightarrow A_{(1)})$ is 1-connective and t -coconnective, as desired.

For $i \geq 1$, suppose that we have constructed $A_{(i)}$ and let $C = \text{cofib}(A' \rightarrow A_{(i)}) \in \text{APerf}_{A'}$. Since C is i -connective and $\pi_0(A') \cong \pi_0(A_{(i)})$, $L_{A_{(i)}/A'}$ is i -connective and $\pi_i(L_{A_{(i)}/A'}) \cong \pi_i(C)$ is a finitely presented $\pi_0(A')$ -module. We define $A_{(i+1)}$ to be the square zero extension of $A_{(i)}$ classified by the map $L_{A_{(i)}/A'} \rightarrow \pi_i(C)[i]$. Then $\text{cofib}(A' \rightarrow A_{(i+1)}) \simeq \tau_{\geq i+1}C$ is $(i+1)$ -connective and t -coconnective. The finitely presented $\pi_0(A')$ -module $\pi_i(C)$ fits into a sequence of extensions

$$\pi_i(C) \rightarrow \pi_i(C)/I^{m-1} \rightarrow \cdots \rightarrow \pi_i(C)/I^2 \rightarrow \pi_i(C) \otimes_{\pi_0(A')} \pi_0(B)$$

by finitely presented discrete $\pi_0(B)$ -modules. This implies that $A_{(i+1)} \rightarrow A_{(i)}$ has property (b).

It now remains to verify by induction that $A_{(i+1)}$ is indeed an almost perfect A' -module for all $i \geq -1$. For $i = -1$, this holds by hypothesis and for $i \geq 0$ we constructed $A_{(i+1)}$ as an iterated extension of $A_{(i)}$ by coherent B -modules, all of which are in APerf_A by inductive hypothesis.

Finally, let us show that if $A' \rightarrow A$ is a map in $\text{Art}_{R/B}$, then $A \in \text{APerf}_{A'}$. To this end, note that $A \rightarrow B$ exhibits B as an almost perfect A -module because $A \rightarrow B$ is an Artinian extension. The previous part of the proof therefore shows that A is an iterated square zero extension of B by coherent B -modules. Because $A' \in \text{Art}_{R/B}$, these are all almost perfect as A' -modules, so that $A \in \text{APerf}_{A'}$ as well. \square

Corollary 4.4. *The subcategory $\text{Art}_{R/B} \subset \text{CAlg}_{R/B}^{\text{an}}$ is the smallest full subcategory satisfying the following two conditions:*

- (1) *It contains the terminal object $B \rightarrow B$.*
- (2) *Given $M \in \text{Coh}_{B, \geq 0}$ and a pullback square exhibiting A_η as the square zero extension of A by M*

$$(9) \quad \begin{array}{ccc} A_\eta & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \xrightarrow{\eta} & B \oplus M[1] \end{array}$$

we have that $A_\eta \in \text{Art}_{R/B}$ as soon as $A \in \text{Art}_{R/B}$.

Furthermore, for each diagram $A_1 \rightarrow A_0 \leftarrow A_2$ in $\text{Art}_{R/B}$ where one arrow is a nilpotent extension, the pullback is again contained in $\text{Art}_{R/B}$.

Proof. The first assertion follows from Proposition 4.3 applied to $A = B$: this shows that $A' \rightarrow B$ is an Artinian extension if and only if it decomposes into a finite sequence of square zero extensions by connective coherent B -modules. The second assertion follows from Proposition 4.3 by decomposing the nilpotent extension into square zero extensions by coherent B -modules and using the first assertion. \square

Corollary 4.5. *Let B be a coherent animated ring and let $A \rightarrow B$ be an Artinian extension. Then A is coherent as well.*

Proof. By Proposition 4.3, it suffices to treat the case where A is a square zero extension of B by a coherent B -module. In this case, Lemma 2.20 implies that the truncation of an almost perfect A -module remains almost perfect. \square

Corollary 4.6. *Let B be a coherent animated ring and let $A \rightarrow A'$ be a map of Artinian extensions of B . Then A' is almost finitely presented as an animated A -algebra.*

Proof. By Proposition A.19, it suffices to verify that A' is an almost perfect A -module. Since A' arises from B by square zero extensions by coherent B -modules, it suffices to verify that B is almost perfect as an A -module, which holds by assumption (and Proposition A.19). \square

The ∞ -category of almost Artinian extensions $\mathbb{A}\text{Art}_{R/B}$ is essentially obtained from the ∞ -category of Artinian extensions by adding limits of relative Postnikov towers.

Lemma 4.7. *Let $\text{CAlg}_{R/B}^{\text{an,sur}} \subseteq \text{CAlg}_{R/B}^{\text{an}}$ denote the full subcategory of animated surjections $A \rightarrow B$. For each $(A \rightarrow B) \in \text{CAlg}_{R/B}^{\text{an,sur}}$, there exists an initial map $(A \rightarrow B) \rightarrow (\tau_{\leq n}(A/B) \rightarrow B)$ to an animated surjection with n -coconnective fibre.*

We will refer to $\tau_{\leq n}(A/B)$ as the *fibrewise n -truncation* of A over B .

Proof. Consider the full subcategory $\mathcal{C} \subset \text{Fun}(\Delta^1, \text{CAlg}_R^{\text{an}})$ spanned by the animated surjections, and recall that \mathcal{C} has compact projective generators of the form $R[x_1, \dots, x_{n+m}] \rightarrow R[x_1, \dots, x_n]$. Using this, one sees that an object $(A \rightarrow B)$ of \mathcal{C} is n -truncated if and only if A and $\text{fib}(A \rightarrow B)$ are n -truncated (which implies that B is $(n+1)$ -truncated). Furthermore, each animated surjection $(A \rightarrow B)$ admits an n -truncation $\tau_{\leq n}(A \rightarrow B)$ whose domain and fibre are $\tau_{\leq n}A$ and $\tau_{\leq n} \text{fib}(A \rightarrow B)$; write $\tau'_{\leq n}B$ for the codomain, which need not be the n -truncation of B . Given an animated surjection $A \rightarrow B$, we then define

$$\tau_{\leq n}(A/B) = \tau_{\leq n}A \times_{\tau'_{\leq n}B} B.$$

This determines a functor $\tau_{\leq n}: \text{CAlg}_{R/B}^{\text{an,sur}} \rightarrow \text{CAlg}_{R/B}^{\text{an,sur}}$ equipped with a natural transformation $\text{id} \rightarrow \tau_{\leq n}$. By construction, the map $\text{fib}(A \rightarrow B) \rightarrow \text{fib}(\tau_{\leq n}(A/B) \rightarrow B)$ exhibits its target as the n -truncation of the domain. In particular, this implies that $\tau_{\leq n}$ is a localisation and that the $\tau_{\leq n}$ -local objects are the animated surjections with n -coconnective fibre. \square

Lemma 4.8. *Let $A \rightarrow B$ be an animated surjection. Then the following are equivalent:*

- (1) $A \rightarrow B$ is an almost Artinian extension.
- (2) For each $n \geq 0$, the fibrewise n -truncation $\tau_{\leq n}(A/B) \rightarrow B$ is an Artinian extension.

Consequently, $\mathbb{A}\text{Art}_{B/R} \subseteq \text{CAlg}_{R/B}^{\text{an}}$ is closed under

Proof. Note that $\pi_i(A) \cong \pi_i(\tau_{\leq n}(A/B))$ for all $i \leq n$. Assuming (2), it therefore follows that $A \rightarrow B$ is a nilpotent extension and that A is a coherent animated ring. Since each $\pi_i(B)$ is a finitely presented discrete module over $\pi_0(A) \cong \pi_0(\tau_{\leq 0}(A/B))$, we conclude that B is an almost perfect A -module, so that $A \rightarrow B$ is almost finitely presented.

Assuming (1), the argument from Proposition 4.3 shows that there exists a tower $\cdots \rightarrow A_n \rightarrow A_{n-1} \rightarrow \cdots \rightarrow B$ of Artinian extensions such that $\text{fib}(A \rightarrow A_n)$ becomes increasingly connective. In particular, choosing n large enough we see that $\pi_0(A) \cong \pi_0(A_n)$ is a coherent ring and that each $\pi_i(A) \cong \pi_i(A_n)$ is a finitely presented $\pi_0(A)$ -module. It follows that A is coherent, so that each truncation $\tau_{\leq n} \text{fib}(A \rightarrow B)$ is an almost perfect A -module. Using this and Proposition 2.14, one sees that each $A \rightarrow \tau_{\leq n}(A/B)$ is almost finitely presented. It follows that $\tau_{\leq n}(A/B) \rightarrow B$ is almost finitely presented, and it is clearly a nilpotent extension with n -coconnective fibre. \square

Definition 4.9. Let $\text{Art}_R \hookrightarrow \text{AArt}_R \hookrightarrow \text{Fun}(\Delta^1, \text{CAlg}_R^{\text{an}})$ denote the subcategories whose:

- (1) objects are (almost) Artinian extensions $A \rightarrow B$ where B is coherent.
- (2) morphisms $(\tilde{f}, f): (A \rightarrow B) \rightarrow (A' \rightarrow B')$ are commuting squares in which f is étale.

$$\begin{array}{ccc} A & \xrightarrow{\tilde{f}} & A' \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & B' \end{array}$$

Proposition 4.10. Let $\text{CAlg}_R^{\text{an,coh,et}} \hookrightarrow \text{CAlg}_R^{\text{an}}$ denote the subcategory of coherent animated R -algebras and étale maps between them. Then the codomain projections define cocartesian fibrations

$$\text{Art}_R \longrightarrow \text{CAlg}_R^{\text{an,coh,et}} \qquad \text{AArt}_R \longrightarrow \text{CAlg}_R^{\text{an,coh,et}}.$$

Furthermore, for each étale map $f: B \rightarrow B'$, the induced functor $f^*: \text{AArt}_{R/B} \rightarrow \text{AArt}_{R/B'}$ preserves pullbacks along nilpotent extensions, as well as limits of fibrewise Postnikov towers.

Proof. A commuting square as in Definition 4.9 defines a locally cocartesian arrow in AArt_R if $A \rightarrow A'$ is étale. Indeed, any map of Artinian extensions $(A \rightarrow B) \rightarrow (A'' \rightarrow B')$ covering the map $B \rightarrow B'$ factors uniquely over $(A' \rightarrow B')$ because the étale map $A \rightarrow A'$ has the unique lifting property against the nilpotent extension $A'' \rightarrow B'$. It is clear that these types of locally cocartesian arrows are stable under composition.

It therefore remains to verify that for every Artinian extension $(A \rightarrow B)$ and every étale map $f: B \rightarrow B'$, there exists a lift $(\tilde{f}, f): (A \rightarrow B) \rightarrow (A' \rightarrow B')$ such that $\tilde{f}: A \rightarrow A'$ is étale. For every animated ring B , the functor $B' \mapsto \pi_0(B) \otimes_B B'$ defines an equivalence between the categories of étale B -algebras and étale $\pi_0(B)$ -algebras. In turn, the functor $B'' \mapsto B'' \otimes_{\pi_0(B)} \pi_0(B)/I$ determines an equivalence between the (ordinary) categories of étale $\pi_0(B)$ -algebras and $\pi_0(B)/I$ -algebras whenever I is a nilpotent ideal [Sta19, Tag 039R]. Using this, one sees that there is a unique lift of f to an étale map $\tilde{f}: A \rightarrow A'$.

To see that the resulting map $A' \rightarrow B'$ is (almost) Artinian, note that it is almost of finite presentation since both A' and B' are almost of finite presentation over A . Furthermore, $\ker(\pi_0(A') \rightarrow \pi_0(B')) \cong \pi_0(A') \otimes_{\pi_0(A)} \ker(\pi_0(A) \rightarrow \pi_0(A'))$ is the base change of a nilpotent ideal along a flat map, and hence a nilpotent ideal itself. Likewise, $\text{fib}(A' \rightarrow B') \simeq A' \otimes_A \text{fib}(A \rightarrow B)$ is eventually coconnective if $\text{fib}(A \rightarrow B)$ is eventually coconnective, as $A \rightarrow A'$ is flat.

Finally, consider the change of fibre functor $f^*: \text{AArt}_{B/R} \rightarrow \text{AArt}_{B'/R}$. For any map $A_1 \rightarrow A_2$ of (almost) Artinian extensions of B , we have that $A_i \rightarrow f^*(A_i)$ is the unique étale map covering

f . This implies that $f^*(A_2) \simeq f^*(A_1) \otimes_{A_1} A_2$. Using this, one readily verifies that f^* preserves pullbacks along nilpotent extensions. Furthermore, the fact that $A \rightarrow f^*(A)$ is flat implies that $f^*(A) \otimes_A \tau_{\leq n}(A/B) \simeq \tau_{\leq n}(f^*(A)/B')$, so that f^* preserves limits of fibrewise Postnikov towers as well. \square

4.2. Formal moduli problems. We recall the notion of a formal moduli problem [SAG, Chapter 12]:

Definition 4.11. A functor $X: \text{Art}_{R/B} \rightarrow \mathcal{S}$ is a *formal moduli problem* if it satisfies the following two conditions:

- (1) $X(B) \simeq *$.
- (2) For any pullback square in $\text{Art}_{R/B}$ of the form

$$\begin{array}{ccc} A & \longrightarrow & A_0 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A_{01} \end{array}$$

where $A_0 \rightarrow A_{01}$ is a nilpotent extension, $X(A) \rightarrow X(A_0) \times_{X(A_{01})} X(A_1)$ is an equivalence. We will write $\text{Moduli}_{B/R}$ for the ∞ -category of formal moduli problems.

Remark 4.12. Using Proposition 4.3, one sees that $X: \text{Art}_{R/B} \rightarrow \mathcal{S}$ is a formal moduli problem if and only if $X(B) \simeq *$ and X preserves each pullback square (9) describing a square zero extension of $A \in \text{Art}_{R/B}$ by a coherent connective B -module.

Example 4.13. Let $B' \rightarrow B$ be a map of derived R -algebras. We will write $\text{Spf}(B')$ for the corresponding *corepresentable formal moduli problem*

$$\text{Spf}(B'): \text{Art}_{R/B} \longrightarrow \mathcal{S}; \quad A \longmapsto \text{Map}_{\text{CAlg}_{R/B}^{\text{der}}} (B', A).$$

This defines the right adjoint in an adjoint pair

$$\mathcal{O}: \text{Moduli}_{B/R} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \text{CAlg}_{R/B}^{\text{der,op}} : \text{Spf}$$

where $\mathcal{O}(X)$ can be viewed as the global sections of the formal moduli problem X .

Lemma 4.14. *Restriction along $i: \text{Art}_{R/B} \hookrightarrow \mathbb{A}\text{Art}_{R/B}$ defines an equivalence between $\text{Moduli}_{B/R}$ and the ∞ -category of functors $X: \mathbb{A}\text{Art}_{R/B} \rightarrow \mathcal{S}$ satisfying the following conditions:*

- (1) $X(B) \simeq *$.
- (2) X preserves pullbacks along nilpotent extensions in $\mathbb{A}\text{Art}_{R/B}$.
- (3) X preserves limits of fibrewise Postnikov towers.

Proof. Lemma 4.8 implies that for any functor $Y: \text{Art}_{R/B} \rightarrow \mathcal{S}$, its right Kan extension along i is given by $i_*Y(A) = \lim_n Y(\tau_{\leq n}(A/B))$. The result follows readily from this. \square

Recall that every formal moduli problem has a tangent complex:

Construction 4.15. Let $\text{Der}_{B/R, \geq 0}^{\text{coh}}$ denote the ∞ -category of derivations $\alpha: L_{B/R}[-1] \rightarrow M$ where M is a connective coherent B -module.

Given a formal moduli problem $X: \text{Art}_{R/B} \rightarrow \mathcal{S}$, write

$$T_{B/X}[1]: \text{Der}_{B/R, \geq 0}^{\text{coh}} \longrightarrow \mathcal{S}; \quad (M, \alpha) \longmapsto X(B \oplus_{\alpha} M)$$

for the restriction of X to the square zero extensions of B . Because X is a formal moduli problem, this functor preserves pullbacks of π_0 -surjections. Consequently, Proposition 3.15 and Lemma A.42 (or Example 4.25 below) imply that $T_{B/X}[1]$ is classified by an object of $\mathrm{QC}_{B/T_{B/R}[1]}^\vee$ that we will refer to as the *tangent fibre* of the formal moduli problem X , and denote by $T_{B/X}[1]$ as well.

Example 4.16. Let $B' \rightarrow B$ be a map of derived R -algebras and let $\mathrm{Spf}(B')$ be the associated corepresentable formal moduli problem (Example 4.13). The tangent fibre of $\mathrm{Spf}(B')$ is then given by the functor $\mathrm{Der}_{B'/R, \geq 0}^{\mathrm{coh}} \rightarrow \mathcal{S}$ corepresented by the cotangent complex $L_{B'/R}[-1] \rightarrow L_{B'/B'}[-1]$. Consequently, $T_{B'/\mathrm{Spf}(B')} \simeq L_{B'/B'}^\vee[1]$ is the pro-coherent dual of the cotangent fibre.

Finally, we record the functoriality of the ∞ -category $\mathrm{Moduli}_{B/R}$ with respect to étale maps:

Construction 4.17. Write $\mathcal{A} \rightarrow \mathcal{B}$ for the cocartesian fibration $\mathrm{Art}_R \rightarrow \mathrm{CAlg}_R^{\mathrm{an,coh,et}}$ from Proposition 4.10 and consider the relative functor ∞ -category $\mathrm{Fun}_{/\mathcal{B}}(\mathcal{A}, \mathcal{S}) \rightarrow \mathcal{B}$. Recall that this is the ∞ -category over \mathcal{B} defined by the universal property that functors $\mathcal{D} \rightarrow \mathrm{Fun}_{/\mathcal{B}}(\mathcal{A}, \mathcal{S})$ are naturally equivalent to pairs consisting of a functor $\mathcal{D} \rightarrow \mathcal{B}$ and a functor $\mathcal{A} \times_{\mathcal{B}} \mathcal{D} \rightarrow \mathcal{S}$. Then projection $\mathrm{Fun}_{/\mathcal{B}}(\mathcal{A}, \mathcal{S}) \rightarrow \mathcal{B}$ is a cartesian and cocartesian fibration, classifying the functor sending $B \mapsto \mathrm{Fun}(\mathrm{Art}_{R/B}, \mathcal{S})$ and a map $f: B \rightarrow B'$ to restriction and left Kan extension along $\mathrm{Art}_{R/B} \rightarrow \mathrm{Art}_{R/B'}$ [GHN17, Proposition 7.3]. We write

$$\mathrm{Moduli}_{/R} \subseteq \mathrm{Fun}_{/\mathcal{B}}(\mathcal{A}, \mathcal{S})$$

for the full subcategory spanned by the tuples $(B, X: \mathrm{Art}_{R/B} \rightarrow \mathcal{S})$ where X is a formal moduli problem. Since each $\mathrm{Art}_{R/B} \rightarrow \mathrm{Art}_{R/B'}$ preserves pullbacks along nilpotent extensions, the projection $\mathrm{Moduli}_{/R} \rightarrow \mathrm{CAlg}_R^{\mathrm{an,coh,et}}$ is a cartesian and cocartesian fibration.

4.3. Deformation theory. Let B be a coherent animated R -algebra. Our goal will be to show that taking the tangent complex of a formal moduli problem (Construction 4.15) refines to an equivalence

$$T_{B/-}[1]: \mathrm{Moduli}_{B/R} \xrightarrow{\sim} \mathrm{LieAlg}_{\Delta, B/R}^\pi.$$

In fact, we will construct a version of this equivalence that is functorial in B with respect to étale maps. To this end, we introduce the following notation:

Notation 4.18. Consider the functor $\mathrm{CAlg}_R^{\mathrm{an,coh,et}} \rightarrow \mathrm{Cat}_\infty$ sending $B \mapsto \mathrm{LieAlg}_{\Delta, B/R}^\pi$ and sending each étale map $f: B \rightarrow B'$ to $f^*: \mathrm{LieAlg}_{\Delta, B/R}^\pi \rightarrow \mathrm{LieAlg}_{\Delta, B'/R}^\pi$. This classifies a cartesian fibration and a cocartesian fibration that we will denote by

$$p: \mathrm{LieAlg}_{\Delta/R}^{\pi, \mathrm{et}} \longrightarrow \mathrm{CAlg}_R^{\mathrm{an,coh,et,op}} \quad p^\vee: \mathrm{LieAlg}_{\Delta/R}^{\Delta/R} \longrightarrow \mathrm{CAlg}_R^{\mathrm{an,coh,et}}.$$

More precisely, p is the restriction of the cartesian fibration from Proposition 3.25 to the subcategory of étale maps, and we define p^\vee to be the cocartesian fibration classified by the same diagram of ∞ -categories. Part (2) of Proposition 3.25 implies that p^\vee is a cartesian fibration as well.

Theorem 4.19. *There is an equivalence of ∞ -categories*

$$\begin{array}{ccc} \mathrm{LieAlg}_{\pi, \mathrm{et}}^{\Delta/R} & \xrightarrow{\sim} & \mathrm{Moduli}_{/R} \\ & \searrow & \swarrow \\ & \mathrm{CAlg}_R^{\mathrm{an,coh,et}} & \end{array}$$

Furthermore, the induced equivalence between fibres over a coherent animated R -algebra B fits into a commuting diagram

$$\begin{array}{ccc} \mathrm{LieAlg}_{\Delta, B/R}^{\pi} & \xrightarrow{\sim} & \mathrm{Moduli}_{B/R} \\ \mathrm{forget} \downarrow & & \downarrow T_{B/-}[1] \\ \mathrm{QC}_{B/T_{B/R}[1]}^{\vee} & \hookrightarrow & \mathrm{Fun}(\mathrm{Der}_{B/R, \geq 0}^{\mathrm{coh}}, \mathcal{S}). \end{array}$$

We will first construct the functor $\mathrm{LieAlg}_{\pi, \mathrm{et}}^{\Delta/R} \rightarrow \mathrm{Moduli}_{/R}$ and then use the language of deformation theories from [SAG, Chapter 12] to prove that it is an equivalence.

Construction 4.20. If an animated surjection $A \rightarrow B$ is an (almost) Artinian extension, then it is in particular complete almost finitely augmented (Definition 2.15). Using Corollary 3.13, we thus obtain a composite functor

$$\mathfrak{D}: \mathrm{Art}_R \xrightarrow{\mathrm{cot}} \mathrm{Coalg}_{\mathrm{coLie}_{\Delta}}(\mathrm{Der}_{/R, \geq 0}^{\mathrm{aperf}}) \xrightarrow{(-)^{\vee}} (\mathrm{LieAlg}_{\Delta, /R}^{\pi})^{\mathrm{op}}.$$

sending each Artinian extension $A \rightarrow B$ to the partition Lie algebroid $\mathfrak{D}(A) = \mathrm{cot}(A)^{\vee}$ over B . Since the maps in Art_R all cover étale maps $B \rightarrow B'$, this assembles into a functor

$$\begin{array}{ccc} \mathrm{Art}_R & \xrightarrow{\mathfrak{D}} & (\mathrm{LieAlg}_{\Delta, /R}^{\pi, \mathrm{et}})^{\mathrm{op}} \\ & \searrow & \swarrow \\ & \mathrm{CAlg}_R^{\mathrm{an, coh, et}} & \end{array}$$

The functor \mathfrak{D} preserves cocartesian arrows: indeed, for any cocartesian arrow $(A \rightarrow B) \rightarrow (A' \rightarrow B')$ in Art_R covering an étale map $f: B \rightarrow B'$, we have that $B' \simeq A' \otimes_A B$ so that $\mathfrak{D}(A') \simeq f^* \mathfrak{D}(A)$. The functor \mathfrak{D} induces a fully faithful functor on fibres by Corollary 3.13 and is hence fully faithful itself, since it preserves cocartesian arrows.

Proposition 4.21. *Let B be a coherent animated ring and let $A \rightarrow B \oplus M[1]$ be a map in $\mathrm{Art}_{R/B}$ classifying a square zero extension $A_{\eta} \rightarrow A$ by a coherent connective B -module. Then induced map of partition Lie algebroids over B*

$$(10) \quad \mathfrak{D}(A) \amalg_{\mathfrak{D}(B \oplus M[1])} \mathfrak{D}(B) \longrightarrow \mathfrak{D}(A_{\eta})$$

is an equivalence.

Lemma 4.22. *Let B be a coherent animated ring and let $F^*M \in \mathrm{FilMod}_B$ be an almost perfect filtered B -module. Then the diagram of pro-coherent duals*

$$(F^0M/F^1M)^{\vee} \longrightarrow (F^0M/F^2M)^{\vee} \longrightarrow \dots \longrightarrow (F^0M)^{\vee}$$

is a colimit diagram in QC_B^{\vee} .

Proof. Recall that the pro-coherent dual of a B -module M can be described by the left exact functor $\mathrm{Coh}_B \rightarrow \mathcal{S}$ corepresented by M . It therefore suffices to verify that for any coherent B -module N , the map $\mathrm{colim} \mathrm{Map}_B(F^0M/F^nM, N) \rightarrow \mathrm{Map}_B(F^0M, N)$ is an equivalence of spaces. Suppose that N is k -coconnective. Because F^*M is almost perfect, there exists an m such that F^nM is k -connective for all $n \geq m$. It follows that $\mathrm{Map}_B(F^0M/F^nM, N) \rightarrow \mathrm{Map}_B(F^0M, N)$ is an equivalence for all $n \geq m$. \square

Proof of Proposition 4.21. Let us say that an object $A \in \text{CAlg}_{R/B}^{\text{der}, \wedge, \text{aft}}$ is *adequate* if for every $N \in \text{APerf}_{B, \geq 0}$, the natural map

$$\mathfrak{D}(A) \sqcup \mathfrak{D}(B \oplus N) \longrightarrow \mathfrak{D}(A \oplus N)$$

is an equivalence, where $B \oplus N$ and $A \oplus N = A \times_B (B \oplus N)$ denote the trivial square zero extensions by N . To prove the proposition, it will suffice to verify the following assertions:

- (1) Every square zero extension $B \oplus_\alpha M$ of B by $M \in \text{APerf}_{B, \geq 0}$ is adequate.
- (2) Suppose that A is adequate and consider a pullback square (9) classifying a square zero extension $A_\eta \rightarrow A$ by $M \in \text{APerf}_{B, \geq 0}$. Then the following hold:
 - (2a) the map (10) is an equivalence.
 - (2b) A_η is adequate.

Indeed, Corollary 4.4 then shows by induction that every $A \in \text{Art}_{R/B}$ is adequate, so that \mathfrak{D} sends every pullback square (9) to a pushout of partition Lie algebroids.

Recall that for any square zero extension $B \oplus_\alpha M$ classified by $\alpha: L_{B/R}[-1] \rightarrow M$, the partition Lie algebroid $\mathfrak{D}(B \oplus_\alpha M)$ is the *free* partition Lie algebroid on $\alpha^\vee: M^\vee \rightarrow T_{B/R}[1]$. Consequently, $\mathfrak{D}((B \oplus_\alpha M) \oplus N)$ is the free partition Lie algebroid on $(\alpha^\vee, 0): M^\vee \oplus N \rightarrow T_{B/R}[1]$. Assertion (1) then follows from the fact that taking free Lie algebroids is a left adjoint.

We will simultaneously verify assertions (2a) and (2b). To this end, suppose that A is adequate and let $A_\eta \rightarrow A$ be a square zero extension of A classified by $\alpha: A \rightarrow B \oplus M[1]$. Let $A \rightarrow A^\bullet \rightarrow B \oplus M[1]$ be the cosimplicial resolution provided by Corollary 2.30. Given $N \in \text{APerf}_{B, \geq 0}$, we then obtain a diagram of complete almost finitely augmented algebras over B

$$(11) \quad \begin{array}{ccccc} A_\eta \oplus N & \longrightarrow & A_\eta^\bullet \oplus N & \longrightarrow & B \oplus N \\ \downarrow & & \downarrow & & \downarrow (\text{id}, 0) \\ A & \longrightarrow & A^\bullet & \longrightarrow & B \oplus M[1]. \end{array}$$

where the top row is obtained from the bottom row by base change. This gives rise to a diagram of partition Lie algebroids of the form

$$(12) \quad \begin{array}{ccc} |\mathfrak{D}(A^\bullet) \amalg_{\mathfrak{D}(B \oplus M[1])} \mathfrak{D}(B \oplus N)| & \longrightarrow & \mathfrak{D}(A) \amalg_{\mathfrak{D}(B \oplus M[1])} \mathfrak{D}(B \oplus N) \\ \downarrow & & \downarrow \\ |\mathfrak{D}(A_\eta^\bullet \oplus N)| & \longrightarrow & \mathfrak{D}(A_\eta \oplus N). \end{array}$$

It suffices to verify that the right vertical map is an equivalence: if this is the case, then taking $N = 0$ yields assertion (2a) and assertion (2b) then follows from the equivalences

$$\begin{array}{ccc} (\mathfrak{D}(A) \amalg_{\mathfrak{D}(B \oplus M[1])} \mathfrak{D}(B)) \amalg \mathfrak{D}(B \oplus N) & \xrightarrow{\sim} & \mathfrak{D}(A) \amalg_{\mathfrak{D}(B \oplus M[1])} \mathfrak{D}(B \oplus N) \\ \downarrow \sim & & \downarrow \sim \\ \mathfrak{D}(A_\eta) \amalg \mathfrak{D}(B \oplus N) & \longrightarrow & \mathfrak{D}(A_\eta \oplus N). \end{array}$$

To see that the right vertical map in (12) is an equivalence, we will show that each of the other three maps is an equivalence.

Top horizontal map. To see that the top horizontal map in (12) is an equivalence, it suffices to show that the map $|\mathfrak{D}(A^\bullet)| \rightarrow \mathfrak{D}(A)$ is an equivalence. Since geometric realisations of partition

Lie algebroids are computed in $(\mathrm{QC}_B^\vee)_{/T_{B/R}[1]}$, it suffices to verify this at the level of pro-coherent modules. Now recall that the cosimplicial resolution $A \rightarrow A^\bullet$ from Corollary 2.30 has the property that the augmented cosimplicial diagram $\mathrm{cot}(A) \rightarrow \mathrm{cot}(A^\bullet)$ in $\mathrm{Der}_{B/R, \geq 0}^{\mathrm{aperf}}$ is split. Taking pro-coherent duals, one obtains that the augmented simplicial diagram $\mathfrak{D}(A^\bullet) \rightarrow \mathfrak{D}(A)$ is split in $(\mathrm{QC}_B^\vee)_{/T_{B/R}[1]}$, so that $|\mathfrak{D}(A^\bullet)| \simeq \mathfrak{D}(A)$.

Left vertical map. It suffices to verify that each square of partition Lie algebroids

$$\begin{array}{ccc} \mathfrak{D}(B \oplus M[1]) & \longrightarrow & \mathfrak{D}(B \oplus N) \\ \downarrow & & \downarrow \\ \mathfrak{D}(A^i) & \longrightarrow & \mathfrak{D}(A_\eta^i \oplus N) \end{array}$$

is a pushout. The cosimplicial resolution A^\bullet from Corollary 2.30 has the property that $A^i \simeq B \oplus_\alpha K$ is a square zero extension of B and that the map $A^i \simeq B \oplus_\alpha K \rightarrow B \oplus M[1]$ arises from a map $e: K \rightarrow M[1]$ in $\mathrm{Der}_{B/R, \geq 0}^{\mathrm{aperf}}$. It follows that the above square is the image of a pushout square in $(\mathrm{QC}_B^\vee)_{/T_{B/K}[1]}$ under the free partition Lie algebroid functor, and hence a pushout square itself.

Bottom horizontal map. It suffices to check that the augmented simplicial diagram $\mathfrak{D}(A_\eta^\bullet \oplus N) \rightarrow \mathfrak{D}(A_\eta \oplus N)$ is a colimit diagram in QC_B^\vee . Recall that this is the pro-coherent dual of the augmented cosimplicial diagram of B -modules $\mathrm{cot}(A_\eta \oplus N) \rightarrow \mathrm{cot}(A_\eta^\bullet \oplus N)$. We will refine this to a diagram of filtered B -modules as follows. The diagram (11) of derived R -algebras over B lifts to a diagram of filtered R -algebras over B : we endow $B \oplus N$, A and A^\bullet with the filtration where $F^i = 0$ for all $i \geq 1$, and $B \oplus M[1]$ with the filtration

$$F^1(B \oplus M[1]) = M[1] \quad F^i(B \oplus M[1]) = 0 \quad \text{for } i \geq 2.$$

Taking fibre products, we then obtain (finite) filtrations on $A_\eta \oplus N$ and $A_\eta^\bullet \oplus N$.

The maps of animated filtered algebras $F^*(A_\eta \oplus N) \rightarrow B$ and $F^*(A_\eta^\bullet \oplus N) \rightarrow B$ are almost of finite presentation. Indeed, each $A^i \rightarrow B$ (concentrated in weight 0) is almost of finite presentation and $A_\eta^i \oplus N \rightarrow A^i$ is a square zero extension by an almost perfect filtered A^i -module, and thus almost of finite presentation by Lemma 2.20. By Proposition 2.14, we therefore obtain an augmented cosimplicial diagram of almost perfect filtered B -modules

$$\mathrm{cot}(F^*(A_\eta \oplus N)) \longrightarrow \mathrm{cot}(F^*(A_\eta^\bullet \oplus N)).$$

At the level of F^0 , this is given by $\mathrm{cot}(A_\eta \oplus N) \rightarrow \mathrm{cot}(A_\eta^\bullet \oplus N)$ and at the level of the associated graded, it is given by

$$\mathrm{cot}(A \oplus M(1) \oplus N) \longrightarrow \mathrm{cot}(A^\bullet \oplus M(1) \oplus N).$$

Here $A \oplus M(1) \oplus N$ denotes the derived graded R -algebra given by the trivial square zero extension of A by N (in weight 0) and M (in weight 1).

Using Lemma 4.22, we therefore obtain a certain augmented simplicial diagram of pro-coherent B -modules equipped with an *increasing* filtration

$$(13) \quad F_\star \mathfrak{D}(A_\eta^\bullet \oplus N) \longrightarrow F_\star \mathfrak{D}(A_\eta \oplus N)$$

At the level of the underlying objects (i.e. taking the colimit as $\star \rightarrow \infty$), this yields the augmented simplicial diagram $\mathfrak{D}(A_\eta^\bullet \oplus N) \rightarrow \mathfrak{D}(A_\eta \oplus N)$. To see that this is a colimit diagram, it now suffices

to show that (13) is a colimit diagram at the level of the associated graded. The associated graded of (13) is equivalent to the augmented simplicial diagram (with a certain weight grading that will not play a role)

$$\mathfrak{D}(A^\bullet \oplus M \oplus N) \rightarrow \mathfrak{D}(A \oplus M \oplus N).$$

Since A and all A^i (which are of the form $B \oplus_\alpha I$) are adequate, this is equivalent to the map

$$\mathfrak{D}(A^\bullet) \amalg \mathfrak{D}(B \oplus M \oplus N) \rightarrow \mathfrak{D}(A) \amalg \mathfrak{D}(B \oplus M \oplus N).$$

This is an equivalence because (as we already saw), $|\mathfrak{D}(A^\bullet)| \simeq \mathfrak{D}(A)$. \square

By Proposition 4.21, any partition Lie algebroid \mathfrak{g} over B determines a formal moduli problem

$$(14) \quad \Psi_{\mathfrak{g}}: \text{Art}_{R/B} \rightarrow \mathcal{S}; \quad \Psi_{\mathfrak{g}}(A) = \text{Map}_{\text{LieAlgd}_{\Delta, B/R}^\pi}(\mathfrak{D}(A), \mathfrak{g}).$$

Using this, we will now construct the putative equivalence of Theorem 4.19.

Construction 4.23. Consider the functor

$$\text{Art}_R \times_{\text{CAlg}_R^{\text{an, coh, et}}} \text{LieAlgd}_{\pi, \text{et}}^{\Delta/R} \xrightarrow{(\mathfrak{D}, \text{id})} (\text{LieAlgd}_{\Delta/R}^{\pi, \text{et}})^{\text{op}} \times_{\text{CAlg}_R^{\text{an, coh, et}}} \text{LieAlgd}_{\pi, \text{et}}^{\Delta/R} \xrightarrow{\text{Map}} \mathcal{S}$$

where the last functor is the fibrewise mapping space functor. Using Construction 4.17, this is adjoint to a functor $\text{LieAlgd}_{\pi, \text{et}}^{\Delta/R} \rightarrow \text{Fun}_{/\text{CAlg}_R^{\text{an, coh, et}}}(\text{Art}_R, \mathcal{S})$. Unravelling the definitions, this sends each $\mathfrak{g} \in \text{LieAlgd}_{\Delta, B/R}^\pi$ to the formal moduli problem $\Psi_{\mathfrak{g}}$ from (14). We thus obtain a functor

$$\begin{array}{ccc} \text{LieAlgd}_{\pi, \text{et}}^{\Delta/R} & \xrightarrow{\Psi} & \text{Moduli}/R \\ & \searrow & \swarrow \\ & \text{CAlg}_R^{\text{an, coh, et}} & \end{array}$$

Because \mathfrak{D} preserved cocartesian fibrations, the functor Ψ is a map of cartesian fibrations that preserves cartesian arrows.

It now remains to verify that Ψ restricts to an equivalence on fibres. For this we will use Lurie's formalism of deformation theories, slightly reformulated as follows:

Theorem 4.24 (cf. [SAG, Theorem 12.3.3.5]). *Let \mathcal{B} be a presentable ∞ -category equipped with a set of adjunctions $L_\alpha: \mathcal{S}^p \rightleftarrows \mathcal{B} : R_\alpha$ such that each R_α preserves sifted colimits and the R_α are jointly conservative. Let $\mathcal{B}_0 \subseteq \mathcal{B}$ be the smallest full subcategory of \mathcal{B} satisfying the following conditions:*

- (1) *It contains the initial object 0.*
- (2) *If $K \in \mathcal{B}_0$, then for each pushout square in \mathcal{B} of the form*

$$(15) \quad \begin{array}{ccc} L_\alpha(\mathbb{S}[n]) & \longrightarrow & K \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & K' \end{array}$$

for some α and $n < 0$, the object K' is contained in \mathcal{B}_0 as well.

*Then the restricted Yoneda embedding $h: \mathcal{B} \rightarrow \mathcal{P}(\mathcal{B}_0); K \mapsto \text{Map}_{\mathcal{B}}(-, K)$ is fully faithful, with essential image consisting of those presheaves X on \mathcal{B}_0 such that $X(0) \simeq *$ and such that (15) is sent to a pullback diagram of spaces.*

Proof. Consider the adjoint pair $\mathfrak{D}^* : \mathcal{B} \rightleftarrows \mathcal{P}(\mathcal{B}_0)^{\text{op}} : \mathfrak{D}_*$ where $\mathfrak{D}^*(K) = \text{Map}_{\mathcal{B}}(K, -)$. The conditions imply that this is a deformation theory in the sense of [SAG, Definition 12.3.3.2], and the result follows from [SAG, Theorem 12.3.3.5]. \square

Example 4.25. Let B be a coherent animated R -algebra. For each $M \in \text{Coh}_{B, \geq 0}$, consider

$$R_M : \text{QC}_{B/T_{B/R}[1]}^{\vee} \longrightarrow \text{Sp}; \quad (\rho : N \rightarrow T_{B/R}[1]) \longmapsto \text{Hom}_{\text{QC}^{\vee}(B)}(M^{\vee}, \text{fib}(\rho)).$$

This is a right adjoint functor preserving sifted colimits since $M^{\vee} \in \text{QC}_B^{\vee}$ is compact. Since the collection of M^{\vee} with $M \in \text{Coh}_{B, \geq 0}$ generates QC_B^{\vee} , this satisfies the conditions of Theorem 4.24. The full subcategory $\mathcal{B}_0^{\text{lin}} \subseteq \text{QC}_{B/T_{B/R}[1]}^{\vee}$ appearing in Theorem 4.24 can then be identified as follows. Note that

$$L_M(\mathbb{S}) \simeq (0 : M^{\vee} \rightarrow T_{B/R}[1]).$$

For any $\rho : N^{\vee} \rightarrow T_{B/R}[1]$ with $N \in \text{Coh}_{B, \geq 0}$, we then have that:

- (a) For any $n < 0$ and $M^{\vee}[n] \rightarrow N^{\vee}$, the cofibre $\text{fib}(N \rightarrow M[-n])^{\vee}$ is dual to a coherent connective B -module.
- (b) it can be obtained as a pushout of a diagram $0 \leftarrow L_N(\mathbb{S}[-1]) \rightarrow 0$ in $\text{QC}_{B/T_{B/R}[1]}^{\vee}$.

It follows that $\mathcal{B}_0^{\text{lin}}$ is the full subcategory of $\rho : N^{\vee} \rightarrow T_{B/R}[1]$ where N^{\vee} is the pro-coherent dual of a coherent connective B -module. Using Lemma 2.16, we can identify $\mathcal{B}_0^{\text{lin,op}} \simeq \text{Der}_{B/R, \geq 0}^{\text{coh}}$ with the full subcategory of derivations $\alpha : L_{B/R}[-1] \rightarrow M$ for which $M \in \text{Coh}_{B, \geq 0}$. Consequently, there is a fully faithful functor

$$\text{QC}_{B/T_{B/R}}^{\vee} \hookrightarrow \text{Fun}(\text{Der}_{B/R, \geq 0}^{\text{coh}}, \mathbb{S})$$

whose essential image consists of functors X such that $X(0) \simeq *$ and preserving pullbacks along all maps $0 \rightarrow (M[1], \alpha)$ in $\text{Der}_{B/R, \geq 0}^{\text{coh}}$ with $M \in \text{Coh}_{B, \geq 0}$.

Proof of Theorem 4.19. We will mimic the strategy from Example 4.25. For each $M \in \text{Coh}_{B, \geq 0}$, consider the functor

$$R_M : \text{LieAlgd}_{\Delta, B/R}^{\pi} \longrightarrow \text{Sp}; \quad (\rho : \mathfrak{g} \rightarrow T_{B/R}[1]) \longmapsto \text{Hom}_{\text{QC}_B^{\vee}}(M^{\vee}, \text{fib}(\rho)).$$

Proposition 3.8 implies that the forgetful functor $\text{LieAlgd}_{\Delta, B/R}^{\pi} \rightarrow \text{QC}_{B/T_{B/R}[1]}^{\vee}$ is a right adjoint that preserves sifted colimits and detects equivalences. Using this and Example 4.25, it follows that the set of functors R_M satisfies the conditions of Theorem 4.24.

To identify the full subcategory $\mathcal{B}_0 \subseteq \text{LieAlgd}_{\Delta, B/R}^{\pi}$, note that $L_M(\mathbb{S}) \simeq \text{Lie}_{\Delta, B/R}^{\pi}(0 : M^{\vee} \rightarrow T_{B/R})$ is the free partition Lie algebroid generated by the zero map from M^{\vee} . By Proposition 3.8, this means that there are natural equivalences

$$L_M(\mathbb{S}) \simeq \text{Lie}_{\Delta, B/R}^{\pi}(0 : M^{\vee} \rightarrow T_{B/R}) \simeq \mathfrak{D}(B \oplus M)$$

for all $M \in \text{Coh}_{B, \geq 0}$. It follows that $\mathcal{B}_0 \subseteq \text{LieAlgd}_{\Delta, B/R}^{\pi}$ is the smallest full subcategory containing the initial partition Lie algebroid $0 \simeq \mathfrak{D}$ which is closed under pushouts along the maps $\mathfrak{D}(B \oplus M[1]) \rightarrow \mathfrak{D}(B)$ for all $M \in \text{Coh}_{B, \geq 0}$. Corollary 4.4 and Proposition 4.21 now imply that there is a

commuting diagram

$$\begin{array}{ccccc}
 \mathrm{Der}_{B/R, \geq 0}^{\mathrm{coh}} & \xrightarrow[\sim]{(-)^\vee} & \mathcal{B}_0^{\mathrm{lin}, \mathrm{op}} & \hookrightarrow & (\mathrm{QC}_{B/T_{B/R}[1]}^\vee)^{\mathrm{op}} \\
 \mathrm{sqz} \downarrow & & \downarrow & & \downarrow \mathrm{Lie}_{\Delta, B/R}^\pi \\
 \mathrm{Art}_{R/B} & \xrightarrow[\sim]{\mathfrak{D}} & \mathcal{B}_0^{\mathrm{op}} & \hookrightarrow & (\mathrm{LieAlgd}_{\Delta, B/R}^\pi)^{\mathrm{op}}
 \end{array}$$

where equivalence \mathfrak{D} sends each pullback square (9) to the opposite of a pushout square of the form (15). Using this, Theorem 4.24 then implies that $\Psi: \mathrm{LieAlgd}_{\pi, \mathrm{et}}^{\Delta/R} \rightarrow \mathrm{Moduli}_{/R}$ restricts to an equivalence $\mathrm{LieAlgd}_{\Delta, B/R}^\pi \simeq \mathrm{Moduli}_{B/R}$ on fibres with the desired properties. \square

Remark 4.26. Combining the equivalence of Theorem 4.19 with the adjoint pair from Example 4.13, we obtain an adjoint pair

$$C^*: \mathrm{LieAlgd}_{\Delta, B/R}^\pi \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \mathrm{CAlg}_{R/B}^{\mathrm{der}, \mathrm{op}} : \mathfrak{D}$$

where $\mathfrak{D}(B') = T_{B/\mathrm{Spf}(B')}$ with the partition Lie algebroid structure arising from Theorem 4.19. Example 4.16 implies that the underlying object of $\mathfrak{D}(B')$ in $\mathrm{QC}_{B/T_{B/R}[1]}^\vee$ is naturally equivalent to $L_{B/B'}^\vee[1] \rightarrow L_{B/R}^\vee[1] = T_{B/R}[1]$.

The right adjoint C^* can be thought of as a version of the *Chevalley–Eilenberg complex* for partition Lie algebroids. Explicitly, $C^*(\mathfrak{g})$ is the derived algebra of functions on the formal moduli problem classified by \mathfrak{g} . One can now check that the adjoint pair (C^*, \mathfrak{D}) also defines a deformation theory in the sense of [SAG, Definition 12.3.3.2].

Remark 4.27. Suppose that X is a locally coherent qcqs derived scheme. By Remark 3.30, the ∞ -category of partition Lie algebroids on X can be identified with the ∞ -category of dotted sections

$$\begin{array}{ccccc}
 \mathrm{Open}^{\mathrm{aff}}(X)^{\mathrm{op}} & \xrightarrow{\mathfrak{g}} & \mathrm{LieAlgd}_{\pi, \mathrm{et}}^{\Delta/R} & \xrightarrow{\sim} & \mathrm{Moduli}_{/R} \\
 & \searrow \mathcal{O} & \downarrow p^\vee & \swarrow & \\
 & & \mathrm{CAlg}_R^{\mathrm{an}, \mathrm{coh}, \mathrm{et}} & &
 \end{array}$$

sending each map to a cocartesian arrow. Using Theorem 4.19 and unravelling Construction 4.17, one sees that the ∞ -category of partition Lie algebroids on X is equivalent to the ∞ -category of functors $F: \mathrm{Art}_R \times_{\mathrm{CAlg}_R^{\mathrm{an}, \mathrm{coh}, \mathrm{et}}} \mathrm{Open}^{\mathrm{aff}}(X)^{\mathrm{op}} \rightarrow \mathcal{S}$ satisfying the following two properties:

- (1) For each affine open $U \subseteq X$, F restricts to a formal moduli problem $F_U: \mathrm{Art}_{R/\mathcal{O}(U)} \rightarrow \mathcal{S}$.
- (2) For each inclusion of affine opens $V \subseteq U \subseteq X$, F_V is the initial formal moduli problem equipped with a map from F_U to its restriction along $\mathrm{Art}_{R/\mathcal{O}(U)} \rightarrow \mathrm{Art}_{R/\mathcal{O}(V)}$.

4.4. Underlying partition Lie algebra. Suppose that R is coherent and that B is a coherent animated R -algebra. We will use Theorem 4.19 to show that the underlying pro-coherent R -module of a partition Lie algebroid \mathfrak{g} over B can be endowed with a (R -linear) partition Lie algebra structure. To this end, we begin by recalling the following properties of the ∞ -categories $\mathrm{CAlg}_A^{\mathrm{an}}$ of animated A -algebras.

Proposition 4.28. *Given a cartesian square of animated rings on the left in which $f: A_0 \rightarrow A_{01}$ induces a surjection on π_0 , the right square of ∞ -categories is cartesian as well:*

$$\begin{array}{ccc} A & \xrightarrow{g'} & A_0 \\ f' \downarrow & & \downarrow f \\ A_1 & \xrightarrow{g} & A_{01} \end{array} \quad \begin{array}{ccc} \mathrm{CAlg}_A^{\mathrm{an}} & \xrightarrow{g'^*} & \mathrm{CAlg}_{A_0}^{\mathrm{an}} \\ f'^* \downarrow & & \downarrow f^* \\ \mathrm{CAlg}_{A_1}^{\mathrm{an}} & \xrightarrow{g^*} & \mathrm{CAlg}_{A_{01}}^{\mathrm{an}}. \end{array}$$

Proof. Write $h: A \rightarrow A_{01}$ for the composite map and consider the following diagram of ∞ -categories:

$$\begin{array}{ccc} \mathrm{CAlg}_A^{\mathrm{an}} & \xrightleftharpoons{(g'^*, f'^*)} & \mathrm{CAlg}_{A_0}^{\mathrm{an}} \times_{\mathrm{CAlg}_{A_{01}}^{\mathrm{an}}} \mathrm{CAlg}_{A_1}^{\mathrm{an}} \\ \mathrm{forget} \downarrow & & \downarrow \mathrm{forget} \\ \mathrm{Mod}_{A, \geq 0} & \xrightleftharpoons{(g'^*, f'^*)} & \mathrm{Mod}_{A_0, \geq 0} \times_{\mathrm{Mod}_{A_{01}, \geq 0}} \mathrm{Mod}_{A_1, \geq 0}. \end{array}$$

Here the rows are both adjoint pairs: the left adjoint (g'^*, f'^*) sends an animated A -algebra (resp. A -module) B to the matching triple of objects $(A_0 \otimes_A B, A_{01} \otimes_A B, A_1 \otimes_A B)$ and the right adjoint sends a matching triple of animated algebras (resp. modules) (B_0, B_{01}, B_1) to the fibre product $g'_* B_0 \times_{h_* B_{01}} f'_* B_1$ of A -algebras (resp. A -modules). In particular, the vertical forgetful functors commute both with the left adjoints and with the right adjoints. Since the forgetful functor detects equivalences, the unit and counit of the top adjunction are an equivalence if the unit and counit of the bottom adjunction are equivalences. This follows from [SAG, Theorem 16.2.0.2], so that the top adjunction is an equivalence. \square

Definition 4.29. Suppose that R is coherent and let $B \in \mathrm{CAlg}_R^{\mathrm{an}}$ be a R -algebra. We will write $\mathrm{Def}_{B/R}$ for the full subcategory of $\mathrm{Fun}(\Delta^1, \mathrm{CAlg}_R^{\mathrm{an}})_{/R \rightarrow B}$ spanned by those commuting squares

$$\begin{array}{ccc} A & \longrightarrow & R \\ \downarrow & & \downarrow \\ B_A & \longrightarrow & B \end{array}$$

that are cocartesian and in which $A \rightarrow R$ is an Artinian extension. We will typically write $A \rightarrow B_A$ for an object in $\mathrm{Def}_{B/R}$, omitting the data of the cocartesian square.

There is a canonical functor $\mathrm{Def}_{B/R} \rightarrow \mathrm{Art}_{R/R}$ sending each such square to $A \rightarrow R$. We then have the following corollary of Proposition 4.28:

Lemma 4.30. *Suppose that R is coherent and let B be an animated R -algebra. Then the functor $\mathrm{Def}_{B/R} \rightarrow \mathrm{Art}_{R/R}$ is a left fibration, classified by a formal moduli problem $\mathrm{def}_B: \mathrm{Art}_{R/R} \rightarrow \mathcal{S}$.*

Proof. The functor $\mathrm{Def}_{B/R} \rightarrow \mathrm{Art}_{R/R}$ is a cocartesian fibration, classified by $\mathrm{def}_B: \mathrm{Art}_{R/R} \rightarrow \mathrm{Cat}_\infty$ sending $A \mapsto \mathrm{CAlg}_A^{\mathrm{an}} \times_{\mathrm{CAlg}_R^{\mathrm{an}}} \{B\}$. Proposition 4.28 implies that def_B is a formal moduli problem. In particular, $\mathrm{def}_B(R \oplus M) \simeq \Omega \mathrm{def}_B(R \oplus M[1])$ is a space for every coherent R -module M . By Artinian induction, this implies that $\mathrm{def}_B(A)$ is a space for all A , so that the result follows. \square

Notation 4.31. Write $\Gamma(T_{B/R}[1]) \in \mathrm{LieAlg}_{\Delta, R}^\pi$ for the partition Lie algebra classified by the formal moduli problem def_B .

One can identify the complex underlying $\Gamma(T_{B/R}[1])$ as follows.

Construction 4.32. Given a map of coherent animated rings $f: R \rightarrow B$, consider the functor $f^*: \text{Coh}_{R, \geq 0} \rightarrow \text{Mod}_B$ and define

$$\text{Def}_{B/R}^{\text{sqz}} = \text{Coh}_{R, \geq 0} \times_{\text{Mod}_B} (\text{Mod}_B)_{L_{B/R}[-1]}$$

to be the ∞ -category of tuples (M, α) where $M \in \text{Coh}_{R, \geq 0}$ and $\alpha: L_{B/R}[-1] \rightarrow f^*M$ is a derivation. This fits into a commuting square of ∞ -categories

$$(16) \quad \begin{array}{ccc} \text{Def}_{B/R}^{\text{sqz}} & \xrightarrow{\text{sqz}} & \text{Def}_{B/R} \\ \downarrow & & \downarrow \\ \text{Coh}_{R, \geq 0} & \xrightarrow{\text{triv}} & \text{Art}_{R/R} \end{array}$$

in which the vertical functors are left fibrations. The bottom horizontal functor sends $M \in \text{Coh}_{R, \geq 0}$ to the trivial square zero extension $R \oplus M$ and the top horizontal map sends (M, α) to $(R \oplus M \rightarrow B \oplus_\alpha f^*M)$. This indeed defines a deformation of B over $R \oplus M$: it is obtained as the pullback of the diagram of deformations

$$(R \rightarrow B) \xrightarrow{\alpha} (R \oplus M[1] \rightarrow B \oplus f^*M[1]) \xleftarrow{0} (R \rightarrow B)$$

where the left map arises from the derivation $\alpha: B \rightarrow B \oplus f^*M[1]$ and the right map from the zero derivation. By the proof of Proposition 4.28, the fibre product then defines a deformation of B as well.

Lemma 4.33. *Let $f: R \rightarrow B$ be a map of coherent animated rings. Then there is an equivalence of pro-coherent R -modules $\Gamma(T_{B/R}[1]) \simeq f_*T_{B/R}[1]$.*

Proof. By construction, the left fibration $\text{Def}_{B/R}^{\text{sqz}} \rightarrow \text{Coh}_{R, \geq 0}$ is classified by the functor $\text{Coh}_{R, \geq 0} \rightarrow \mathcal{S}$ sending M to the space of B -linear maps $\text{Map}_B(L_{B/R}[-1], f^*M)$. By Lemma A.42, this functor is classified by a pro-coherent R -module; unravelling the definitions using Example A.32 and Definition A.37, one sees that this pro-coherent R -module is precisely $f_*T_{B/R}[1]$.

It now suffices to verify that this pro-coherent R -module is equivalent to the module underlying the partition Lie algebra $\Gamma(T_{B/R}[1])$. In terms of formal moduli problems, this is equivalent to the square (16) being cartesian. Since the base change

$$P = \text{Def}_{B/R} \times_{\text{Art}_{R/R}} \text{Coh}_{R, \geq 0} \longrightarrow \text{Coh}_{R, \geq 0}$$

is a left fibration classified by a reduced excisive functor $\text{Coh}_{R, \geq 0} \rightarrow \mathcal{S}$, it suffices to verify the following: for each $M \in \text{Coh}_{R, \geq 0}$, the induced map on fibres $\text{Map}_B(L_{B/R}[-1], f^*M) \rightarrow P_M$ induces an equivalence on loop spaces at the canonical (zero) basepoint. This map of loop spaces can be identified with the canonical map

$$\text{Map}_B(L_{B/R}, f^*M) \longrightarrow \text{Map}_{\text{CAlg}_{R \oplus M/B}^{\text{an}}} (B \oplus f^*M, B \oplus f^*M) \simeq \text{Map}_{\text{CAlg}_{R/B}^{\text{an}}} (B, B \oplus f^*M)$$

which is an equivalence by definition of the cotangent complex. \square

Remark 4.34. Suppose that R is coherent and let $\mathfrak{g} \in \text{LieAlg}_{\Delta, R}^\pi$ be a partition Lie algebra. Write $X: \text{Art}_{R/R} \rightarrow \mathcal{S}$ for the formal moduli problem classified by \mathfrak{g} and $\int X \rightarrow \text{Art}_{R/R}$ for its unstraightening. We then have fully faithful inclusions

$$(\text{LieAlg}_{\Delta, R}^\pi)_{/\mathfrak{g}} \xrightarrow{\simeq} (\text{Moduli}_{R/R})_{/X} \hookrightarrow \text{Fun}(\text{Art}_{R/R}, \mathcal{S})_{/X} \xrightarrow{\simeq} \text{Fun}(\int X, \mathcal{S}).$$

The last equivalence follows from the fact that, under unstraightening, both ∞ -categories are equivalent to the ∞ left fibrations over $\int X$. Unravelling the definitions, one sees that the essential image of the above inclusion consists of those functors $\int X \rightarrow \mathcal{S}$ that preserve the terminal object, as well as every pullback square in $\int X$ whose image in $\text{Art}_{R/R}$ is as in Definition 4.11.

Construction 4.35. Let B be a coherent R -algebra. Given an object $A \rightarrow B_A$ in $\text{Def}_{B/R}$, the map $B_A \rightarrow B$ is an almost Artinian extension. Indeed, it is the base change of the almost finitely presented map $A \rightarrow R$, hence almost finitely presented, and the kernel of $\pi_0(B_A) \rightarrow \pi_0(B)$ is generated by the images of the (nilpotent) generators of the kernel of $\pi_0(A) \rightarrow \pi_0(R)$. We thus obtain a functor $t: \text{Def}_{B/R} \rightarrow \text{AArt}_{R/B}$ given by $t(A \rightarrow B_A) = B_A$. Now suppose that $A_0 \rightarrow A_{01}$ is a nilpotent extension and consider a pullback square in $\text{Def}_{B/R}$

$$\begin{array}{ccc} (A \rightarrow B_A) & \longrightarrow & (A_0 \rightarrow B_{A_0}) \\ \downarrow & & \downarrow \\ (A_1 \rightarrow B_{A_1}) & \longrightarrow & (A_{01} \rightarrow B_{A_{01}}). \end{array}$$

Then $B_{A_0} \rightarrow B_{A_{01}}$ is a nilpotent extension as well, and the proof of Proposition 4.28 shows that $B_A \simeq B_{A_0} \times_{B_{A_{01}}} B_{A_1}$. It follows that restriction along t preserves formal moduli problems. Using Remark 4.34, we therefore obtain a functor

$$\Gamma: \text{LieAlg}_{\Delta, B/R}^{\pi} \xrightarrow{\sim} \text{Moduli}_{B/R} \xrightarrow{t^*} (\text{Moduli}_{R/R})_{/\text{def}_B} \xrightarrow{\sim} (\text{LieAlg}_{\Delta, R}^{\pi})_{/\Gamma(T_{B/R}[1])}.$$

Proposition 4.36. *Let $f: R \rightarrow B$ be a map of coherent animated rings. Then there is a commuting diagram of right adjoint functors*

$$(17) \quad \begin{array}{ccc} \text{LieAlg}_{\Delta, B/R}^{\pi} & \xrightarrow{\Gamma} & (\text{LieAlg}_{\Delta, R}^{\pi})_{/\Gamma(T_{B/R}[1])} \\ \text{forget} \downarrow & & \downarrow \\ (\text{QC}_B^{\vee})_{/T_{B/R}[1]} & \xrightarrow{f_*} & (\text{QC}_R^{\vee})_{/\Gamma(T_{B/R}[1])}. \end{array}$$

In particular, $\Gamma(T_{B/R}[1]) \simeq f_(T_{B/R}[1])$ as pro-coherent R -modules.*

Proof. Consider the commuting square of ∞ -categories

$$\begin{array}{ccc} \text{Def}_{B/R}^{\text{sqz}} & \xrightarrow{t'} & \text{Der}_{B/R, \geq 0}^{\text{aperf}} \\ \downarrow & & \downarrow \\ \text{Def}_{B/R} & \xrightarrow{t} & \text{AArt}_{R/B}. \end{array}$$

where the top functor t' sends $(M, \alpha: L_{B/R}[-1] \rightarrow f^*M)$ to (f^*M, α) . By restriction, this induces a commuting square of ∞ -categories of formal moduli problems and right adjoint functors between them. Using the equivalences from Example 4.25, Theorem 4.19 and Remark 4.34, this square of ∞ -categories of formal moduli problems is equivalent to a commuting square of the form (17). Indeed, the functor Γ is equivalent to restriction along t by definition, and one readily verifies that restriction along the functor t' sending (M, α) to (f^*M, α) corresponds to f_* at the level of pro-coherent modules. \square

5. FORMALLY INTEGRATING PARTITION LIE ALGEBROIDS

Let X be a locally coherent qcqs derived scheme over a coherent ground ring R . In this section, we will formally integrate partition Lie algebroids

$$\mathfrak{g} \rightarrow T_{X/R}[1]$$

on locally coherent qcqs derived schemes X over animated rings R . The result of our integration procedure will be *formal moduli stacks*

$$X \rightarrow Y$$

under X , where Y can be thought of as a formal leaf space. More precisely, we define:

Definition 5.1 (Formal moduli stacks). A *formal moduli stack* under X is a map of R -prestacks

$$X \longrightarrow Y$$

satisfying the following conditions:

- (1) Y has deformation theory, which means that it preserves limits of Postnikov towers and pullbacks along nilpotent extensions of R -algebras, see Definition B.6;
- (2) $X \rightarrow Y$ is locally almost of finite presentation ('laft'), see Definition B.3;
- (3) $X \rightarrow Y$ is a nil-isomorphism. Equivalently, it restricts to an equivalence on all reduced affine schemes, see Remark B.13.

Let $\text{ModuliStk}_{X/R} \subset (\text{PrStk}_R)_{X/}$ be the full subcategory spanned by all formal moduli stacks under X .

Remark 5.2. Note that Definition 5.1 also makes sense for X a general prestack.

In Section 5.1 below, we will link formal moduli stacks under *affine* schemes to formal moduli problems in the sense of Definition 4.11, and hence to partition Lie algebroids. In Section 5.2, we will then prove our main integration equivalence in Theorem 5.26.

5.1. Formal moduli problems and formal moduli stacks. Throughout this section, let $X = \text{Spec}(B)$ be a coherent affine derived scheme over R . We will show:

Theorem 5.3. *Let $X = \text{Spec}(B)$ be a coherent affine derived R -scheme. Then there is an equivalence between the ∞ -category $\text{ModuliStk}_{X/R}$ of formal moduli stacks under X and the ∞ -category of formal moduli problems*

$$F: \text{Art}_{B/R} \longrightarrow \mathcal{S}$$

in the sense of Definition 4.11. If $X \rightarrow Y$ is a formal moduli stack, then the corresponding formal moduli problem $\text{Art}_{B/R} \rightarrow \mathcal{S}$ sends $A \rightarrow B$ to the space of diagonal lifts

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & \searrow \text{dotted} & \downarrow \\ \text{Spec}(A) & \longrightarrow & \text{Spec}(R). \end{array}$$

The proof closely follows (a simple version of) the argument in [GR17a] and proceeds in two steps.

Notation 5.4. Throughout, we write

$$X_{\text{prored}} = \text{Spec}(B_{\text{prored}}), \quad X_{\text{inf}} := (X/R)_{\text{inf}}$$

for the pro-reduction (Definition B.8) and infinitesimal stack of the affine derived scheme X . We will write \sqcup for the pushout of (pro-)affine derived schemes (i.e. the fibre product of (ind-)rings) and consider the following categories:

- (1) Write $\text{Aff}_{X/R}^{\text{laft-nil},+}$ for the ∞ -category of affine R -schemes equipped with a nilpotent embedding $X \rightarrow S$ that exhibits X as an almost finitely presented S -scheme, such that $\mathcal{O}(S) \rightarrow \mathcal{O}(X) = B$ has eventually coconnective fibre. Equivalently, we simply have

$$\text{Aff}_{X/R}^{\text{laft-nil},+} = \text{Art}_{B/R}^{\text{op}}.$$

In particular, each $S \in \text{Aff}_{X/R}^{\text{laft-nil},+}$ is a coherent affine derived scheme as well (Corollary 4.5).

- (2) Let $\text{Aff}_{X/R}^{\text{nil},+}$ be the ∞ -category of affine R -schemes equipped with a nilpotent embedding $X \rightarrow S$ such that $\mathcal{O}(S) \rightarrow \mathcal{O}(X)$ has eventually coconnective fibre.
- (3) Let $\text{Aff}_{X_{\text{inf}}}$ be the ∞ -category of affine schemes over X_{inf} , i.e. $S \in \text{Aff}_R$ together with a map $S \leftarrow S_{\text{prored}} \rightarrow X$.

Our goal will be to first identify formal moduli problems with certain functors on the ∞ -category $\text{Aff}_{X/R}^{\text{nil},+}$. Next, we will relate functors on $\text{Aff}_{X/R}^{\text{nil},+}$ with prestacks $X \rightarrow Y \rightarrow X_{\text{inf}}$ in between X and its infinitesimal prestack. The ∞ -category of formal moduli stacks can then be identified with a natural subcategory of this ∞ -category.

Definition 5.5. We will say that a functor $F: (\text{Aff}_{X/R}^{\text{nil},+})^{\text{op}} \rightarrow \mathcal{S}$ has deformation theory if:

- (1) its value on the initial object is contractible,
- (2) it sends pushouts along nilpotent embeddings to pullbacks of spaces.

Let $u: \text{Aff}_{X/R}^{\text{laft-nil},+} \hookrightarrow \text{Aff}_{X/R}^{\text{nil},+}$ be the evident fully faithful embedding and consider the induced adjoint pair between ∞ -categories of presheaves:

$$u_!: \mathcal{P}(\text{Aff}_{X/R}^{\text{laft-nil},+}) \begin{array}{c} \xleftarrow{\quad} \\ \longleftarrow \end{array} \mathcal{P}(\text{Aff}_{X/R}^{\text{nil},+}): u^*.$$

The main technical step in the proof of Theorem 5.3 is the following:

Proposition 5.6. *Let $F: (\text{Aff}_{X/R}^{\text{laft-nil},+})^{\text{op}} \rightarrow \mathcal{S}$ be a formal moduli problem. Then the left Kan extension $u_!F: (\text{Aff}_{X/R}^{\text{nil},+})^{\text{op}} \rightarrow \mathcal{S}$ has deformation theory.*

The proof will require some preliminaries:

Lemma 5.7. *Let $F: (\text{Aff}_{X/R}^{\text{laft-nil},+})^{\text{op}} \rightarrow \mathcal{S}$ be a formal moduli problem and consider a pushout diagram in $\text{Aff}_{X/R}^{\text{nil},+}$*

$$\begin{array}{ccc} S_0 & \xrightarrow{f} & S'_0 \\ \downarrow & & \downarrow \\ S & \longrightarrow & S' \end{array}$$

such that S_0 and S'_0 are contained in $\text{Aff}_{X/R}^{\text{laft-nil},+}$ and $S_0 \rightarrow S$ is a square zero extension. Then the induced map

$$u_!F(S') \longrightarrow u_!F(S) \times_{u_!F(S_0)} u_!F(S'_0)$$

is an equivalence.

Proof. We will compare the fibres over a fixed $x: S'_0 \rightarrow u_1F$. Suppose $S_0 \rightarrow S$ is a square zero extension by an (necessarily eventually coconnective) ideal I and let $S_{0,I[1]}$ be the trivial square zero extension of S_0 by its suspension. Since $S_0 \in \text{Aff}_{X/R}^{\text{lft-nil},+}$ is a coherent affine derived scheme (Corollary 4.5), we can choose a filtered system $I_*: \mathcal{K} \rightarrow \text{Coh}(S_0)_{\geq 0}$ with $\text{colim}_{\alpha} I_{\alpha} = I$.

Having fixed this, let \mathcal{C}' be the ∞ -category of factorizations

$$\begin{array}{ccccc} S'_0 & \xrightarrow{=} & S'_0 & & \\ \downarrow & & \downarrow & \searrow x & \\ S' & \cdots \rightarrow & T' & \cdots \rightarrow & u_1F \end{array}$$

with $T' \in \text{Aff}_{X/R}^{\text{lft-nil},+}$. A map is a natural transformation that restricts to the identity on the solid part of the diagram. The geometric realization of this ∞ -category computes the fibre $u_1F(S'_0)_x$. Likewise, let \mathcal{C} be the ∞ -category of factorizations

$$\begin{array}{ccccc} S_0 & \xrightarrow{=} & S_0 & & \\ \downarrow & & \downarrow & \searrow x|_{S_0} & \\ S & \cdots \rightarrow & T & \cdots \rightarrow & u_1F \end{array}$$

which computes the fibre $u_1F(S)_x$. The restriction $u_1F(S'_0)_x \rightarrow u_1F(S)_x$ then arises as the realization of the functor $\pi: \mathcal{C}' \rightarrow \mathcal{C}$ that restricts a factorization along $S \rightarrow S'$.

We therefore have to prove that π is a weak equivalence. To do this, we will replace \mathcal{C}' and \mathcal{C} be weakly equivalent categories \mathcal{D}' and \mathcal{D} . First, let us consider the functor $\mathcal{K} \rightarrow \text{Cat}$ sending each $\alpha \in \mathcal{K}$ to the ∞ -category $\mathcal{D}(\alpha)$ of diagrams of the form

$$(18) \quad \begin{array}{ccccccc} & & S_{0,I[1]} & \xrightarrow{\quad} & S_0 & & \\ & \swarrow & \downarrow & & \downarrow & \searrow & \\ S_{0,I_{\alpha}[1]} & \cdots \rightarrow & S_0 & \xrightarrow{\quad} & S & & \\ \downarrow & & \downarrow & & \downarrow & \searrow x|_{S_0} & \\ S_0 & \cdots \rightarrow & T & \cdots \rightarrow & u_1F & & \end{array}$$

where $T \in \text{Aff}_{X/R}^{\text{lft-nil},+}$ and a morphism is a natural transformation which is the identity on the solid part of the diagram. This depends functorially on α by restriction along the map of trivial square zero extensions $S_{0,I[1]} \rightarrow S_{0,I_{\beta}[1]} \rightarrow S_{0,I_{\alpha}[1]}$ for each map of ideals $I_{\alpha} \rightarrow I_{\beta}$. We let $\mathcal{D} = \int_{\alpha \in \mathcal{K}} \mathcal{D}(\alpha)$ be the (cocartesian) unstraightening.

There is a natural functor $q: \mathcal{D} \rightarrow \mathcal{C}$ restricting to the right face of the cube in (18). This map is a cocartesian fibration (by postcomposition with $T \rightarrow T'$) whose fibre over a factorization $S \rightarrow T \rightarrow u_1F$ is given as follows. Note that the map $S \rightarrow T$ exhibits S as a square zero extension of S_0 by I relative to T , i.e. it is classified by a map $L_{S_0/T} \rightarrow I[1]$. One can then identify

$$q^{-1}(T) = \int_{\alpha \in \mathcal{K}} \text{Map}_{\text{QC}(S_0)/I[1]}(L_{S_0/T}, I_{\alpha}[1])$$

with the unstraightening of the functor sending each α to the space of lifts of the map $L_{S_0/T} \rightarrow I[1]$ to $I_\alpha[1]$. Since S_0 and T are in $\text{Aff}_{X/R}^{\text{laft-nil},+}$, the map $S_0 \rightarrow T$ is almost finitely presented (Corollary 4.6). Using that $L_{S_0/T}$ is an almost perfect S_0 -module, we find that

$$|q^{-1}(T)| \simeq \text{colim}_{\alpha \in K} \text{Map}_{\text{QC}(S_0)/I[1]}(L_{S_0/T}, I_\alpha[1]) \simeq \text{Map}_{\text{QC}(S_0)/I[1]}(L_{S_0/T}, \text{colim}_{\alpha} I_\alpha[1]) = *.$$

We conclude that $q: \mathcal{D} \rightarrow \mathcal{C}$ is a weak equivalence.

Next, we define \mathcal{D}' to be the ∞ -category of diagrams of the form

$$(19) \quad \begin{array}{ccccccc} & & S_{0,I[1]} & \longrightarrow & S_0 & \longrightarrow & S'_0 \\ & \swarrow & \downarrow & & \downarrow & & \downarrow \\ S_{0,I_\alpha[1]} & \cdots & S_0 & \longrightarrow & S'_0 & & \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S_0 & \xrightarrow{\cong} & S_0 & \longrightarrow & S & \longrightarrow & S' \\ & & & & \downarrow & & \downarrow \\ & & & & T' & \longrightarrow & u_1 F \end{array}$$

x

where again, $T' \in \text{Aff}_{X/R}^{\text{laft-nil},+}$ and a morphism is a natural transformation which is the identity on the solid part of the diagram. Restriction to the right face determines a cocartesian fibration $q': \mathcal{D}' \rightarrow \mathcal{C}'$ whose fibre over T' satisfies

$$|q'^{-1}(T')| \simeq \text{colim}_{\alpha \in K} \text{Map}_{\text{QC}(S_0)/I[1]}(L_{S_0/T'}, I_\alpha[1]) \simeq \text{Map}_{\text{QC}(S_0)/I[1]}(L_{S_0/T'}, \text{colim}_{\alpha} I_\alpha[1]) = *.$$

We thus obtain a commuting square

$$\begin{array}{ccc} \mathcal{D}' & \longrightarrow & \mathcal{C}' \\ \pi' \downarrow & & \downarrow \pi \\ \mathcal{D} & \longrightarrow & \mathcal{C} \end{array}$$

where the horizontal functors are weak equivalences and π' forgets the copies of S'_0 and S' in Diagram (19). It therefore remains to verify that π' is a weak equivalence.

Now let $\mathcal{E} \hookrightarrow \mathcal{D}$ and $\mathcal{E}' \hookrightarrow \mathcal{D}'$ be the full subcategories of diagrams of the form (18), resp. (19), whose front face is a pushout in $\text{Aff}_{X/R}^{\text{laft-nil},+}$. These inclusions admit left adjoints L, L' that replace the object T by the pushout of the front face; in particular, \mathcal{E} and \mathcal{E}' are weakly equivalent to \mathcal{D} and \mathcal{D}' . Furthermore, there is a natural functor $\phi: \mathcal{E} \rightarrow \mathcal{E}'$ sending each T as in (18) to $T' = S'_0 \sqcup_{S_0} T$, which fits naturally into a diagram of the form (19).

Now note that for any $T \in \mathcal{E} \subseteq \mathcal{D}$, there is a natural transformation $T \rightarrow \pi' \circ \phi(T) = S'_0 \sqcup_{S_0} T$. In particular, the map $\pi': |\mathcal{D}'| \rightarrow |\mathcal{D}|$ admits a right inverse (presented by ϕ). Conversely, for every object $T' \in \mathcal{D}'$ as in (19), there is a natural map

$$\phi \circ L \circ \pi'(T) = S'_0 \sqcup_{S_{0,I_\alpha[1]}} S_0 \rightarrow T$$

which shows that $\pi': |\mathcal{D}'| \rightarrow |\mathcal{D}|$ admits a left inverse (presented by $\phi \circ L$). We conclude that π' , and therefore the map $u_1 F(S')_x \rightarrow u_1 F(S)_{x|S_0}$ is an equivalence. \square

Lemma 5.8. *Let $F: (\text{Aff}_{X/R}^{\text{laft-nil},+})^{\text{op}} \rightarrow \mathcal{S}$ be a formal moduli problem and consider a pushout square*

$$\begin{array}{ccc} S_0 & \xrightarrow{f} & S'_0 \\ \downarrow & & \downarrow \\ S & \longrightarrow & S' \end{array}$$

in $\text{Aff}_{X/R}^{\text{nil},+}$ such that $S'_0 \in \text{Aff}_{X/R}^{\text{laft-nil},+}$ and $S_0 \rightarrow S$ is a square zero extension. Then the induced map

$$u_!F(S') \longrightarrow u_!F(S) \times_{u_!F(S_0)} u_!F(S'_0)$$

is an equivalence.

Proof. We fix $x: S \rightarrow u_!F$ with $x_0 = x|_{S_0}$. Consider the ∞ -category \mathcal{D} of diagrams of the form

$$(20) \quad \begin{array}{ccccc} S_0 & \cdots \rightarrow & S'_0 & & \\ \downarrow & & \downarrow & & \\ S & \cdots \rightarrow & S' & \xrightarrow{x'} & u_!F \\ & \searrow \scriptstyle x & \nearrow & & \end{array}$$

where the square is cocartesian and $S'_0 \in \text{Aff}_{X/R}^{\text{laft-nil},+}$. A map is a natural transformation that is constant on the solid part of the diagram. Likewise, let \mathcal{C}_0 be the ∞ -category of factorizations

$$S_0 \cdots \rightarrow S'_0 \xrightarrow{x'_0} u_!F$$

where $S'_0 \in \text{Aff}_{X/R}^{\text{laft-nil},+}$. There is an evident right fibration $\pi: \mathcal{D} \rightarrow \mathcal{C}_0$ sending $x' \mapsto x'|_{S'_0}$. The lemma asserts that the fibres of π are contractible.

By Lemma 5.7, the functor π is also a left fibration. Furthermore, the ∞ -category \mathcal{C}_0 is contractible by definition of $u_!F$ as a left Kan extension. Consequently, it will suffice to prove that the ∞ -category \mathcal{D} itself is contractible.

To see this, let \mathcal{D}' be the ∞ -category of diagrams of the form (20) where the square is not necessarily cocartesian, but where S' is contained in $\text{Aff}_{X/R}^{\text{laft-nil},+}$. There is an evident functor $\pi: \mathcal{D}' \rightarrow \mathcal{D}$ sending a factorization $x': S' \rightarrow u_!F$ with $S' \in \text{Aff}_{X/R}^{\text{laft-nil},+}$ to $S \sqcup_{S_0} S'_0 \rightarrow S' \rightarrow u_!F$. To prove the lemma it will thus suffice to show that:

- (a) \mathcal{D}' is weakly contractible.
- (b) $\pi: \mathcal{D}' \rightarrow \mathcal{D}$ is a weak equivalence.

Let us start with (a): notice that restriction of the diagram (20) to the bottom row defines a functor $\phi: \mathcal{D}' \rightarrow \mathcal{C}'$ to the ∞ -category of factorizations

$$S \cdots \rightarrow S' \xrightarrow{x'} u_!F$$

with $S' \in \text{Aff}_{X/R}^{\text{laft-nil},+}$. Note that \mathcal{C}' is contractible by definition of $u_!F$ as a left Kan extension and that ϕ admits a right adjoint sending each factorization $S \rightarrow S' \rightarrow u_!F$ to

$$\begin{array}{ccccc} S_0 & \cdots & S' & & \\ \downarrow & & \parallel & & \\ S & \cdots & S' & \xrightarrow{x'} & u_!F. \\ & \searrow x & & & \end{array}$$

This implies that \mathcal{D}' is weakly contractible.

For (b), pick a factorization $x': S' = S \sqcup_{S_0} S'_0 \rightarrow u_!F$ in \mathcal{D} and consider the comma ∞ -category x'/\mathcal{D}' . Unraveling the definitions, one sees that x'/\mathcal{D}' is just the ∞ -category \mathcal{D}' considered in our proof of (a), but with $S_0 \rightarrow S$ replaced $S'_0 \rightarrow S'$. The proof of (a) then shows that x'/\mathcal{D}' is contractible, so that π is a weak equivalence. \square

Corollary 5.9. *Let $F: (\text{Aff}_{X/R}^{\text{laft-nil},+})^{\text{op}} \rightarrow \mathcal{S}$ be a formal moduli problem and consider a pushout square in $\text{Aff}_{X/R}^{\text{nil},+}$*

$$\begin{array}{ccc} S_0 & \xrightarrow{f} & S'_0 \\ \downarrow & & \downarrow \\ S & \longrightarrow & S' \end{array}$$

such that $S_0 \rightarrow S$ is a square zero extension. Then the induced map

$$u_!F(S') \longrightarrow u_!F(S) \times_{u_!F(S_0)} u_!F(S'_0)$$

is an equivalence.

Proof. Let us fix a map $x: S'_0 \rightarrow u_!F$. It suffices to show that the induced map on fibres $u_!F(S')_x \rightarrow u_!F(S)_{x|_{S_0}}$ is an equivalence. Now x factors as $S'_0 \rightarrow T_0 \xrightarrow{t_0} u_!F$ with $T_0 \in \text{Aff}_{X/R}^{\text{laft-nil},+}$. Letting $T = T_0 \sqcup_{S_0} S$, we then have maps

$$u_!F(T)_{t_0} \longrightarrow u_!F(S')_{t_0|_{S'_0}} \longrightarrow u_!F(S)_{t_0|_{S_0}}.$$

The left map and the composite are equivalences by Lemma 5.8, so that the right map is an equivalence as well. \square

Proof of Proposition 5.6. Every nilpotent embedding of eventually coconnective affine derived schemes can be decomposed into a finite sequence of square zero extensions, so this follows from Corollary 5.9. \square

Definition 5.10. Let $f: G \rightarrow G'$ be a natural transformation between functors $(\text{Aff}_{X/R}^{\text{nil},+})^{\text{op}} \rightarrow \mathcal{S}$. We will say that f is *locally almost finitely presented* if for any $n \in \mathbb{N}$ and any cofiltered diagram $S_*: \mathcal{K} \rightarrow \text{Aff}_{X/R}^{\text{nil},+}$ with limit S such that all $\mathcal{O}(S_\alpha) \rightarrow \mathcal{O}(X) = B$ have $(n-1)$ -coconnective fibre, the map

$$\text{colim}_{\alpha \in \mathcal{K}^{\text{op}}} G(S_\alpha) \rightarrow G(S) \times_{G'(S)} \left(\text{colim}_{\alpha \in \mathcal{K}^{\text{op}}} G'(S_\alpha) \right)$$

is an equivalence.

Remark 5.11. Suppose that $S_* : \mathcal{K} \rightarrow \text{Aff}_{X/R}^{\text{nil},+}$ is a cofiltered diagram of affine schemes under X such that all $\mathcal{O}(S_\alpha) \rightarrow \mathcal{O}(X)$ have $(n-1)$ -coconnective fibre for some fixed n . Then

$$S_\alpha \simeq S_\alpha^{\leq n} \sqcup_{X^{\leq n}} X$$

where we write $X^{\leq n} = \text{Spec}(\tau_{\leq n} B)$ for the corresponding truncated affine derived scheme.

Corollary 5.12. *The adjoint pair $(u_!, u^*)$ restricts to an equivalence between the full subcategories of:*

- (1) *formal moduli problems $F : (\text{Aff}_{X/R}^{\text{laft-nil},+})^{\text{op}} \rightarrow \mathcal{S}$.*
- (2) *functors $G : (\text{Aff}_{X/R}^{\text{nil},+})^{\text{op}} \rightarrow \mathcal{S}$ with deformation theory such that the canonical map $X \rightarrow G$ is locally almost finitely presented.*

Proof. It will suffice to show that $X \rightarrow u_!(F)$ is indeed locally almost finitely presented. Assuming this, one has that $u_!$ is fully faithful because it takes the left Kan extension along the fully faithful functor u . A map $G \rightarrow G'$ between functors with deformation theory is an equivalence if and only if it is for every square zero extension $X \rightarrow X_I$ with $I \in \text{QC}(X)_{\geq 0}^+$, the induced map

$$G(X_I) \simeq G(X) \times_{G(X_{I[1]})} G(X) \longrightarrow G'(X) \times_{G'(X_{I[1]})} G'(X) = G'(X_I)$$

is an equivalence. If $X \rightarrow G$ and $X \rightarrow G'$ are both locally almost finitely presented, it suffices to verify this when $I \in \text{Coh}(X)$, in which case $X_I \in \text{Aff}_{X/R}^{\text{laft-nil},+}$. It follows that u^* detects equivalences such functors, so that $(u_!, u^*)$ restricts to the desired equivalence.

To see that $X \rightarrow u_!(F)$ is indeed locally almost finitely presented, let $S_* : \mathcal{K} \rightarrow \text{Aff}_{X/R}^{\text{nil},+}$ be a cofiltered diagram as in Definition 5.10, with limit S . By Remark 5.11, we can assume that $S_* = S_*^{\leq n} \sqcup_{X^{\leq n}} X$ arises as the pushout of the cofiltered diagram of n -coconnective schemes $S_\alpha^{\leq n}$ under $X^{\leq n}$. Consequently, we have natural equivalences $S_\alpha \simeq S_\alpha^{\leq n} \sqcup_{S^{\leq n}} S$. We will need to prove that the square

$$\begin{array}{ccc} \text{colim}_\alpha X(S_\alpha) & \longrightarrow & X(S) \\ \downarrow & & \downarrow \\ \text{colim}_\alpha u_!F(S_\alpha) & \longrightarrow & u_!F(S) \end{array}$$

is cartesian. To this end, note that $u_!F(S)$ arises as the classifying space of the ∞ -category \mathcal{C} of factorizations $S \rightarrow T \rightarrow j_!F$ with $T \in \text{Aff}_{X/R}^{\text{laft-nil},+}$. Likewise, for each α let $\mathcal{C}(\alpha)$ be the ∞ -category of factorizations $S_\alpha \rightarrow T \rightarrow u_!F$. This determines a diagram of categories $\mathcal{K} \rightarrow \text{Cat}$ whose unstraightening

$$\mathcal{C}' = \int_{\alpha \in \mathcal{K}} \mathcal{C}(\alpha)$$

has the property that $|\mathcal{C}'| = \text{colim}_\alpha u_!F(S_\alpha)$.

We now consider the functor $\pi : \mathcal{C}' \rightarrow \mathcal{C}$ sending each $S_\alpha \rightarrow T \rightarrow u_!F$ to the composite $S \rightarrow S_\alpha \rightarrow T \rightarrow u_!F$. This is a cocartesian fibration (by postcomposition with $T \rightarrow T'$) whose fibre over $x = (S \rightarrow T \rightarrow u_!F) \in \mathcal{C}$ is given by the ∞ -category

$$\pi^{-1}(x) = \int_{\alpha \in \mathcal{K}} \text{Map}_{S'/} (S_\alpha, T) \simeq \int_{\alpha \in \mathcal{K}} \text{Map}_{S^{\leq n}/} (S_\alpha^{\leq n}, T).$$

The classifying space of this ∞ -category can be identified with $|\pi^{-1}(x)| = \operatorname{colim}_\alpha \operatorname{Map}_{S \leq n} (S_\alpha^{\leq n}, T)$. For every map in \mathcal{C} of the form

$$S \rightarrow T \xrightarrow{f} T' \rightarrow u_! F$$

the induced change-of-fibres map $f_* : \pi^{-1}(x) \rightarrow \pi^{-1}(x')$ is then given at the level of classifying spaces by

$$\operatorname{colim}_\alpha \operatorname{Map}_{S \leq n} (S_\alpha^{\leq n}, T) \xrightarrow{f \circ -} \operatorname{colim}_\alpha \operatorname{Map}_{S \leq n} (S_\alpha^{\leq n}, T').$$

This map is a weak equivalence since every $T \rightarrow T'$ in $\operatorname{Aff}_{X/R}^{\text{laft-nil},+}$ is locally almost finitely presented (Corollary 4.6). Consequently, given a point $x \in |\mathcal{C}| = u_! F(S)$, corresponding to a factorization $x : S \rightarrow T \rightarrow u_! F$, there is an equivalence

$$\left(\operatorname{colim}_\alpha u_! F(S_\alpha) \right) \times_{u_! F(S)} \{x\} \simeq |\pi^{-1}(x)|.$$

Applying this to each point $x : S \rightarrow X \rightarrow u_! F$, one sees that

$$\left(\operatorname{colim}_\alpha u_! F(S_\alpha) \right) \times_{u_! F(S)} \{x\} \simeq \operatorname{colim}_\alpha \operatorname{Map}_{S'}(S_\alpha, X) \simeq \left(\operatorname{colim}_\alpha X(S_\alpha) \right) \times_{X(S)} \{x\}$$

which yields the result. \square

Next, let $\operatorname{PrStk}_{X_{\text{inf}}}^{\text{conv}}$ be the ∞ -category of convergent prestacks over X_{inf} . Every $S \in \operatorname{Aff}_{X/R}^{\text{nil},+}$ in gives rise to a convergent prestack $S \rightarrow X_{\text{inf}}$ over X_{inf} , where the map is adjoint to the canonical map $S_{\text{prored}} = X_{\text{prored}} \rightarrow X$. Let us now consider the adjoint pairs

$$v_! : \mathcal{P}(\operatorname{Aff}_{X/R}^{\text{nil},+}) \xrightarrow{\quad} (\operatorname{PrStk}_{X_{\text{inf}}}^{\text{conv}}) : v^* \qquad \tilde{v}_! : \mathcal{P}(\operatorname{Aff}_{X/R}^{\text{nil},+}) \xrightarrow{\quad} (\operatorname{PrStk}_{X_{\text{inf}}}^{\text{conv}})_{X'} : \tilde{v}^*$$

where $v_!$ and $\tilde{v}_!$ are the unique colimit-preserving functors sending each nilpotent embedding $S \in \operatorname{Aff}_{X/R}^{\text{nil},+}$ to the diagrams of convergent prestacks $S \rightarrow X_{\text{inf}}$ and $X \rightarrow S \rightarrow X_{\text{inf}}$, respectively. At the level of the right adjoints, given a diagram of convergent R -prestacks $X \rightarrow Y \rightarrow X_{\text{inf}}$, the presheaf $\tilde{v}^* Y$ sends each nilpotent embedding $X \hookrightarrow S$ in $\operatorname{Aff}_{X/R}^{\text{nil},+}$ to the space of dotted lifts

$$(21) \quad \begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & \searrow \text{dotted} & \downarrow \\ S & \longrightarrow & X_{\text{inf}} \end{array}$$

in $\operatorname{PrStk}_R^{\text{conv}}$. Let us point out that, since $X \hookrightarrow S$ is a nilpotent embedding, this space is equivalent to the space of dotted lifts where we forget compatibility with the projection to X_{inf} .

Remark 5.13. The functor $v_!$ can be computed explicitly as follows: if $x : S \rightarrow X_{\text{inf}}$ is a map from an *eventually coconnective* affine derived scheme, then

$$\operatorname{Map}_{/X_{\text{inf}}} (S, v_! F) \simeq F(X \sqcup_{S_{\text{prored}}} S) = \operatorname{colim}_\alpha F(S \amalg_{S_\alpha} X)$$

with the colimit taken over the filtered ∞ -category of nilpotent ideals in $\pi_0(\mathcal{O}(S))$. Indeed, both of these functors preserve colimits, as colimits of convergent prestacks are computed pointwise on eventually coconnective affine derived schemes. Furthermore, they are naturally equivalent for

functors F represented by $T \in \text{Aff}_{X/R}^{\text{nil},+}$, because $\text{Map}_{/X_{\text{inf}}}(S, v_!T)$ is naturally equivalent to the diagram of dotted extensions in Aff_R

$$\begin{array}{ccc} S_{\text{prored}} & \xrightarrow{x} & X \\ \downarrow & & \downarrow \\ S & \cdots\cdots\cdots & T. \end{array}$$

Lemma 5.14. *The adjoint pair $(\tilde{v}_!, \tilde{v}^*)$ restricts to an equivalence between the full subcategories of:*

- (1) functors $F: (\text{Aff}_{X/R}^{\text{nil},+})^{\text{op}} \rightarrow \mathcal{S}$ with deformation theory.
- (2) diagrams of nil-isomorphic convergent prestacks $X \rightarrow Y \rightarrow X_{\text{inf}}$ where Y has deformation theory.

Proof. Let us first prove that $\tilde{v}_!$ and \tilde{v}^* preserve these two subcategories. First, given $X \rightarrow Y \rightarrow X_{\text{inf}}$ where Y has deformation theory, the diagram (21) directly shows that $\tilde{v}^*(Y)$ sends pushouts in $\text{Aff}_{X/R}^{\text{nil},+}$ along nilpotent embeddings to pullbacks of spaces and sends X to a contractible space.

Next, we consider the functor

$$\mathcal{P}(\text{Aff}_{X/R}^{\text{nil},+}) \xrightarrow{\tilde{v}_!} (\text{PrStk}_{X_{\text{inf}}}^{\text{conv}})_{X/} \xrightarrow{\text{forget}} \text{PrStk}_{X_{\text{inf}}}^{\text{conv}}$$

and the unique colimit-preserving functor

$$v_!: \mathcal{P}(\text{Aff}_{X/R}^{\text{nil},+}) \longrightarrow \text{PrStk}_{X_{\text{inf}}}^{\text{conv}}$$

that sends $X \hookrightarrow S$ to the map $S \rightarrow X_{\text{inf}}$ adjoint to $S_{\text{prored}} = X_{\text{prored}} \rightarrow X$. Both functors preserve colimits indexed by contractible diagrams and are naturally equivalent on representables. Consequently, they are naturally equivalent on all presheaves arising as colimits of contractible diagrams of representables. Since every functor $F: \text{Aff}_{X/R}^{\text{nil},+} \rightarrow \mathcal{S}$ with deformation theory has $F(X) \simeq *$, it is such a colimit of a contractible diagram, i.e. we can identify $\tilde{v}_!(F) \rightarrow X_{\text{inf}}$ simply with $v_!(F) \rightarrow X_{\text{inf}}$. Using the formula from Remark 5.13 and Lemma B.7, one now readily verifies that if F has deformation theory, then the convergent prestack $\tilde{v}_!(F) \simeq v_!(F)$ has deformation theory and is nil-isomorphic to X_{inf} (or equivalently, to X).

The adjoint pair $(\tilde{v}_!, \tilde{v}^*)$ therefore restricts to an adjoint pair between functors F with deformation theory and nil-isomorphisms $X \rightarrow Y \rightarrow X_{\text{inf}}$ where Y has deformation theory. When F has deformation theory, the unit $F \rightarrow \tilde{v}^*\tilde{v}_!F$ is given at each nilpotent embedding $X \hookrightarrow S$ as follows. Write $x_0: X \rightarrow F$ for the canonical map and suppose that $\mathcal{O}(S) \rightarrow \mathcal{O}(X)$ has $(n-1)$ -coconnective fibre for some n , so that $S = S^{\leq n} \sqcup_{X^{\leq n}} X$ is the pushout of the corresponding n -coconnective affine derived schemes. Then the unit map can be identified with

$$(22) \quad F(S) \simeq F(S^{\leq n} \sqcup_{X^{\leq n}} X) \longrightarrow F(S^{\leq n} \amalg_{S_{\text{prored}}} X) \times_{F(X^{\leq n} \amalg_{X_{\text{prored}}} X)} \{\nabla^* x_0\}$$

where $f: X^{\leq n} \sqcup_{X_{\text{prored}}} X \rightarrow X$ arises from the codiagonal map. Diagrammatically, this sends the dotted map $x: S \rightarrow F$ in the following diagram to its restrictions:

$$\begin{array}{ccc} X^{\leq n} \amalg_{X_{\text{prored}}} X & \xrightarrow{\nabla} & X \\ \downarrow & & \downarrow \searrow^{x_0} \\ S^{\leq n} \amalg_{S_{\text{prored}}} X & \longrightarrow & S \cdots\cdots\cdots F \end{array}$$

where $X^{\leq n} \amalg_{X_{\text{prored}}} X$ is a pro-object in $\text{Aff}_{X/R}^{\text{nil},+}$ using the inclusion of X as the right component. Using that $X_{\text{prored}} \rightarrow S_{\text{prored}}$ is an equivalence and that $S \simeq S^{\leq n} \sqcup_{X^{\leq n}} X$, one sees that the above square is a pushout square of pro-objects in $\text{Aff}_{X/R}^{\text{nil},+}$. Consequently, the unit map (22) is an equivalence, since F had deformation theory.

On the other hand, for $X \xrightarrow{y_0} Y \rightarrow X_{\text{inf}}$, the counit $\tilde{v}_1 \tilde{v}^* Y \rightarrow Y$ is given at each $x: S \rightarrow X_{\text{inf}}$, where S is an eventually coconnective affine derived scheme, by the map

$$(23) \quad \text{Map}_{/X_{\text{inf}}} (S \amalg_{S_{\text{prored}}} X, Y) \times_{\text{Map}_{/X_{\text{inf}}}(X, Y)} \{y_0\} \longrightarrow \text{Map}_{/X_{\text{inf}}}(S, Y).$$

In terms of diagrams, the domain can be identified with the space of dotted extensions in the following diagram

$$\begin{array}{ccccc} S_{\text{prored}} & \longrightarrow & X & \xrightarrow{y_0} & Y \\ \downarrow & & \downarrow & \nearrow \text{dotted } y & \downarrow \\ S & \longrightarrow & S \sqcup_{S_{\text{prored}}} X & \longrightarrow & X_{\text{inf}}. \end{array}$$

The counit sends such y to its restriction to S . Notice that the pushout $S \sqcup_{S_{\text{prored}}} X$ arises as a pro-system of pushout squares $S \sqcup_{S_\alpha} X$, where each $S_\alpha \rightarrow S$ is a nilpotent embedding. Since Y has deformation theory, it follows that the map

$$Y(X \sqcup_{S_{\text{prored}}} S) \longrightarrow Y(X) \times_{Y(S_{\text{prored}})} Y(S)$$

is an equivalence. Consequently, the counit map (23) can be identified with the map

$$Y(S) \times_{(Y(S_{\text{prored}}) \times_{X(S_{\text{prored}})} \{x\})} \{y_0|_{S_{\text{prored}}}\} \longrightarrow Y(S)$$

projecting onto the first factor. The fact that $Y \rightarrow X_{\text{inf}}$ is a nil-isomorphism is equivalent to the fact that $Y(S_{\text{prored}}) \simeq X(S_{\text{prored}})$, so that $Y(S_{\text{prored}}) \times_{X(S_{\text{prored}})} \{x\}$ is contractible and the above map is indeed an equivalence as desired. \square

Finally, we note:

Lemma 5.15. *The functor $\text{obl}_v: (\text{PrStk}_{/X_{\text{inf}}})_{X/R} \rightarrow \text{PrStk}_{X/R}$ sending each $X \rightarrow Y \rightarrow X_{\text{inf}}$ to $X \rightarrow Y$ restricts to an equivalence between the full subcategories of:*

- (1) *Diagrams $X \rightarrow Y \rightarrow X_{\text{inf}}$ where Y has deformation theory and $X \rightarrow Y$ is a left nil-isomorphism.*
- (2) *Formal moduli stacks $X \rightarrow Y$ in the sense of Definition 5.1.*

Proof. Note that if $f: X \rightarrow Y$ is locally almost finitely presented, then f is a nil-isomorphism if and only if it restricts to an equivalence on reduced schemes, by Remark B.13. The forgetful functor then restricts to an equivalence because for any nil-isomorphism $X \rightarrow Y$, space $\text{Map}_{X/R}(Y, X_{\text{inf}})$ is contractible, as all three functors are naturally equivalent on pro-reduced affine schemes S_{prored} . \square

Proof of Theorem 5.3. Proposition 5.6 and Lemma 5.14 imply that the composite adjunction

$$w_! : \mathcal{P}(\text{Aff}_{X/R}^{\text{laft-nil},+}) \xrightleftharpoons[u^*]{u_!} \mathcal{P}(\text{Aff}_{X/R}^{\text{nil},+}) \xrightleftharpoons[\tilde{v}^*]{\tilde{v}_!} (\text{PrStk}_{X_{\text{inf}}}^{\text{conv}})_{X/R} : w^*$$

restricts to an adjunction $(w_!^{\text{def}}, w_{\text{def}}^*)$ between formal moduli problems on $\text{Aff}_{X/R}^{\text{laft-nil},+} \simeq \text{Art}_{B/R}^{\text{op}}$ and nil-isomorphisms $X \rightarrow Y \rightarrow X_{\text{inf}}$ where Y has deformation theory.

The resulting adjoint pair $(w_!^{\text{def}}, w_{\text{def}}^*)$ has a fully faithful left adjoint: $w_!$ is fully faithful by Corollary 5.12 and $\tilde{v}_!$ was fully faithful on functors with deformation theory by Lemma 5.14. It therefore remains to show that the essential image of $w_!^{\text{def}}$ consists precisely of nil-isomorphisms $X \rightarrow Y \rightarrow X_{\text{inf}}$ which exhibit X as a laft Y -prestack. Lemma 5.15 then implies that w_{def}^* induces an equivalence between formal moduli stacks $X \rightarrow Y$ and formal moduli problems. Unravelling the definitions, one sees that w_{def}^* is precisely the functor from Theorem 5.3.

Let us first verify that $X \rightarrow w_!^{\text{def}}(F)$ is indeed laft, i.e. for each map $T \rightarrow w_!^{\text{def}}(F)$ from an eventually coconnective affine derived R -scheme T , the pullback $X \times_{w_!^{\text{def}}(F)} T$ is a laft T -prestack. Note that $w_!^{\text{def}}(F)$ is the colimit of a contractible diagram of nilpotent extensions $S \in \text{Aff}_{X/R}^{\text{laft-nil,+}}$, and that such colimits in $\text{PrStk}_R^{\text{conv}}$ are computed pointwise on eventually coconnective affine derived schemes. Hence it suffices to treat the case where $T \rightarrow w_!^{\text{def}}(F)$ fits into a diagram

$$\begin{array}{ccc} & X & \\ & \downarrow & \searrow \\ T & \longrightarrow & S \xrightarrow{w_!(x)} w_!^{\text{def}}(F) \end{array}$$

where $X \rightarrow S$ is contained in $\text{Aff}_{X/R}^{\text{laft-nil,+}}$ and $x \in F(S)$. It therefore suffices to treat the case where $T = S$. In that case,

$$X \times_{w_!^{\text{def}}(F)} S = w_!^{\text{def}}(X \times_F S)$$

arises as the image of a formal moduli problem over the base S , i.e. a formal moduli problem $(\text{Aff}_{X/S}^{\text{laft-nil,+}})^{\text{op}} \rightarrow \mathcal{S}$. This uses that $w_!$ preserves pullbacks of formal moduli problems (by Corollary 5.12) and that $v_!$ preserves pullbacks by the formula from Remark 5.13.

Since $X \rightarrow S$ is a nilpotent embedding which is almost finitely presented, each $S' \in \text{Aff}_{X/S}^{\text{laft-nil,+}}$ is almost finitely presented over S (Corollary 4.6). Consequently, the prestack $X \times_{w_!^{\text{def}}(F)} T$ is a colimit of almost finitely presented affine derived T -schemes; this implies that it is a laft T -prestack, as required.

To conclude, it now suffices to verify that w_{def}^* detects equivalences between laft nil-isomorphisms $X \rightarrow Y$ where Y has deformation theory. This follows from the fact that \tilde{v}^* detects weak equivalences by Lemma 5.14 and that u^* detects weak equivalences by Corollary 5.12. \square

5.2. Formal integration of partition Lie algebroids. Let us again fix a coherent animated base ring R and a locally coherent qcqs derived R -scheme X . To prove our main theorem Theorem 5.26, we will proceed in two steps: first, we will show that the shifted tangent complex functor

$$\begin{aligned} \text{ModuliStk}_{X/R} &\longrightarrow (\text{QC}_X^{\vee})_{/T_{X/R}[1]} \\ (X \rightarrow Y) &\longmapsto (T_{X/Y}[1] \rightarrow T_{X/R}[1]) \end{aligned}$$

is monadic; next, we will identify the resulting monad with the partition Lie algebroid monad constructed in Theorem 3.1. In our proof, we will need to relate formal moduli stacks on different schemes.

Construction 5.16 (Functorial moduli stacks and their tangent fibres). Let $X' \rightarrow X$ be a laft map of prestacks with deformation theory. Given a formal moduli stack $X \rightarrow Y$, consider the composite map

$$X' \rightarrow X \rightarrow Y$$

and let $Y_{X'}^\wedge$ be the formal completion of Y along $X' \rightarrow Y$ as defined in Construction B.9. Via the natural map $X' \rightarrow Y_{X'}^\wedge$, the prestack $Y_{X'}^\wedge$ becomes a formal moduli stack under X' by Lemma B.14. The construction $(X \rightarrow Y) \mapsto (X' \rightarrow Y_{X'}^\wedge)$ defines a functor

$$(-)_{X'}^\wedge : \text{ModuliStk}_{X/R} \rightarrow \text{ModuliStk}_{X'/R}.$$

More formally, let us write $\text{PrStk}_R^{\text{def,laft-map}}$ for the ∞ -category of R -prestacks with deformation theory and laft maps between them. Consider the full subcategory

$$\text{ModuliStk}_R \subset \text{Fun}(\Delta^1, \text{PrStk}_R^{\text{def,laft-map}})$$

whose objects are maps of R -prestacks $X \rightarrow Y$ with Y a formal moduli stack under X (Definition 5.1) and whose morphisms are squares

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y' \end{array}$$

for which the vertical maps are laft and have deformation theory. Evaluation at 0 defines a cartesian fibration

$$\text{ModuliStk}_R \rightarrow \text{PrStk}_R^{\text{def,laft-map}},$$

which we straighten to obtain a functor $(\text{PrStk}_R^{\text{def,laft-map}})^{\text{op}} \rightarrow \text{Cat}_\infty$. It sends an object X to $\text{ModuliStk}_{X/R}$ and a morphism $X' \rightarrow X$ to the induced

$$(-)_{X'}^\wedge : \text{ModuliStk}_{X/R} \longrightarrow \text{ModuliStk}_{X'/R}; \quad (X \rightarrow Y) \longmapsto (X' \rightarrow Y_{X'}^\wedge).$$

Construction B.23 and Proposition B.32 then define (after suspending) the functorial shifted relative tangent complex

$$\begin{aligned} T[1] : \text{ModuliStk}_R &\longrightarrow \text{QC}_{/T_{-/R}[1]}^\vee \\ (X \rightarrow Y) &\longmapsto (T_{X/Y}[1] \rightarrow T_{X/R}[1]) \end{aligned}$$

This defines a map of cartesian fibrations by Proposition B.32 and Proposition B.28(4). In other words, we obtain functors $T_{X/-}[1] : \text{ModuliStk}_{X/R} \rightarrow (\text{QC}_X^\vee)_{/T_{X/R}[1]}$ that are natural in X with respect to laft maps, i.e., each laft map $f : X' \rightarrow X$ gives rise to a commutative square

$$\begin{array}{ccc} \text{ModuliStk}_{X/R} & \xrightarrow{T_{X/-}[1]} & (\text{QC}_X^\vee)_{/T_{X/R}[1]} \\ (-)_{X'}^\wedge \downarrow & & \downarrow f^\sharp \\ \text{ModuliStk}_{X'/R} & \xrightarrow{T_{X'/-}[1]} & (\text{QC}_{X'}^\vee)_{/T_{X'/R}[1]} \end{array}$$

where $f^\sharp(\mathcal{F} \rightarrow T_{X/R}[1]) \simeq f^*(\mathcal{F}) \times_{(f^*T_{X/R}^{\text{pre}})_{\text{aft}[1]}} T_{X'/R}[1]$. If f is formally étale, then this simplifies to $f^\sharp(\mathcal{F} \rightarrow T_{X/R}[1]) \simeq f^*\mathcal{F}$ by Proposition B.28(3).

Proposition 5.17 (Conservativity). *Let X be a prestack with deformation theory and $Y_1 \rightarrow Y_2$ a map of formal moduli stacks under X . If $T_{X/Y_1} \rightarrow T_{X/Y_2}$ is an equivalence, then so is $Y_1 \rightarrow Y_2$.*

Proof. Since Y_1 and Y_2 are both convergent, it suffices to verify that $Y_1(A) \rightarrow Y_2(A)$ is an equivalence for every eventually coconnective animated ring A . To see this, consider the ind-ring

$$A_{\text{prored}} = \text{“colim”}_I \pi_0(A)/I$$

indexed by the filtered poset of nilpotent ideals of $\pi_0(A)$. We then have a commuting diagram

$$\begin{array}{ccc} Y_1(A) & \longrightarrow & Y_2(A) \\ \downarrow & & \downarrow \\ X(A_{\text{prored}}) & \xrightarrow{\sim} & Y_1(A_{\text{prored}}) \xrightarrow{\sim} Y_2(A_{\text{prored}}). \end{array}$$

The bottom maps are equivalences since $X \rightarrow Y_1$ and $X \rightarrow Y_2$ are nil-isomorphisms and left (Remark B.13). We therefore have to show that for each point $x \in X(A_{\text{prored}}) = \text{colim}_I X(\pi_0(A)/I)$, the map on fibres $Y_1(A)_x \rightarrow Y_2(A)_x$ is an equivalence. To see this, it will suffice to show that for any nilpotent ideal $I \subseteq \pi_0(A)$ and $x \in X(\pi_0(A)/I)$, the map

$$(24) \quad Y_1(A) \times_{Y_1(\pi_0(A)/I)} \{x\} \rightarrow Y_2(A) \times_{Y_2(\pi_0(A)/I)} \{x\}$$

is an equivalence. Indeed, the map $Y_1(A)_x \rightarrow Y_2(A)_x$ is a filtered colimit of maps of the above form.

Now recall that $A \rightarrow \pi_0(A)/I$ decomposes as a finite sequence $A = A_n \rightarrow A_{n-1} \rightarrow \cdots \rightarrow A_0 = \pi_0(A)/I$ where each $A_k \rightarrow A_{k-1}$ is a square zero extension by an A_0 -module J_k (concentrated in a single degree, although we will not need this). To prove that (24) is an equivalence, we will proceed by induction on this tower, the case of A_0 being evident. For the inductive step, note that there is a natural map of fibre sequences of spaces

$$\begin{array}{ccccc} Y_1(A_k) \times_{Y_1(A_0)} \{x\} & \longrightarrow & Y_1(A_{k-1}) \times_{Y_1(A_0)} \{x\} & \longrightarrow & Y_1(A_0 \oplus J_k[1]) \times_{Y_1(A_0)} \{x\} \\ \downarrow & & \downarrow & & \downarrow \\ Y_2(A_k) \times_{Y_2(A_0)} \{x\} & \longrightarrow & Y_2(A_{k-1}) \times_{Y_2(A_0)} \{x\} & \longrightarrow & Y_2(A_0 \oplus J_k[1]) \times_{Y_2(A_0)} \{x\} \end{array}$$

since Y_1 and Y_2 have deformation theory. By inductive hypothesis the middle vertical map is an equivalence, so it suffices to verify that the right vertical map is an equivalence. Unravelling Definition B.17 and using that the maps $X \rightarrow Y_i$ are left, one sees that the right vertical map fits into a pullback square

$$\begin{array}{ccc} Y_1(A_0 \oplus J_k[1]) \times_{Y_1(A_0)} \{x\} & \longrightarrow & T_{X/Y_1, x}(J_k[2]) \\ \downarrow & & \downarrow \\ Y_2(A_0 \oplus J_k[1]) \times_{Y_2(A_0)} \{x\} & \longrightarrow & T_{X/Y_2, x}(J_k[2]). \end{array}$$

The right vertical map is an equivalence by the assumption that $T_{X/Y_1} \rightarrow T_{X/Y_2}$ was an equivalence, so the result follows. \square

Definition 5.18 (Zariski descent). We say that a prestack $X \in \text{PrStk}$ satisfies *Zariski descent* if for all animated rings $A \in \text{CAlg}^{\text{an}}$ and all families of animated rings

$$\{A \rightarrow A_i\}$$

for which the maps $\text{Spec}(A_i) \rightarrow \text{Spec}(A)$ are jointly surjective open immersions, we have

$$X(A) \xrightarrow{\sim} \lim X(S),$$

where the limit is indexed by all $S \in \text{CAlg}_A^{\text{an}}$ such that $A \rightarrow S$ factors through some A_i .

Theorem 7.3.5.2 in [HTT] implies the following ‘finite’ characterisation of Zariski descent:

Proposition 5.19. *A prestack $X \in \text{PrStk}$ satisfies Zariski descent if and only if the following conditions hold:*

- (1) $X(0) \simeq *$ is contractible;
- (2) Let us say that a square of animated rings

$$\begin{array}{ccc} A & \longrightarrow & A_1 \\ \downarrow & & \downarrow \\ A_2 & \longrightarrow & A_{12} \end{array}$$

is a Zariski square if it is cocartesian ($A_{12} \simeq A_1 \otimes_A A_2$) and the maps $\text{Spec}(A_i) \rightarrow \text{Spec}(A)$ are jointly surjective open immersions. Then X sends each Zariski square to a pullback square.

Remark 5.20. For any Zariski square, the induced square of n -truncations is a Zariski square as well, since $A \rightarrow A_1$ and $A \rightarrow A_2$ are flat. Consequently, each Zariski square is the limit of the Zariski squares of its n -truncations. A convergent prestack therefore satisfies Zariski descent if and only if it sends every Zariski square of eventually coconnective animated rings to a pullback.

Proposition 5.21. *Let X be a prestack with deformation theory that satisfies Zariski descent, and let $X \rightarrow Y$ be a formal moduli stack under X . Then the prestack Y satisfies Zariski descent as well.*

Proof. By Remark 5.20, it suffices to verify that $Y(A) \simeq Y(A_1) \times_{Y(A_{12})} Y(A_2)$ for any Zariski square of eventually coconnective animated rings. Let us fix such a Zariski square, as well as a filtered system of nilpotent ideals $I \subseteq \pi_0(A)$ such that $\text{colim}_I \pi_0(A)/I \cong A_{\text{red}}$. For each such ideal, let us furthermore write $I_i = \pi_0(A_i) \otimes_{\pi_0(A)} I$. For $i = \emptyset, 1, 2, 12$, we then have that $\text{colim}_I \pi_0(A_i)/I_i \cong A_i \otimes_A A_{\text{red}} \cong (A_i)_{\text{red}}$. Since $X \rightarrow Y$ is left, we have a cartesian square

$$\begin{array}{ccc} \text{colim}_I X(\pi_0(A_i)/I_i) & \longrightarrow & \text{colim}_I Y(\pi_0(A_i)/I_i) \\ \downarrow & & \downarrow \\ X(A_{i,\text{red}}) & \longrightarrow & Y(A_{i,\text{red}}) \end{array}$$

for each $i = \emptyset, 1, 2, 12$. The bottom map is an equivalence because $X \rightarrow Y$ is a nil-isomorphism, so that the top map is an equivalence as well. We therefore obtain a diagram of the form

$$\begin{array}{ccc} Y(A) & \longrightarrow & Y(A_1) \times_{Y(A_{12})} Y(A_2) \\ \downarrow & & \downarrow \\ \text{colim}_I Y(\pi_0 A/I) & \xrightarrow{\sim} & \text{colim}_I (Y(\pi_0 A_1/I_1) \times_{Y(\pi_0 A_{12}/I_{12})} Y(\pi_0 A_2/I_2)) \\ \sim \uparrow & & \uparrow \sim \\ \text{colim}_I X(\pi_0 A/I) & \xrightarrow{\sim} & \text{colim}_I (X(\pi_0 A_1/I_1) \times_{X(\pi_0 A_{12}/I_{12})} X(\pi_0 A_2/I_2)) \end{array}$$

in which the bottom map is an equivalence because X satisfies Zariski descent. Given a point $x \in \text{colim}_I X(\pi_0(A)/I)$, we therefore need to check that the induced map on fibres $Y(A)_x \rightarrow Y(A_1)_{x_1} \times_{Y(A_{12})_{x_{12}}} Y(A_2)_{x_2}$ is an equivalence, where x_i denotes the restriction of x to the corresponding Zariski open. For this it suffices to show that for any nilpotent ideal I and any point

$x \in X(\pi_0(A)/I)$, the square

$$(25) \quad \begin{array}{ccc} Y(A) \times_{Y(\pi_0 A/I)} \{x\} & \longrightarrow & Y(A_1) \times_{Y(\pi_0 A_1/I_1)} \{x_1\} \\ \downarrow & & \downarrow \\ Y(A_2) \times_{Y(\pi_0 A_2/I_2)} \{x_2\} & \longrightarrow & Y(A_{12}) \times_{Y(\pi_0 A_{12}/I_{12})} \{x_{12}\} \end{array}$$

is cartesian. Indeed, one then finds that $Y(A)_x \simeq Y(A_1)_{x_1} \times_{Y(A_{12})_{x_{12}}} Y(A_2)_{x_2}$ by taking the filtered colimit over all I .

We now proceed as in the proof of Proposition 5.17: the map $A \rightarrow \pi_0(A)/I$ decomposes as a finite sequence $A = A(n) \rightarrow A(n-1) \rightarrow \cdots \rightarrow A(0) = \pi_0 A/I$ where each $A(k) \rightarrow A(k-1)$ is a square zero extension by an $A(0)$ -module $J(k)$. Localising, we similarly have that $A_i \rightarrow \pi_0(A_i)/I_i$ decomposes as a finite sequence $A_i = A(n)_i \rightarrow \cdots \rightarrow A(0)_i = \pi_0(A_i)/I_i$ of square zero extensions by the $A(0)_i$ -modules $J(k)_i = J(k) \otimes_{A(0)} A(0)_i$. We will prove by induction on this tower that the square (25) is cartesian; when $k = 0$, all spaces are contractible and the assertion is evident.

For the inductive step, note that for each $i = \emptyset, 1, 2, 12$, there is a natural fibre sequence of spaces

$$Y(A(k)_i) \times_{Y(A(0)_i)} \{x_i\} \longrightarrow Y(A(k-1)_i) \times_{Y(A(0)_i)} \{x_i\} \longrightarrow Y(A(0)_i \oplus J(k)_i[1]) \times_{Y(A(0)_i)} \{x_i\}$$

because Y has deformation theory. By inductive hypothesis, the middle terms for $i = \emptyset, 1, 2, 12$ fit into a cartesian square. It therefore suffices to show that the right terms also form a cartesian square. Since Y has deformation theory, there is a natural equivalence

$$Y(A(0) \oplus J(k)_i[1]) \times_{Y(A(0))} \{x\} \simeq Y(A(0)_i \oplus J(k)_i[1]) \times_{Y(A(0)_i)} \{x_i\}.$$

and it suffices to verify that the square

$$\begin{array}{ccc} Y(A(0) \oplus J(k)_1[1]) \times_{Y(A(0))} \{x\} & \longrightarrow & Y(A(0) \oplus J(k)_1[1]) \times_{Y(A(0))} \{x\} \\ \downarrow & & \downarrow \\ Y(A(0) \oplus J(k)_2[1]) \times_{Y(A(0))} \{x\} & \longrightarrow & Y(A(0) \oplus J(k)_2[1]) \times_{Y(A(0))} \{x\} \end{array}$$

is cartesian. The functor $Y(A(0) \oplus -) \times_{Y(A(0))} \{x\}: \text{Mod}_{A, \geq 0} \rightarrow \mathcal{S}$ preserves all pullback diagrams preserved by the inclusion into Mod_A , again because Y has deformation theory. We are therefore left with showing that $J(k)_1[1] \simeq J(k)_1[1] \times_{J(k)_{12}[1]} J(k)_2[1]$. Unravelling the definitions, this is nothing but Zariski descent for (the quasi-coherent sheaf associated to) the $A(0)$ -module $J(k)$. \square

Theorem 5.22 (Zariski descent for formal moduli stacks). *Given a derived scheme X , write \mathfrak{U} for the poset of affine opens of X . Then there is canonical equivalence*

$$\text{ModuliStk}_{X/R} \xrightarrow{\sim} \lim_{U \in \mathfrak{U}^{\text{op}}} \text{ModuliStk}_{U/R}.$$

Proof. Let us consider the cartesian fibration

$$\text{ModuliStk}_{\mathfrak{U}, R} \rightarrow \mathfrak{U}$$

obtained by restricting the cartesian fibration

$$\text{ModuliStk}_R \rightarrow \text{PrStk}_R^{\text{def, left-map}}$$

from Construction 5.16 along the evident functor $\mathfrak{U} \rightarrow \text{PrStk}_R^{\text{def, left-map}}$. By [HTT, Proposition 3.3.3.1], there is an equivalence

$$\lim_{U \in \mathfrak{U}} \text{ModuliStk}_{U/R} \simeq \text{Fun}_{\mathfrak{U}}^{\text{cart}}(\mathfrak{U}, \text{ModuliStk}_{\mathfrak{U}, R}),$$

where the right hand side denotes the ∞ -category of cartesian sections of $\text{ModuliStk}_{\mathfrak{U},R} \rightarrow \mathfrak{U}$.

We have a canonical functor

$$\Psi: \text{ModuliStk}_{X/R} \longrightarrow \text{Fun}_{\mathfrak{U}}^{\text{cart}}(\mathfrak{U}, \text{ModuliStk}_{\mathfrak{U},R})$$

sending $(X \rightarrow Y)$ to

$$\{U \longrightarrow Y_U^\wedge\}_{U \in \mathfrak{U}}.$$

Here the formal completion $Y_U^\wedge \simeq Y \times_{Y^\#} U^\#$ is the prestack defined in Definition B.8, where we have used the notation $F^\#(-) := F(-_{\text{red}})$. Note that each map $U \rightarrow Y$ is laft by [DAG, Proposition 5.3.10], so Y_U^\wedge agrees with $(Y/U)_{\text{inf}}$.

The above functor is well-defined as each $U \longrightarrow Y_U^\wedge$ is a formal moduli stack under U . Indeed, the fact that $Y_U^\wedge \simeq (Y/U)_{\text{inf}}$ has deformation theory follows immediately from the definitions. Since $U \rightarrow Y$ is laft, $U \rightarrow Y_U^\wedge$ is laft by Lemma B.14 and clearly an equivalence on reduced objects.

To construct an inverse to Ψ , we will need some notation. Write $\text{PrStk}_{R,<\infty}$ for the ∞ -category of prestacks defined only on coconnective objects. Let $\text{Stk}_R^{\text{Zar}} \subset \text{PrStk}_R$ ($\text{Stk}_{R,<\infty}^{\text{Zar}} \subset \text{PrStk}_{R,<\infty}$) denote the full subcategories spanned by all prestacks Y satisfying Zariski descent, i.e. satisfying $Y(0) \simeq *$ and sending Zariski squares (of eventually coconnective rings) to pullbacks. By Remark 5.20, the functor

$$\text{conv}: \text{PrStk}_{R,<\infty} \longrightarrow \text{PrStk}_R; \quad X \longmapsto \lim_n X(\tau_{\leq n} -)$$

sends sheaves to sheaves.

We then define

$$\Phi: \text{Fun}_{\mathfrak{U}}^{\text{cart}}(\mathfrak{U}, \text{ModuliStk}_{\mathfrak{U},R}) \longrightarrow \text{ModuliStk}_{X/R}$$

as the composite

$$\text{Fun}_{\mathfrak{U}}^{\text{cart}}(\mathfrak{U}, \text{ModuliStk}_{\mathfrak{U},R}) \subset \text{Fun}(\mathfrak{U}, \text{Fun}(\Delta^1, \text{Stk}_{R,<\infty}^{\text{Zar}})) \xrightarrow{\text{colim}} \text{Fun}(\Delta^1, \text{Stk}_{R,<\infty}^{\text{Zar}}) \xrightarrow{\text{conv}} \text{Fun}(\Delta^1, \text{Stk}_R^{\text{Zar}}).$$

Note that colim is computed by sheafifying the pointwise colimit. We have used Proposition 5.21 to embed formal moduli stacks into Zariski sheaves.

We now want to verify that Φ takes values in formal moduli stacks under X . To this end, let us fix a family

$$\{U \rightarrow Y_U\}_{U \in \mathfrak{U}} \in \text{Fun}_{\mathfrak{U}}^{\text{cart}}(\mathfrak{U}, \text{ModuliStk}_{\mathfrak{U},R})$$

indexed by the poset \mathfrak{U} of affine opens U of X and let $(X \rightarrow Y) \in \text{Fun}(\Delta^1, \text{Stk}_R^{\text{Zar}})$ denote its image under Φ . In a first step, we will prove that the natural map

$$\{U \rightarrow Y_U\}_{U \in \mathfrak{U}} \rightarrow \{U \rightarrow Y_U^\wedge\}_{U \in \mathfrak{U}}$$

is an equivalence, i.e. that for each $U \in \mathfrak{U}$, the following square in $\text{Stk}_{R,<\infty}^{\text{Zar}}$ is cartesian:

$$\begin{array}{ccc} Y_U & \longrightarrow & U^\# \\ \downarrow & & \downarrow \\ \text{colim}_{U \in \mathfrak{U}} Y_U & \longrightarrow & (\text{colim}_{U \in \mathfrak{U}} Y_U)^\# \end{array}$$

Note that the functor $(-)^{\#}: \text{PrStk}_{R,<\infty} \rightarrow \text{PrStk}_{R,<\infty}$ preserves colimits and Zariski sheaves, and commutes with Zariski sheafification. For the last assertion, it suffices to observe that for any covering sieve $S \rightarrow \text{Spec}(A)$ for the Zariski topology and a map $f: \text{Spec}(B) \rightarrow \text{Spec}(A)^{\#}$, corresponding to $f_{\text{red}}: \text{Spec}(B_{\text{red}}) \rightarrow \text{Spec}(A_{\text{red}})$, the base change $f^* S^{\#} \rightarrow \text{Spec}(B)$ is a covering sieve as well: it is

the unique covering sieve whose restriction to $\mathrm{Spec}(B_{\mathrm{red}})$ coincides with $f_{\mathrm{red}}^* S$. Consequently, the natural map of Zariski sheaves $\mathrm{colim}_{U \in \mathfrak{U}} Y_U^\# \rightarrow (\mathrm{colim}_{U \in \mathfrak{U}} Y_U)^\#$ is an equivalence.

To see that the above square is cartesian, it then suffices to consider the natural transformation α of \mathfrak{U} -indexed diagrams in $\mathrm{Stk}_{R, < \infty}^{\mathrm{Zar}}$ given by

$$Y_U \rightarrow (Y_U)^\#$$

Write $\bar{\alpha}$ for the induced transformation between the $\mathfrak{U}^\triangleright$ -indexed colimit cones. The transformation α is cartesian as for each $U \rightarrow V$, we have $Y_U \simeq (Y_V)^\wedge_U \simeq Y_V \times_{V^\#} U^\#$, which implies by [HTT, Theorem 6.1.0.6] that the above square is also cartesian.

We will use this to show that $X \rightarrow Y$ is a formal moduli problem. The prestack Y is convergent by construction and the map $X \rightarrow Y$ is left by (the derived analogue of) [SAG, 17.4.3.3]. To verify it has deformation theory, it suffices to check that for any pullback square of eventually coconnective animated rings

$$\begin{array}{ccc} B' & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & A \end{array}$$

with $B \rightarrow A$ a nilpotent extension and any $f \in Y(B)$ with image $f_A \in Y(A)$, the map

$$Y(\mathcal{O}_{B'}) \times_{Y(\mathcal{O}_B)} \{f\} \rightarrow Y(\mathcal{O}_B \otimes_{B'} A') \times_{Y(\mathcal{O}_B \otimes_B A)} \{f_A\}$$

of sheaves on the topological space $|\mathrm{Spec}(B)| \cong |\mathrm{Spec}(B')|$ is an equivalence on global sections. Taking global sections, this implies that the fibres $Y(B') \times_{Y(B)} \{f\} \rightarrow Y(A') \times_{Y(A)} \{f_A\}$ are equivalent, so that $Y(B') \simeq Y(B) \times_{Y(A)} Y(A')$.

Since Y is a Zariski sheaf, it suffices to check this after restricting to a Zariski cover of $|\mathrm{Spec}(B)|$. Let us pick a cover of $\mathrm{Spec}(B)$ by affine opens $V_i \subset \mathrm{Spec}(B)$ corresponding to affine opens $V_{i, \mathrm{red}} \subset \mathrm{Spec}(B_{\mathrm{red}})$ such that each composite

$$f_{i, \mathrm{red}}: V_{i, \mathrm{red}} \subset \mathrm{Spec}(B_{\mathrm{red}}) \rightarrow Y_{\mathrm{red}} \simeq X_{\mathrm{red}}$$

factors through some affine open $U_i \subset X$. This implies that each restriction $f_i: V_i \rightarrow Y$ of the map f factors through some $Y_{U_i}^\wedge$. The claim follows as $Y_{U_i}^\wedge \simeq Y_{U_i}$ by the above argument, and Y_{U_i} has deformation theory.

Finally, to see that $X \rightarrow Y$ is an equivalence on reduced affine schemes, we note that precomposing with $A \mapsto A_{\mathrm{red}}$ gives rise to a limit-preserving functor $\mathrm{Fun}(\mathrm{CR}^{\mathrm{red}}, \mathcal{S}) \rightarrow \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{an}}, \mathcal{S})$, $F \mapsto F(-_{\mathrm{red}})$ which sends sheaves to sheaves. Here $\mathrm{CR}^{\mathrm{red}} \subset \mathrm{CAlg}^{\mathrm{an}}$ denotes the full subcategory of reduced discrete rings. Passing to left adjoints, we see that restriction to $\mathrm{CR}^{\mathrm{red}}$ commutes with sheafification. As each object $U \rightarrow Y_U$ induces an equivalence on reduced affine schemes, we deduce that $X = \mathrm{colim}_{U \in \mathfrak{U}} U \rightarrow \mathrm{colim}_{U \in \mathfrak{U}} Y_U = Y$ does so as well.

We may therefore conclude that the functor Φ lands in formal moduli stacks, and the above argument also shows that $\Psi \circ \Phi$ is equivalent to the identity.

To see that $\Phi \circ \Psi$ is equivalent to the identity, let us fix a formal moduli stack $X \rightarrow Y$ under X . It suffices to verify that the canonical map

$$\mathrm{colim}_{U \in \mathfrak{U}} Y_U^\wedge \rightarrow Y$$

is an equivalence of sheaves on eventually coconnective objects. Since colimits in the ∞ -topos $\mathrm{Stk}_{R, < \infty}^{\mathrm{Zar}}$ are universal in the sense of [HTT, Definition 6.1.1.2], it suffices to check that for all maps

$\mathrm{Spec}(A) \rightarrow Y$, the induced map

$$\mathrm{colim}_{U \in \mathfrak{U}} (Y_U^\wedge \times_Y \mathrm{Spec}(A)) \rightarrow \mathrm{Spec}(A)$$

is an equivalence. But

$$Y_U^\wedge \times_Y \mathrm{Spec}(A) \simeq \mathrm{Spec}(A) \times_{Y^\#} U^\# \simeq \mathrm{Spec}(A) \times_{X^\#} U^\#$$

and hence the above map is an equivalence since $\mathrm{colim}_{U \in \mathfrak{U}} U^\# \rightarrow X^\#$ is an equivalence. \square

Proposition 5.23 (Monadicity of the tangent fibre). *Let X be a locally coherent qcqs derived scheme over an animated ring R . Then the functor*

$$\begin{aligned} T_{X/-}[1]: \mathrm{ModuliStk}_{X/R} &\longrightarrow (\mathrm{QC}_X^\vee)_{/T_{X/R}[1]} \\ (X \rightarrow Y) &\longmapsto (T_{X/Y}[1] \rightarrow T_{X/R}[1]) \end{aligned}$$

in Construction 5.16 is a monadic right adjoint.

The proof uses the following ∞ -categorical result:

Lemma 5.24. *Let $F: \mathcal{C} \rightleftarrows \mathcal{D}: U$ be a monadic adjunction and let $G: \mathcal{D}' \rightarrow \mathcal{D}$ be a functor from an ∞ -category \mathcal{D}' with geometric realisations. If UG admits a left adjoint, then G admits a left adjoint.*

Proof. As in [HA, Lemma 4.7.3.13], let $\mathcal{X} \subseteq \mathcal{D}$ be the full subcategory of objects $D \in \mathcal{D}$ such that $\mathrm{Map}_{\mathcal{D}}(D, G(-))$ is corepresentable and let $L: \mathcal{X} \rightarrow \mathcal{D}'$ be the corresponding partial left adjoint. For any simplicial diagram D_\bullet in \mathcal{X} , the colimit $|L(D_\bullet)|$ corepresents $\mathrm{Map}_{\mathcal{D}}(|D_\bullet|, G(-))$, so that \mathcal{X} is closed under geometric realisations. Since UG admits a left adjoint, \mathcal{X} contains all the free objects $F(C)$ with $C \in \mathcal{C}$. As every object in \mathcal{D} is the geometric realisation of a (U -split) simplicial diagram of free objects, $\mathcal{X} = \mathcal{D}$. \square

Proof of Proposition 5.23. When $X = \mathrm{Spec}(B)$ is affine, the tangent functor $T_{X/-}[1]$ from Construction 5.16 can be identified with the composite

$$T_{X/-}[1]: \mathrm{ModuliStk}_{X/R} \xrightarrow{\sim} \mathrm{Moduli}_{B/R} \xrightarrow{T_{B/[1]}} (\mathrm{QC}_B^\vee)_{/T_{B/R}[1]}$$

where the first functor is the equivalence from Theorem 5.3, restricting formal moduli stacks to Artinian extensions of B , and the second functor takes the tangent complex of the corresponding formal moduli problem. It then follows from Proposition 3.12 and Theorem 4.19 that $T_{X/-}[1]$ is a monadic right adjoint preserving sifted colimits.

For non-affine X , let us first prove that $\mathrm{ModuliStk}_{X/R}$ has sifted colimits and that $T_{X/-}[1]$ preserves these. To this end, let us again write \mathfrak{U} for the category of Zariski open affine subsets of X . By Proposition B.32, have a commutative square

$$\begin{array}{ccc} \mathrm{ModuliStk}_{X/R} & \longrightarrow & (\mathrm{QC}_X^\vee)_{/T_{X/R}[1]} \\ \simeq \downarrow & & \downarrow \simeq \\ \lim_{U \in \mathfrak{U}} \mathrm{ModuliStk}_{U/R} & \longrightarrow & \lim_{U \in \mathfrak{U}} (\mathrm{QC}_U^\vee)_{/T_{U/R}[1]} \end{array}$$

where the vertical functors are equivalences by Theorem 5.22 and Corollary A.39, respectively. Each functor $\mathrm{ModuliStk}_{U/R} \rightarrow (\mathrm{QC}_U^\vee)_{/T_{U/R}[1]}$ preserves sifted colimits, and this also holds true for the transition maps associated to inclusions $j: U \hookrightarrow V$ of Zariski open subsets of X

$$\mathrm{ModuliStk}_{V/R} \xrightarrow{(-)_{\hat{U}}} \mathrm{ModuliStk}_{U/R} \quad (\mathrm{QC}_V^\vee)_{/T_{V/R}[1]} \xrightarrow{j^*} (\mathrm{QC}_U^\vee)_{/T_{U/R}[1]}.$$

For the functor $(-)\hat{\smile}_U$, we use that the functors $T_{U/[1]}: \text{ModuliStk}_{U/R} \rightarrow (\text{QC}_U^\vee)_{/T_{U/R}[1]}$ are conservative and hence *detect* sifted colimits. It then follows from [HTT, Corollary 5.3.6.10] that $T_{X/-}[1]$ is also a sifted colimit preserving functor between ∞ -categories with sifted colimits.

Next, we will show that $T_{X/-}[1]$ admits a left adjoint. By Lemma 5.24, it suffices to verify that

$$\text{ModuliStk}_{X/R} \xrightarrow{T_{X/-}[1]} (\text{QC}_X^\vee)_{/T_{X/R}[1]} \xrightarrow{\text{fib}} \text{QC}_X^\vee$$

admits a left adjoint, since taking the fibre is a monadic right adjoint. This follows immediately from Proposition B.33, using that the left adjoint $F \mapsto X_F$ constructed there takes values in the full subcategory of formal moduli stacks since $X \rightarrow X_F$ is a nil-isomorphism.

The claim now follows from the Barr–Beck–Lurie theorem [HA, Theorem 4.7.3.5] as the tangent fibre is conservative by Proposition 5.17. \square

Proposition 5.25 (Identification of the monad). *The right adjoint in Proposition 5.23 induces the partition Lie algebroid monad from Definition 3.2 the ∞ -category $(\text{QC}_X^\vee)_{/T_{X/R}[1]}$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccc} \text{ModuliStk}_{X/R} & \xrightarrow{\sim} & \lim_{U \in \mathfrak{U}} \text{ModuliStk}_{U/R} & \xrightarrow{\sim} & \lim_{U \in \mathfrak{U}} \text{Moduli}_{\mathcal{O}(U)/R} & \xrightarrow{\sim} & \lim_{U \in \mathfrak{U}} \text{LieAlg}_{\Delta, \mathcal{O}(U)/R}^\pi \\ T_{X/-}[1] \downarrow & & & \searrow^{T_{U/-}[1]} & \downarrow^{T_{\mathcal{O}(U)/[1]}} & \swarrow^{\text{oblv}} & \\ (\text{QC}_X^\vee)_{/T_{X/R}[1]} & \xrightarrow{\sim} & \lim_{U \in \mathfrak{U}} (\text{QC}_U^\vee)_{/T_{U/R}[1]} & & & & \end{array}$$

where the left square is defined as in Proposition 5.23 and the two triangles arise from Theorem 5.3 and Theorem 4.19. The claim then follows from the construction of the monad $\text{Lie}_{\Delta, X/R}^\pi$ in the proof of Theorem 3.1. \square

We deduce:

Theorem 5.26 (Main theorem). *Let X be a locally coherent qcqs derived R -scheme over an animated ring R .*

(1) *The shifted tangent fibre functor*

$$\begin{array}{ccc} \text{ModuliStk}_{X/R} & \longrightarrow & (\text{QC}_X^\vee)_{/T_{X/R}[1]} \\ (X \rightarrow Y) & \longmapsto & (T_{X/Y}[1] \rightarrow T_{X/R}[1]) \end{array}$$

lifts to an equivalence of ∞ -categories

$$T_{X/-}[1]: \text{ModuliStk}_{X/R} \xrightarrow{\simeq} \text{LieAlg}_{\Delta, X/R}^\pi.$$

Hence every partition Lie algebroid on X integrates uniquely to a formal moduli stack under X .

(2) *This equivalence is functorial with respect to almost finitely presented maps. More precisely, if $X \rightarrow Y$ is a locally almost finitely presented map, there is a commuting diagram*

$$\begin{array}{ccc} \text{ModuliStk}_{Y/} & \xrightarrow{T_{Y/-}[1]} & \text{Lie}_{\Delta, Y/R}^\pi \\ (-)\hat{\smile}_X \downarrow & & \downarrow f^\sharp \\ \text{ModuliStk}_{X/-} & \xrightarrow{T_{X/-}[1]} & \text{Lie}_{\Delta, X/R}^\pi \end{array}$$

where $f^\sharp(\mathfrak{g} \rightarrow T_{Y/R}[1]) = f^ \mathfrak{g} \times_{(f^* L_{Y/R})^\vee[1]} T_{X/R}[1]$ as pro-coherent sheaves.*

APPENDIX A. DERIVED ALGEBRA RECOLLECTIONS

The goal of this section is to introduce some elements of (homological) algebra that will be used throughout the text. We will start in Section A.1 by reviewing the theory of derived commutative rings, the non-connective analogues of animated (or simplicial) rings. We will work in a somewhat general setting that also includes the derived filtered and graded rings studied in Section 2. Section A.2 discusses the theory of *pro-coherent modules* over an animated ring.

A.1. Derived commutative rings. To set the stage, we will briefly recall the method of right-left extending monads from [BM, Section 3.2] and [BCN, Section 2.2] in the generality needed for us, see also [Rak20, Section 4.2] for the case of the commutative algebra monad.

Definition A.1. Let \mathcal{C} be a presentable stable ∞ -category with a left complete t -structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. We call \mathcal{C} is *stably projectively generated* if there exists a full additive subcategory $\text{Vect}^{\text{ft}}(\mathcal{C}) \subseteq \mathcal{C}_{\geq 0}$ whose objects are compact in \mathcal{C} and projective in $\mathcal{C}_{\geq 0}$, generating $\mathcal{C}_{\geq 0}$ under sifted colimits.

In this situation, there is an equivalence

$$\mathcal{C} \simeq \text{Fun}^{\oplus}(\text{Vect}^{\text{ft}}(\mathcal{C})^{\text{op}}, \text{Sp})$$

identifying t -structures [Rak20, Remark 4.2.2]. We define the subcategory of *flat objects* in \mathcal{C} as the ind-completion of $\text{Vect}^{\text{ft}}(\mathcal{C})$:

$$\text{Flat}(\mathcal{C}) := \text{Ind}(\text{Vect}^{\text{ft}}(\mathcal{C})) \subseteq \mathcal{C}.$$

Example A.2. For $A \in \text{Alg}(\text{Sp})$ a connective algebra object in spectra, the stable ∞ -category Mod_A of (left) A -module spectra with its usual t -structure is stably projectively generated. In this case, we will take $\text{Vect}_A^{\text{ft}}$ to be the ∞ -category consisting of finite direct sums of copies of A . Then Flat_A is the ∞ -category of flat A -modules in the sense of [HA, Definition 7.2.2.10], by Lurie's version of Lazard's theorem (cf. Theorem 7.2.2.15 of [op.cit]).

Definition A.3. Let \mathcal{C} be a projectively generated stable ∞ -category with t -structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ and \mathcal{D} a presentable ∞ -category. We will say that a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is *right-left extended* if it preserves sifted colimits and totalisations of finite cosimplicial diagrams with values in $\text{Vect}^{\text{ft}}(\mathcal{C}) \subseteq \mathcal{C}$.

The following follows from [BCN, Remark 2.45] and the definition of $\text{Flat}(\mathcal{C})$ as an ind-completion:

Proposition A.4. *The restriction functor*

$$\text{Fun}_{\text{RL}}(\mathcal{C}, \mathcal{D}) \longrightarrow \text{Fun}^{\omega}(\text{Flat}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \text{Fun}(\text{Vect}^{\text{ft}}(\mathcal{C}), \mathcal{D})$$

is fully faithful. Here $\text{Fun}_{\text{RL}}(\mathcal{C}, \mathcal{D}) \subseteq \text{Fun}(\mathcal{C}, \mathcal{D})$ denotes the full subcategory of right-left extended functors and $\text{Fun}^{\omega}(\text{Flat}(\mathcal{C}), \mathcal{D}) \subset \text{Fun}(\text{Flat}(\mathcal{C}), \mathcal{D})$ is the full subcategory of functors preserving filtered colimits.

Definition A.5. Let us say that a functor $F: \text{Vect}^{\text{ft}}(\mathcal{C}) \rightarrow \mathcal{D}$, or equivalently, a filtered-colimit preserving functor $F: \text{Flat}(\mathcal{C}) \rightarrow \mathcal{D}$, *admits a right-left extension* or is *right-left extendable*, if it is contained in the essential image. In that case, we will refer to its (unique) inverse image $\mathbb{L}F: \mathcal{C} \rightarrow \mathcal{D}$ as the *derived functor*, or *right-left extension*, of F . Note that this coincides with the non abelian derived functor in the sense of Dold–Puppe on the connective part $\mathcal{C}_{\geq 0} \simeq \mathcal{P}_{\Sigma}(\text{Vect}^{\text{ft}}(\mathcal{C}))$.

Example A.6. When \mathcal{D} is stable, every functor $F: \text{Vect}^{\text{ft}}(\mathcal{C}) \rightarrow \mathcal{D}$ which is a filtered colimit of finite degree functors is right-left extendable. See [BCN, Corollary 2.49] for details.

Example A.7. Let $\text{End}_{\text{RL}}^{\text{Flat}(\mathcal{C})}(\mathcal{C}) \subset \text{End}_{\text{RL}}(\mathcal{C})$ be the full subcategory of endofunctors $\mathcal{C} \rightarrow \mathcal{C}$ which preserve the full subcategory $\text{Flat}(\mathcal{C}) \subset \mathcal{C}$. Restriction $\text{End}_{\text{RL}}^{\text{Flat}(\mathcal{C})}(\mathcal{C}) \rightarrow \text{End}^{\omega}(\text{Flat}(\mathcal{C}))$ is a monoidal equivalence onto the full subcategory of $\text{End}^{\omega}(\text{Flat}(\mathcal{C}))$ spanned by right-left extendable functors. Consequently, any right-left extendable and filtered-colimit preserving monad $T: \text{Flat}(\mathcal{C}) \rightarrow \text{Flat}(\mathcal{C})$ induces a *derived monad* $\mathbb{L}T: \mathcal{C} \rightarrow \mathcal{C}$.

As an example of this kind of derived monad, consider the derived symmetric algebra monad. To define this, assume that \mathcal{C} is a derived algebraic context in the sense of [Rak20, Section 4.2]:

Definition A.8. We will say that a stably projectively generated ∞ -category \mathcal{C} is a *derived algebraic context* if it comes equipped with a closed symmetric monoidal structure such that:

- (1) $\text{Vect}^{\text{ft}}(\mathcal{C})$ is stable under tensor products and contains the monoidal unit.
- (2) $\text{Vect}^{\text{ft}}(\mathcal{C})$ is an ordinary category.
- (3) For each $M \in \text{Vect}^{\text{ft}}(\mathcal{C})$, the object $\text{Sym}^n(M) = \tau_{\leq 0}((M_{h\Sigma_n}^{\otimes n}))$ is contained in $\text{Vect}^{\text{ft}}(\mathcal{C})$.

Condition (1) implies that $\mathcal{C}_{\geq 0}$ is stable under the tensor product. We include condition (2) for simplicity; it implies that $\text{Flat}(\mathcal{C})$ is an ordinary category and that \mathcal{C}^{\heartsuit} is a symmetric monoidal abelian category with enough projectives. Extending the functors $\text{Sym}^n: \text{Vect}^{\text{ft}}(\mathcal{C}) \rightarrow \text{Vect}^{\text{ft}}(\mathcal{C})$ by filtered colimits and taking the sum over all $n \geq 0$, one then obtains a monad $\text{Sym}: \text{Flat}(\mathcal{C}) \rightarrow \text{Flat}(\mathcal{C})$ preserving filtered colimits. Since each $\text{Sym}^n: \text{Vect}^{\text{ft}}(\mathcal{C}) \rightarrow \text{Vect}^{\text{ft}}(\mathcal{C})$ is of degree $\leq n$, this admits a derived monad $\mathbb{L}\text{Sym}: \mathcal{C} \rightarrow \mathcal{C}$.

Definition A.9. If \mathcal{C} is a derived algebraic context, we will refer to $\mathbb{L}\text{Sym}$ -algebras as *derived commutative algebras* and to connective $\mathbb{L}\text{Sym}$ -algebras as *animated commutative algebras* in \mathcal{C} , and write $\text{CAlg}^{\text{an}}(\mathcal{C}_{\geq 0}) \subseteq \text{CAlg}^{\text{der}}(\mathcal{C})$ for the corresponding ∞ -categories.

The (ordinary) category $\text{CAlg}^{\text{der}}(\mathcal{C})^{\heartsuit}$ of discrete derived algebras is equivalent to the category of commutative algebras in \mathcal{C}^{\heartsuit} .

Example A.10. Let $\mathcal{C} = \text{Mod}_{\mathbb{Z}}$ with the usual t -structure, so that $\text{Vect}^{\text{ft}}(\mathcal{C}) = \text{Vect}_{\mathbb{Z}}^{\text{ft}}$ is the (ordinary) category of finite free abelian groups and $\text{Flat}(\mathcal{C}) = \text{Flat}_{\mathbb{Z}}$ is the (ordinary) category of flat abelian groups. This is a derived algebraic context, and the resulting monad $\text{Sym}: \text{Flat}_{\mathbb{Z}} \rightarrow \text{Flat}_{\mathbb{Z}}$ is the usual symmetric algebra monad. One thus obtains the ∞ -categories $\text{CAlg}^{\text{an}} = \text{CAlg}^{\text{an}}(\text{Mod}_{\mathbb{Z}})$ and $\text{CAlg}^{\text{der}} = \text{CAlg}^{\text{der}}(\text{Mod}_{\mathbb{Z}})$ of animated and derived rings.

Example A.11. Let \mathcal{C} be a derived algebraic context and let \mathcal{J} be a small symmetric monoidal (ordinary) category. Then $\text{Fun}(\mathcal{J}, \mathcal{C})$ is a derived algebraic context as well with respect to the Day convolution product and the t -structure in which a diagram is (co)connective if it is pointwise (co)connective in \mathcal{C} .

Remark A.12. There is a natural equivalence $\mathbb{L}\text{Sym}(V) \otimes \mathbb{L}\text{Sym}(W) \simeq \mathbb{L}\text{Sym}(V \oplus W)$ for all $V, W \in \mathcal{C}$: this follows by right-left extension, using that there is such a natural equivalence for $V, W \in \text{Flat}(\mathcal{C})$ by construction. Consequently, the coproduct of two derived algebras is simply given by their tensor product. Since $\text{CAlg}^{\text{der}}(\mathcal{C}) \rightarrow \mathcal{C}$ preserves sifted colimits by construction, it follows that pushouts of derived algebras are computed as relative tensor products, as usual.

Remark A.13. For every $M \in \text{Vect}^{\text{ft}}(\mathcal{C})$, there is a natural map $(M^{\otimes n})_{h\Sigma_n} \rightarrow \text{Sym}^n(M)$ from the coinvariants in \mathcal{C} . By right-left extension, this induces a map $\text{Free}_{\mathbb{E}_{\infty}} \rightarrow \mathbb{L}\text{Sym}$ from the \mathbb{E}_{∞} -monad. In particular, any derived commutative algebra has an underlying \mathbb{E}_{∞} -algebra and a module over a derived commutative algebra is simply a module over its underlying \mathbb{E}_{∞} -algebra.

Definition A.14. For $A \in \mathrm{CAlg}^{\mathrm{der}}(\mathcal{C})$, we define the ∞ -category of *derived (commutative) A -algebras* to be the under-category

$$\mathrm{CAlg}_A^{\mathrm{der}}(\mathcal{C}) = \mathrm{CAlg}^{\mathrm{der}}(\mathcal{C})_{A/}.$$

The forgetful functor $\mathrm{CAlg}_A^{\mathrm{der}}(\mathcal{C}) \rightarrow \mathrm{Mod}_A(\mathcal{C})$ is monadic and we will write $\mathbb{L}\mathrm{Sym}_A: \mathrm{Mod}_A(\mathcal{C}) \rightarrow \mathrm{Mod}_A(\mathcal{C})$ for the associated monad.

Remark A.15. If $A \in \mathrm{CAlg}^{\mathrm{der}}(\mathcal{C})$ is connective, then $\mathrm{Mod}_A(\mathcal{C})$ is stably projectively generated for the t -structure inherited from \mathcal{C} . The monad $\mathbb{L}\mathrm{Sym}_A$ is then derived from the restricted monad $\mathbb{L}\mathrm{Sym}_A: \mathrm{Flat}(\mathrm{Mod}_A(\mathcal{C})) \rightarrow \mathrm{Flat}(\mathrm{Mod}_A(\mathcal{C}))$.

The usual cotangent complex formalism for animated rings carries over to this setting. Below we will recall one way to see this, using an abstract method for constructing functors out of the ∞ -category of T -algebras over some monad T (see also [Hol23]).

Proposition A.16. *Let \mathcal{C} be a stably projectively generated stable ∞ -category. Suppose that $T: \mathrm{Flat}(\mathcal{C}) \rightarrow \mathrm{Flat}(\mathcal{C})$ is a filtered colimit preserving and right-left extendable monad, and let $\mathbb{L}T: \mathcal{C} \rightarrow \mathcal{C}$ be the derived monad T . Let $\mathrm{Kl}_T(\mathrm{Flat}(\mathcal{C}))$ denote the full subcategory of $\mathrm{Alg}_T(\mathrm{Flat}(\mathcal{C}))$ spanned by the free algebras. Assume that*

$$F: \mathrm{Kl}_T(\mathrm{Flat}(\mathcal{C})) \longrightarrow \mathcal{D}$$

is a functor such that

- (1) the composite $F \circ \mathrm{Free}_T: \mathrm{Flat}(\mathcal{C}) \rightarrow \mathcal{D}$ preserves filtered colimits and is right-left extendable,
- (2) F preserves geometric realizations of simplicial objects in $\mathrm{Kl}_T(\mathrm{Flat}(\mathcal{C}))$ that become split after applying Forget_T .

Then there exists a unique sifted colimit preserving functor $\mathbb{L}F: \mathrm{Alg}_T(\mathcal{C}) \rightarrow \mathcal{D}$ that extends F , together with a natural equivalence $\mathbb{L}F \circ \mathrm{Free} \simeq \mathbb{L}(F \circ \mathrm{Free})$.

Proof. Let $\mathrm{Fun}_{\mathrm{RL}}^{\omega}(\mathrm{Flat}(\mathcal{C}), \mathcal{D})$ denote the ∞ -category of filtered-colimit preserving functors that are right-left extendable, and notice that there is a right action

$$\mathrm{Fun}_{\mathrm{RL}}^{\omega}(\mathrm{Flat}(\mathcal{C}), \mathcal{D}) \curvearrowright \mathrm{Fun}_{\mathrm{RL}}^{\omega}(\mathrm{Flat}(\mathcal{C}), \mathrm{Flat}(\mathcal{C})).$$

In the above situation, we have an algebra object $T \in \mathrm{Fun}_{\mathrm{RL}}^{\omega}(\mathrm{Flat}(\mathcal{C}), \mathrm{Flat}(\mathcal{C}))$, and the functor $F \circ \mathrm{Free}_T \in \mathrm{Fun}_{\mathrm{RL}}^{\omega}(\mathrm{Flat}(\mathcal{C}), \mathcal{D})$ is a right module for T . Using the same logic as [Hol23, Proposition 2.2.15], taking right-left extension is lax monoidal and consequently, the functor $\mathbb{L}(F \circ \mathrm{Free}_T): \mathcal{C} \rightarrow \mathcal{D}$ is a right module for the derived monad $\mathbb{L}T$. On the other hand, $\mathrm{Forget}_T: \mathrm{Alg}_T(\mathcal{C}) \rightarrow \mathcal{C}$ is a left T -module. Since \mathcal{D} admits sifted colimits, we can define $\mathbb{L}F$ by the formula:

$$\mathbb{L}F := |\mathrm{Bar}(\mathbb{L}(F \circ \mathrm{Free}_T), T, \mathrm{Forget}_T)|: \mathrm{Alg}_T(\mathcal{C}) \longrightarrow \mathcal{D}.$$

One readily verifies all the desired properties. \square

Construction A.17 (Cotangent complex as a derived functor). Let \mathcal{C} be a derived algebraic context and write $\mathrm{Mod}(\mathcal{C})$ for the ∞ -category of tuples (A, M) consisting of a derived commutative algebra A and an A -module M in \mathcal{C} . Notice that Mod arises as the ∞ -category of algebras over a derived monad T_{mod} on \mathcal{C} , such that $\mathrm{Kl}_{T_{\mathrm{mod}}}(\mathcal{C})$ is simply the ordinary ∞ -category of free derived algebras $A = \mathrm{Sym}(V)$ generated by $V \in \mathrm{Flat}(\mathcal{C})$, together with a free A -module $A \otimes W$ on $W \in \mathrm{Flat}(\mathcal{C})$. We now have:

- (1) A functor between ordinary categories $\text{triv}: \text{Kl}_{T_{\text{mod}}}(\text{Flat}(\mathcal{C})) \rightarrow \text{CAlg}^{\text{der}}(\mathcal{C})$ sending each (A, M) to the trivial square zero extension $A \oplus M$.
- (2) A functor between ordinary categories $\Omega^1: \text{Kl}_{\text{Sym}}(\text{Flat}(\mathcal{C})) \rightarrow \text{Kl}_{T_{\text{mod}}}(\mathcal{C})$ defined by the universal property that $\text{Map}(\Omega_A^1, (B, M)) \simeq \text{Map}(A, B \oplus M)$. One readily verifies that such a universal object exists for each free algebra $A = \text{Sym}(V)$ and is given by the tuple $(\text{Sym}(V), \text{Sym}(V) \otimes V)$, which is indeed in $\text{Kl}_{T_{\text{mod}}}(\mathcal{C})$.

Both of these functors satisfy the conditions of Proposition A.16 and hence admit derived functors $L: \text{CAlg}^{\text{der}}(\mathcal{C}) \rightarrow \text{Mod}(\mathcal{C})$ and $\text{triv}: \text{Mod}(\mathcal{C}) \rightarrow \text{CAlg}^{\text{der}}(\mathcal{C})$. The universal property of Ω^1 then implies that L is left adjoint to triv . We refer to $L_A \in \text{Mod}_A(\mathcal{C})$ as the *cotangent complex* of A .

As usual, every map $f: A \rightarrow B$ in $\text{CAlg}^{\text{der}}(\mathcal{C})$ induces a map $(A, L_A) \rightarrow (B, L_B)$ in $\text{Mod}(\mathcal{C})$. Since the projection $\text{Mod}(\mathcal{C}) \rightarrow \text{CAlg}^{\text{der}}(\mathcal{C})$ is a cocartesian fibration, as $\mathcal{C} \in \text{CAlg}(\text{Pr}^{\text{L}})$, this is equivalent to a map $f^*L_A \rightarrow L_B$ in Mod_B , whose cofibre is the *relative cotangent complex* $L_{B/A}$:

$$f^*L_A \longrightarrow L_B \longrightarrow L_{B/A}.$$

If $B = \mathbb{L}\text{Sym}_A(M)$ is the free derived A -algebra on $M \in \text{Mod}_A(\mathcal{C})$, then $L_{B/A} \simeq B \otimes_A M$.

We conclude with some observations about finite type conditions on maps between animated commutative algebras, well-known for animated rings.

Definition A.18. If \mathcal{C} is a derived algebraic context and $A \in \text{CAlg}^{\text{an}}(\mathcal{C})$, then we will say that:

- (1) a map $A \rightarrow B$ in $\text{CAlg}^{\text{an}}(\mathcal{C})$ is *almost of finite presentation* if B is an almost compact object of $\text{CAlg}_A^{\text{an}}(\mathcal{C})$, that is, each $\tau_{\leq n}B$ is a compact object of $\tau_{\leq n}\text{CAlg}_A^{\text{an}}(\mathcal{C})$ [HA, Definition 7.2.4.8].
- (2) an A -module $M \in \text{Mod}_A(\mathcal{C})$ is *almost perfect* if $M \in \text{Mod}_A(\mathcal{C})_{\geq k}$ and M defines an almost compact object of $\text{Mod}_A(\mathcal{C})_{\geq k}$.

Proposition A.19. Let \mathcal{C} be a derived algebraic context and $f: A \rightarrow B$ be a map of animated commutative algebras in \mathcal{C} . Then the following are equivalent:

- (1) f is almost of finite presentation.
- (2) The map $\pi_0(A) \rightarrow \pi_0(B)$ of commutative algebras in \mathcal{C}^{\heartsuit} is of finite presentation and $L_{B/A}$ is an almost perfect B -module.

In addition, the following assertions hold:

- (3) If B is almost perfect as an A -module, then f is almost of finite presentation.
- (4) If f is almost of finite presentation and $\pi_0(A) \rightarrow \pi_0(B)$ is surjective, then B is almost perfect as an A -module.

The proof of Proposition A.19 requires some preliminaries about cell decompositions and the resulting filtrations.

Construction A.20. Let $A \rightarrow B$ be a map of animated commutative algebras in \mathcal{C} . By a *cell decomposition* of B , we will mean a sequence of animated filtered A -algebras $A = B_{-1} \rightarrow B_0 \rightarrow B_1 \rightarrow \cdots \rightarrow B$ constructed inductively according to the following procedure:

- Having constructed B_n , choose a map $\chi_n: M_n \rightarrow (B/B_n)[-1]$ from a connective B_n -module M_n such that the cofibre is $(n+1)$ -connective.

- Define B_{n+1} as the pushout of derived filtered algebras

$$\begin{array}{ccc} \mathbb{L}\mathrm{Sym}_{B_n}(M_n) & \xrightarrow{0} & B_n \\ \chi \downarrow & & \downarrow \\ B_n & \xrightarrow{\iota} & B_{n+1} \end{array}$$

where the top map sends M_n to zero. The map $B_n \rightarrow B$ then factors canonically over $\iota: B_n \rightarrow B_{n+1}$.

Let us record some properties of such cell decompositions:

- (a) If $\pi_0(A) \rightarrow \pi_0(B)$ is surjective, one can take $B_0 = B_{-1} = A$.
- (b) Each B_{n+1} admits an increasing filtration whose associated graded is $\mathbb{L}\mathrm{Sym}_{B_n}(M_n[1])$: this filtration is the base change along χ of the filtration on B_n obtained by applying $\mathbb{L}\mathrm{Sym}_{B_n}$ to the zero object with increasing filtration given by $F_0(0) = M_n$ and $F_i(0) = 0$ for $i > 0$. Here we use Example A.11 in the case $\mathcal{J} = (\mathbb{Z}, \leq)$.
- (c) Each cofibre B/B_n is $(n+1)$ -connective and hence each M_n is n -connective. To see this, we proceed by induction, the case $n = -1$ being evident. For the inductive step, note that the map χ_n factors as

$$\chi_n: M_n[1] \rightarrow B_{n+1}/B_n \rightarrow B/B_n.$$

Since the cofibre of the composite is $(n+2)$ -connective by construction, to show that the cofibre B/B_{n+1} of the second map is $(n+2)$ -connective it will suffice to show that $M_n[1] \rightarrow B_{n+1}/B_n$ has $(n+2)$ -connective cofibre. Using the filtration from (b), this map decomposes as

$$M_n[1] \simeq F_1(B_{n+1})/F_0(B_{n+1}) \rightarrow F_2(B_{n+1})/F_0(B_{n+1}) \rightarrow \dots \rightarrow B_{n+1}/F_0(B_{n+1}).$$

It therefore suffices to verify that the cofibre of each of these maps is $(n+2)$ -connective. These cofibres are given by $\mathbb{L}\mathrm{Sym}_{B_n}^m(M_n[1])$ with $m \geq 2$, which fit into a cofibre sequence

$$(M_n[1] \otimes_{B_n} \cdots \otimes_{B_n} M_n[1])_{h\Sigma_m} \rightarrow \mathbb{L}\mathrm{Sym}_{B_n}^m(M_n[1]) \rightarrow F_m(M_n[1]).$$

The result now follows because the outer terms are both $(n+2)$ -connective. For the right term, this follows because F_m is right-left extended from a functor on $\mathrm{Vect}^{\mathrm{ft}}(\mathcal{C})$ with 1-connective values.

- (d) For each n , consider the natural B_{n+1} -linear maps

$$B_{n+1} \otimes_{B_n} M_n[1] \rightarrow B_{n+1} \otimes_{B_n} (B_{n+1}/B_n) \rightarrow L_{B_{n+1}/B_n}$$

where the first map is induced by χ and the last map arises from the universal derivation $B_{n+1} \rightarrow B_{n+1} \oplus L_{B_{n+1}/B_n}$. The composite map is an equivalence of $(n+1)$ -connective modules and the cofibre of the first map is $(n+2)$ -connective, so that the cofibre of the natural map $B_{n+1} \otimes_{B_n} (B_{n+1}/B_n) \rightarrow L_{B_{n+1}/B_n}$ is $(n+3)$ -connective. An inductive argument then shows that each $B \otimes_{B_n} (B/B_n) \rightarrow L_{B/B_n}$ has $(n+3)$ -connective cofibre as well.

Lemma A.21. *Let \mathcal{C} be a derived algebraic context and let $\mathrm{Fun}((\mathbb{Z}_{\geq 0}, =), \mathcal{C})$ be the ∞ -category of non-negatively graded objects in \mathcal{C} , as in Example A.11. Given a graded algebra $A := A^\bullet$ and a graded module $M := M^\bullet$, the following hold:*

- (1) *If $A^0 \otimes_A M \simeq 0$, then $M \simeq 0$.*
- (2) *If A and M are connective and $A^0 \otimes_A M$ is an almost perfect A^0 -module, then M is almost perfect.*

Proof. For (1), we prove the claim by induction on the weight. Clearly, $M^0 = 0$. Assuming that $M^i = 0$ for $i < n$, we have $M^n \simeq (A^0 \otimes_A M)^n$, so that $M^n = 0$ as well. For (2), note that

$$\pi_0(A^0) \otimes_{\pi_0(A)} \pi_0(M) \cong \pi_0(A^0 \otimes_A M)$$

is the module of indecomposables associated to the graded $\pi_0(A)$ -module $\pi_0(M)$. Since this module is finitely generated, it follows that $\pi_0(M)$ is a finitely generated $\pi_0(A)$ -module. Lifting the generators provides a map $P \rightarrow M$ from a finite free graded A -module, which is surjective on π_0 . The cofibre N is then a 1-connective A -module such that $A^0 \otimes_A N$ is an almost perfect A^0 -module. Proceeding by induction, one finds that M is indeed almost perfect. \square

Lemma A.22. *Let \mathcal{C} be a derived algebraic context and let $\text{Fun}((\mathbb{Z}, \leq), \mathcal{C})$ be the ∞ -category of objects in \mathcal{C} with an unbounded increasing filtration, as in Example A.11. Let $A = F_\star A$ be a connective filtered algebra which is complete, i.e. $\lim_{n \rightarrow -\infty} F_n A = 0$. Then a filtered A -module $M = F_\star M$ is almost perfect if and only if it is complete and $\text{Gr}(M)$ is an almost perfect left $\text{Gr}(A)$ -module.*

Proof. If M is almost perfect, then $\text{Gr}(M)$ is almost perfect because taking the associated graded preserves compact objects and is t -exact up to a shift. To see that M is complete, let $n \geq 0$ and pick a perfect filtered A -module N_n with a map $N_n \rightarrow M$ whose cofibre is n -connective. Since A is complete, N_n is complete and hence $\lim F^i M \simeq \lim F^i(M/N_n)$. By the Milnor sequence, the latter is $(n-1)$ -connective. Since n was arbitrary, we conclude that $\lim F^i M = 0$.

For the converse, it suffices that if M is n -connective, there exists a cofibre sequence

$$\bigoplus_{i=1}^k A \otimes_{F_{\geq w_i}} V_i[n] \xrightarrow{f} M \longrightarrow M'$$

where M' is $(n+1)$ -connective and the first term consists of free filtered A -modules with $V_i \in \text{Vect}^{\text{ft}}(\mathcal{C})$, concentrated in filtration weight $\geq w_i$. Indeed, this implies that M' is a complete $(n+1)$ -connective module such that $\text{Gr}(M')$ is almost perfect; it follows by induction that each $\tau_{\leq m}(M)$ is equivalent to the m -truncation of a perfect A -module, so that M is almost perfect.

Since $\text{Gr}(M)$ is an almost perfect n -connective $\text{Gr}(A)$ -module, the desired cofibre sequence exists at the graded level. In particular, there exists a map $f' : \bigoplus_{i=1}^n \text{Gr}(A) \otimes V_i(w_i)[n] \rightarrow \text{Gr}(M)$, where the domain is a sum of free $\text{Gr}(A)$ -modules in various weights w_i . Since the $V_i \in \text{Vect}^{\text{ft}}(\mathcal{C})$ are projective in $\mathcal{C}_{\geq 0}$, we can lift f' to a map f of filtered modules. The cofibre M' of f is then a complete module with $(n+1)$ -connective associated graded, and is thus itself $(n+1)$ -connective by the Milnor sequence. \square

Proof of Proposition A.19. For (1) \Rightarrow (2), note that $\pi_0(B)$ is a compact object in $\pi_0(A)$ -algebras and hence finitely presented. The cotangent complex $L_{B/A}$ is almost perfect because for any filtered system of k -coconnective B -modules M_α , the map $\text{colim} \text{Map}_B(L_{B/A}, M_\alpha) \rightarrow \text{Map}_B(L_{B/A}, \text{colim} M_\alpha)$ is the base change of the equivalence

$$\text{colim} \text{Map}_{\text{CAlg}_A^{\text{an}}(\mathcal{C})}(B, B \oplus M_\alpha) \rightarrow \text{Map}_{\text{CAlg}_A^{\text{an}}(\mathcal{C})}(B, \text{colim}(B \oplus M_\alpha)).$$

For (2) \Rightarrow (1), we claim that there exists a cell decomposition for B with the property that each B_n is almost of finite presentation over A . It then follows that B is the colimit of a sequence of almost finitely presented algebras that stabilises upon k -truncation for any k . This implies that $A \rightarrow B$ is almost finitely presented as well.

To construct this cell decomposition, we fix a finite presentation of $\pi_0(A) \rightarrow \pi_0(B)$. We can then choose a map $M_0 = A \otimes V_0 \rightarrow B$ from a free A -module on $V_0 \in \text{Vect}^{\text{ft}}(\mathcal{C})$ such that

$\pi_0(M_0) \rightarrow \pi_0(B)$ picks out the generators of $\pi_0(B)$. We then set $B_0 = \mathbb{L}\mathrm{Sym}_A(M_0)$. Next, we choose a map $M_1 = A \otimes V_1 \rightarrow \mathrm{fib}(B_0 \rightarrow B)$ from a free B_0 -module on $V_1 \in \mathrm{Vect}^{\mathrm{ft}}(\mathcal{C})$ such that $\pi_0(M_1) \rightarrow \pi_0(B_0) = \mathrm{Sym}_{\pi_0(A)}(\pi_0(M_0))$ picks out the relations of this presentation. The resulting B_1 is almost finitely presented and $\pi_0(B_1) = \pi_0(B)$. Now suppose we have constructed the desired cell decomposition up to B_n , for some $n \geq 1$. Since $A \rightarrow B_n$ is almost of finite presentation, the first part of the proof shows that $L_{B_n/A}$ is an almost perfect B_n -module. Consequently, L_{B/B_n} is an almost perfect B -module. Since

$$B \otimes_{B_n} (B/B_n) \rightarrow L_{B/B_n}$$

has $(n+3)$ -connective cofibre, it follows that we can choose an n -connective free B_n -module $M_n = B_n \otimes V_n[n]$ with $V_n \in \mathrm{Vect}^{\mathrm{ft}}(\mathcal{C})$, together with a map $\chi: M_n \rightarrow (B/B_n)[-1]$ that is surjective on π_n . The resulting map $B_n \rightarrow B_{n+1}$ is then almost finitely presented, so that the composite $A \rightarrow B_{n+1}$ is almost finitely presented as well.

For assertion (4), suppose that f is almost finitely presented and that $\pi_0(A) \rightarrow \pi_0(B)$ is surjective, and consider the above cell decomposition. Then $B_0 = A$, and we claim that each B_n is an almost perfect A -module. The colimit B is then an almost perfect A -module as well, because the cofibre of each $B_n \rightarrow B_{n+1}$ is $(n+1)$ -connective. Note that by part (b) of Construction A.20, B_{n+1} comes with an increasing filtration by B_n -modules whose associated graded $\mathbb{L}\mathrm{Sym}_{B_n}(M_n[1])$ is almost perfect: indeed, each $\mathbb{L}\mathrm{Sym}_{B_n}^k(M_n[1])$ is almost perfect and (at least) k -connective, since M_n was a connective almost perfect B_n -module. It follows that B_{n+1} is an almost perfect B_n -module, so that an inductive argument shows that $B_n \in \mathrm{APerf}_A(\mathcal{C})$ (the case $n=0$ being evident).

Finally, let us prove assertion (3). Consider the cell decomposition for B defined inductively by setting $B_{-1} = A$ and

$$M_n = (B/B_n)[-1].$$

We claim that B is almost perfect as a B_n -module for each B_n . Assuming this, it follows by induction that each $B_n \rightarrow B_{n+1}$ is almost finitely presented, so that $A \rightarrow B_{n+1}$ is almost finitely presented as well. Since the sequence of truncations $\tau_{\leq k} B_n$ is eventually constant, the colimit B is then an almost finitely presented A -algebra as well.

We prove our claim by induction, the case $n = -1$ being evident. Assuming B is an almost perfect B_n -module, we endow B_{n+1} with the increasing filtration from part (b) of Construction A.20 and B with the increasing filtration where $F_0 B = B_n$ and $F_i B = B$ for $i \geq 1$. Then $F_* B$ is an $F_* B_{n+1}$ -module, and it suffices to verify that it is almost perfect as a filtered module. By Lemma A.21, it suffices to verify that the associated graded is an almost perfect module over $\mathrm{Gr}(B_{n+1}) = \mathbb{L}\mathrm{Sym}_{B_n}(B/B_n)$. We then have a cofibre sequence of $\mathrm{Gr}(B_{n+1})$ -modules $B_n(0) \rightarrow \mathrm{Gr}(B) \rightarrow (B/B_n)(1)$ where the outer terms are concentrated in weight 0 and 1. By Lemma A.21, it therefore suffices to verify that the graded B_n -modules

$$\begin{aligned} B_n \otimes_{\mathbb{L}\mathrm{Sym}_{B_n}(B/B_n)} B_n &\simeq \mathbb{L}\mathrm{Sym}_{B_n}(B/B_n[1]) \\ B_n \otimes_{\mathbb{L}\mathrm{Sym}_{B_n}(B/B_n)} (B/B_n) &\simeq \mathbb{L}\mathrm{Sym}_{B_n}(B/B_n[1]) \otimes_{B_n} B/B_n \end{aligned}$$

are almost perfect B_n -modules. This follows from the fact that B/B_n is an almost perfect B_n -module and that $\mathbb{L}\mathrm{Sym}_{B_n}(B/B_n[1])$ is almost perfect, because each $\mathbb{L}\mathrm{Sym}_{B_n}^k(B/B_n[1])$ is almost perfect and k -connective. \square

A.2. Pro-coherent modules. As highlighted in [GR17a], the usual ∞ -category of quasi-coherent sheaves on a derived scheme (or prestack) is not very well-adapted to deformation theory. For schemes that are locally almost finitely presented over a field, it is better to consider their categories

of ind-coherent sheaves instead. In this situation, Serre duality provides a natural identification between ind-coherent sheaves and the dual notion of pro-coherent sheaves: on an almost finitely presented scheme X over a field k , this is simply given by $\mathrm{Ind}(\mathrm{Coh}_X) \simeq \mathrm{Ind}(\mathrm{Coh}_X^{\mathrm{op}})$. To also deal with situations without Serre duality, where we cannot directly use the machinery of ind-coherent sheaves, let us introduce the following provisional modification of the ∞ -category of pro-coherent sheaves over an affine derived scheme:

Definition A.23. Given an animated ring A , we will write QC_A^\vee for the ∞ -category of exact functors $F: \mathrm{Mod}_A \rightarrow \mathrm{Sp}$ satisfying the following two conditions:

- (1) They are *convergent*, in the sense that they preserve limits of Postnikov towers in Mod_A .
- (2) They are *almost finitely presented*, in the sense that for each $n \in \mathbb{N}$ and each filtered diagram of n -coconnective A -modules M_α , the map $\mathrm{colim}_\alpha F(M_\alpha) \rightarrow F(\mathrm{colim} M_\alpha)$ is an equivalence.

Warning A.24. For a general animated ring, the ∞ -category QC_A^\vee is not very manageable and one should view this as an ad-hoc definition. The definition will essentially only become useful when A is a coherent animated ring: in that case $\mathrm{QC}_A^\vee \simeq \mathrm{Ind}(\mathrm{Coh}_A^{\mathrm{op}})$ can be identified with the compactly generated ∞ -category of pro-coherent sheaves on A .

Keeping this warning in mind, let us nonetheless record some formal properties of QC_A^\vee .

Notation A.25. If \mathcal{C} is a stable ∞ -category with a t -structure, we let $\mathcal{C}^+ = \bigcup_n \mathcal{C}_{\leq n}$, $\mathcal{C}^- = \bigcup_n \mathcal{C}_{\geq n}$ and $\mathcal{C}^b = \mathcal{C}^+ \cap \mathcal{C}^-$ be the full subcategories of eventually coconnective, eventually connective and bounded objects [HA].

Lemma A.26. Let A be an animated ring. Then the following ∞ -categories are naturally equivalent:

- (1) The ∞ -category QC_A^\vee .
- (2) The ∞ -category of exact functors $\mathrm{Mod}_A^+ \rightarrow \mathrm{Sp}$ that are almost finitely presented.
- (3) The ∞ -category of left exact functors $F: \mathrm{Mod}_A^+ \rightarrow \mathcal{S}$ that are almost finitely presented.
- (4) The ∞ -category of functors $F: \mathrm{Mod}_{A, \geq 0}^+ \rightarrow \mathcal{S}$ that are locally almost finitely presented and preserve the terminal object and pullbacks along a π_0 -surjection.
- (5) The ∞ -category of functors $F: \mathrm{Mod}_{A, \geq 0} \rightarrow \mathcal{S}$ preserving limits of almost eventually constant towers and satisfying the conditions from (4).

Proof. The equivalence between (1) and (2) is given by restriction and right Kan extension along $j: \mathrm{Mod}_A^+ \hookrightarrow \mathrm{Mod}_A$. Recall that the right Kan extension is given explicitly by $j_*F(M) = \lim F(\tau_{\leq n}M)$. The equivalence between (2) and (3) is given by postcomposition with Ω^∞ . The equivalence between (3) and (4) is given by restriction along the inclusion $i_1: \mathrm{Mod}_A^b \hookrightarrow \mathrm{Mod}_A^+$ and $i_2: \mathrm{Mod}_{A, \geq 0}^+ \hookrightarrow \mathrm{Mod}_A^b$ of bounded (resp. bounded connective) objects. The inverse is given by right Kan extension along i_2 followed by left Kan extension along i_1 . The equivalence between (4) and (5) follows by restriction and right Kan extension along $\mathrm{Mod}_{A, \geq 0}^+ \hookrightarrow \mathrm{Mod}_{A, \geq 0}$. \square

Lemma A.27. Let A be an animated ring and let κ be a regular cardinal such that $\tau_{\leq n}: \mathrm{Mod}_A \rightarrow \mathrm{Mod}_A$ preserves κ -compact objects. Then restriction and left Kan extension along $i: \mathrm{Mod}_A^{+, \kappa} \hookrightarrow \mathrm{Mod}_A^+$ induces an equivalence between QC_A^\vee and the ∞ -category of exact functors $F: \mathrm{Mod}_A^{+, \kappa} \rightarrow \mathrm{Sp}$ preserving colimits of κ -small filtered diagrams of n -coconnective objects, for each n .

Proof. Since $\tau_{\leq n}$ preserves κ -small objects, for any n -coconnective A module M the inclusion $(\mathrm{Mod}_{A, \leq n}^\kappa)_{/M} \hookrightarrow (\mathrm{Mod}_A^{+, \kappa})_{/M}$ is cofinal. Using that $(\mathrm{Mod}_{A, \leq n}^\kappa)_{/M}$ is filtered, one readily sees that $i_*i^*F \simeq F$ for any $F \in \mathrm{QC}_A^\vee$. The identification of its essential image follows from the fact that any filtered colimit can be written as a κ -filtered colimit of κ -small filtered colimits. \square

Corollary A.28. *Let A be an animated ring. Then QC_A^\vee is a stable presentable ∞ -category.*

Proof. By Lemma A.27, $\mathrm{QC}_A^\vee \subseteq \mathrm{Fun}(\mathrm{Mod}_A^{+, \kappa}, \mathrm{Sp})$ is the full subcategory of functors preserving certain types of colimits. It is hence presentable by the proof of [HTT, Proposition 5.5.3.8]. \square

Corollary A.29. *Let $F: \mathrm{Mod}_A \rightarrow \mathrm{Sp}$ be an exact functor which is accessible and convergent and consider the ∞ -category $(\mathrm{QC}_A^\vee)_{/F}$ of functors in QC_A^\vee equipped with a natural transformation to F . This ∞ -category admits a terminal object F^{aft} , which we will refer to as the almost finitely presented approximation to F .*

Proof. Since F is convergent and κ -accessible for some uncountable κ , it suffices to verify that there exists a universal object $F^{\mathrm{aft}} \in \mathrm{QC}_A^\vee$ equipped with a natural transformation to the restriction $F: \mathrm{Mod}_A^{+, \kappa} \rightarrow \mathrm{Sp}$. The previous proof shows that for large enough κ , the inclusion $\mathrm{QC}_A^\vee \subseteq \mathrm{Fun}(\mathrm{Mod}_A^{+, \kappa}, \mathrm{Sp})$ is accessible and preserves colimits, and hence admits a right adjoint. \square

For any A , there is an adjoint pair

$$\iota: \mathrm{Mod}_A \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \mathrm{QC}_A^\vee : \mathbf{v}$$

where the left adjoint sends an A -module M to the exact functor $M \otimes_A (-): \mathrm{Mod}_A^+ \rightarrow \mathrm{Sp}$ and the right adjoint sends any $F: \mathrm{Mod}_A^+ \rightarrow \mathrm{Sp}$ to the A -module $\mathbf{v}(F) = \lim_n F(\tau_{\leq n} A)$.

Observation A.30. *The ∞ -category QC_A^\vee comes equipped with a left complete t -structure whose connective part consists of those $F: \mathrm{Mod}_A^+ \rightarrow \mathrm{Sp}$ that are right t -exact. This induces an equivalence*

$$\mathrm{Mod}_{A, \geq 0} \simeq \mathrm{QC}_{A, \geq 0}^\vee.$$

Indeed, the equivalences of Lemma A.26 identify $\mathrm{QC}_{A, \geq 0}^\vee$ with the ∞ -category of colimit-preserving functors $F: \mathrm{Mod}_{A, \geq 0} \rightarrow \mathcal{S}$, which is equivalent to $\mathrm{Mod}_{A, \geq 0}$ via sending such a functor F to the A -module $F(A) \in \mathrm{Mod}_{A, \geq 0}$.

Consequently, Mod_A is the right completion of QC_A^\vee . In particular, we will typically identify eventually connective A -modules with the corresponding object in QC_A^\vee . If A is eventually coconnective, then $\mathrm{QC}_A \rightarrow \mathrm{QC}_A^\vee$ is fully faithful on all modules (not just the eventually connective ones).

Let A be an animated ring and $F \in \mathrm{QC}_A^\vee$. For each $N \in \mathrm{Mod}_A^+$, the spectrum $F(N)$ inherits a natural A -module structure from the A -module structure on N . More precisely, one can describe this A -module structure on $F(N)$ by the exact functor $\mathrm{Perf}(A)^{\mathrm{op}} \rightarrow \mathrm{Sp}$ sending M to $F(M^\vee \otimes_A N)$. Using this, the ∞ -category QC_A^\vee is tensored over Mod_A via

$$(26) \quad \otimes_A: \mathrm{QC}_A^\vee \times \mathrm{Mod}_A \longrightarrow \mathrm{QC}_A^\vee; \quad (M \otimes_A F)(N) = M \otimes_A (F(N))$$

for each $N \in \mathrm{Mod}_A^+$. This preserves colimits in each variable and the functor $\iota: \mathrm{Mod}_A \rightarrow \mathrm{QC}_A^\vee$ is compatible with the tensoring.

Definition A.31. Let $M \in \mathrm{Mod}_A$ be an A -module. Then the tensoring $M \otimes_A -: \mathrm{QC}_A^\vee \rightarrow \mathrm{QC}_A^\vee$ admits a right adjoint. We define the *pro-coherent dual* M^\vee of M to be the value of this right adjoint on A .

Example A.32. If M is a perfect A -module, then the adjoint to $M \otimes_A -: \mathrm{QC}_A^\vee \rightarrow \mathrm{QC}_A^\vee$ is given by tensoring with the dual perfect module. Consequently, the pro-coherent dual of a perfect A -module is simply its usual A -linear dual. In terms of functors, one can also identify this with the functor $\mathrm{Hom}_A(M, -): \mathrm{Mod}_A \rightarrow \mathrm{Sp}$. Using this and the fact that $(-)^\vee$ sends colimits in Mod_A^- to limits

in QC^\vee , it follows that for any eventually connective $M \in \mathrm{Mod}_A^-$, its pro-coherent dual M^\vee is the almost finitely presented approximation to $\mathrm{Hom}_A(M, -)$ in the sense of Corollary A.29.

We will be particularly interested in the pro-coherent duals of *almost perfect* A -modules: in this case, M^\vee is given by $\mathrm{Hom}_A(M, -): \mathrm{Mod}_A \rightarrow \mathrm{Sp}$, which is already almost finitely presented.

Definition A.33. We will say that a pro-coherent module $F \in \mathrm{QC}_A^\vee$ is *dually almost perfect* if $F: \mathrm{Mod}_A \rightarrow \mathrm{Sp}$ is corepresented by an almost perfect A -module. Taking pro-coherent duals therefore determines an equivalence

$$(-)^\vee: \mathrm{APerf}_A^{\mathrm{op}} \xrightarrow{\sim} \mathrm{dAPerf}_A$$

between almost perfect A -modules and dually almost perfect pro-coherent A -modules.

Lemma A.34. *Let $M \in \mathrm{Mod}_A$ be an A -module and let M^\vee be its pro-coherent A -linear dual. Then pro-coherent duality determines an equivalence*

$$(27) \quad (-)^\vee: ((\mathrm{APerf}_A)_{M/})^{\mathrm{op}} \xrightarrow{\sim} (\mathrm{dAPerf}_A)_{/M^\vee}.$$

Proof. The functor (27) defines a map of right fibrations covering the equivalence $\mathrm{APerf}_A^{\mathrm{op}} \simeq \mathrm{dAPerf}_A$. It therefore suffices to verify that it induces an equivalence on fibres, i.e. that the map $\mathrm{Map}_{\mathrm{Mod}_A}(M, N) \rightarrow \mathrm{Map}_{\mathrm{QC}_A^\vee}(N^\vee, M^\vee)$ is an equivalence for every $N \in \mathrm{APerf}_A$. To see this, recall that $N^\vee: \mathrm{Mod}_A \rightarrow \mathrm{Sp}$ is corepresented by N and $M^\vee: \mathrm{Mod}_A \rightarrow \mathrm{Sp}$ is the almost finitely presented approximation to the functor corepresented by M , so that $\mathrm{Map}_{\mathrm{QC}_A^\vee}(N^\vee, M^\vee)$ is equivalent to the space of maps between the functors corepresented by N and M . Unravelling the definitions, one then sees that $\mathrm{Map}_{\mathrm{Mod}_A}(M, N) \rightarrow \mathrm{Map}_{\mathrm{QC}_A^\vee}(N^\vee, M^\vee)$ is an equivalence by the Yoneda lemma. \square

Definition A.35. Let A be an animated ring. For all $-\infty \leq a \leq b \leq \infty$, we will say that an object $F \in \mathrm{QC}_A^\vee$ has *tor-amplitude* in $[a, b]$ if for any discrete A -module $M \in \mathrm{Mod}_A^\heartsuit$,

$$\pi_n F(M) \neq 0 \quad \implies \quad a \leq n \leq b.$$

Write $\mathrm{QC}_A^{\vee, [a, b]} \subseteq \mathrm{QC}_A^\vee$ for the full subcategory on the objects with tor-amplitude in $[a, b]$.

Example A.36. We have the following special cases:

- (1) A pro-coherent module F has tor-amplitude in $[0, \infty]$ if and only if F is connective. In particular, $\mathrm{QC}_A^{\vee, [0, \infty]} \simeq \mathrm{Mod}_{A, \geq 0}$ is equivalent to the ordinary ∞ -category of connective A -modules.
- (2) $\mathrm{QC}_A^{\vee, [0, 0]} \simeq \mathrm{Flat}_A$ coincides with the full subcategory of $\mathrm{Mod}_{A, \geq 0}$ spanned by the *flat* A -modules.
- (3) Let $F = M^\vee$ be dually almost perfect. Then M^\vee has tor-amplitude in $[-b, -a]$ if and only if M has tor-amplitude in $[a, b]$.

Let us now turn to the functoriality of QC_A^\vee in the animated ring A .

Definition A.37. Let $f: A \rightarrow B$ be a map of rings. We will write $f^*: \mathrm{QC}_A^\vee \rightleftarrows \mathrm{QC}_B^\vee : f_*$ for the adjoint pair whose left adjoint is given by precomposition with $f_*: \mathrm{Mod}_B^+ \rightarrow \mathrm{Mod}_A^+$

Note that f^* fits in a commuting square

$$\begin{array}{ccc} \mathrm{Mod}_A & \xrightarrow{f^*} & \mathrm{Mod}_B \\ \downarrow \iota & & \downarrow \iota \\ \mathrm{QC}_A^\vee & \xrightarrow{f^*} & \mathrm{QC}_B^\vee. \end{array}$$

In the presence of Serre duality, f^* is Serre dual to the $f^!$ -functor on ind-coherent sheaves [Gai11, Section 9.2.3]

Lemma A.38. *If $f: A \rightarrow B$ has finite tor-amplitude, then the right adjoint f_* preserves colimits. Furthermore, for any pullback square*

$$\begin{array}{ccc} \mathrm{Spec}(B') & \xrightarrow{f'} & \mathrm{Spec}(B) \\ g' \downarrow & & \downarrow g \\ \mathrm{Spec}(A') & \xrightarrow{f} & \mathrm{Spec}(A). \end{array}$$

where f has finite tor-amplitude, the Beck–Chevalley map $g^*f_* \rightarrow f'_*g'^*$ is an equivalence.

Proof. If f has finite tor-amplitude, then f_* is simply given by restriction along $f^*: \mathrm{Mod}_A^+ \rightarrow \mathrm{Mod}_B^+$. Using this, one sees that for every $F \in \mathrm{QC}_{A'}^\vee$, and $M \in \mathrm{Mod}_B^+$, the Beck–Chevalley map $(g^*f_*F)(M) \rightarrow (f'_*g'^*F)(M)$ is given by the image under F of the Beck–Chevalley map $g^*f_*M \rightarrow f'_*g'^*M$, which is an equivalence. \square

Corollary A.39. *The functor $\mathrm{QC}^\vee: \mathrm{CAlg}_R^{\mathrm{an}} \rightarrow \mathrm{Pr}^{\mathrm{L}}$ satisfies descent with respect to universal descent morphisms of finite tor-amplitude, in the sense of [SAG, Definition D.3.1.1]. In particular, it satisfies fppf descent.*

Proof. Using that $A \mapsto \mathrm{Mod}_A$ preserves products, one readily sees that QC^\vee preserves finite products of animated rings. Next, let $f: A \rightarrow B$ be a universal descent morphism of finite tor-amplitude. Let B^\bullet be the corresponding Čech nerve and consider the map $\theta: \mathrm{QC}_A^\vee \rightarrow \lim \mathrm{QC}_{B^\bullet}^\vee$. Using Lemma A.38 and [HA, Corollary 4.7.5.3], we see that θ is an equivalence if $f^*: \mathrm{QC}_A^\vee \rightarrow \mathrm{QC}_B^\vee$ is conservative. To see this, take $F \in \mathrm{QC}_A^\vee$ such that $f^*F \simeq 0$. The objects $M \in \mathrm{Mod}_A$ such that $F(M) \simeq 0$ form a stable subcategory of Mod_A that contains $f_*(\mathrm{Mod}_B)$ and is closed under retracts; since f was a universal descent morphism, this means that it contains every A -module. \square

We now specialise to the case where A is a coherent animated ring.

Definition A.40. An animated ring A is said to be *coherent* if the standard t -structure on Mod_A restricts to a t -structure on APerf_A . Equivalently, $\pi_0(A)$ is coherent and each $\pi_n(A)$ is a finitely presented $\pi_0(A)$ -module.

Definition A.41. Assume A is a coherent animated ring. An almost perfect A -module M is *coherent* if it eventually connective and coconnective. Equivalently, an A -module M is coherent if all homotopy groups $\pi_n M$ are finitely presented over $\pi_0 A$, and only finitely many of $\pi_n M$ are non-zero. We denote $\mathrm{Coh}_A \subset \mathrm{APerf}_A$ the full subcategory of coherent modules.

Lemma A.42. *If $A \in \mathrm{CAlg}^{\mathrm{an}}$ is coherent, then QC_A^\vee is equivalent to the categories of:*

- (1) *Exact functors $F: \mathrm{Coh}_A \rightarrow \mathrm{Sp}$.*
- (2) *Functors $F: \mathrm{Coh}_{A, \geq 0} \rightarrow \mathcal{S}$ that preserve the terminal object and pullbacks along π_0 -surjections.*
- (3) *Functors $F: \mathrm{APerf}_{A, \geq 0} \rightarrow \mathcal{S}$ that preserve almost eventually constant towers, the terminal object and pullbacks along π_0 -surjections.*

In particular, $\mathrm{QC}_A^\vee \simeq \mathrm{Ind}(\mathrm{Coh}_A^{\mathrm{op}})$ is compactly generated.

Proof. This follows from Lemma A.27 with $\kappa = \omega$, as well as the argument from Lemma A.26. \square

Remark A.43. Let A be a coherent animated ring with dualising complex ω_A . Then the t -structure on QC_A^\vee from Observation A.30 is Serre dual to the t -structure on $\mathrm{Ind}(\mathrm{Coh}_A)$ whose connective part is generated by ω_A .

Corollary A.44. *The following holds for a map of coherent animated rings $f: A \rightarrow B$:*

- (1) *If f has finite tor-amplitude, then the functor $f^*: \mathrm{QC}_A^\vee \rightarrow \mathrm{QC}_B^\vee$ coincides with $\mathrm{Ind}(f^*)$, for $f^*: \mathrm{Coh}_A^{\mathrm{op}} \rightarrow \mathrm{Coh}_B^{\mathrm{op}}$.*
- (2) *If f is finite, then f^* has a left adjoint $f_!: \mathrm{QC}^\vee(B) \rightarrow \mathrm{QC}^\vee(A)$ given by the ind-completion of $f_*: \mathrm{Coh}_B^{\mathrm{op}} \rightarrow \mathrm{Coh}_A^{\mathrm{op}}$.*

Corollary A.45. *Let A be a coherent animated ring. Then QC_A^\vee carries a unique closed symmetric monoidal structure whose restriction to $\mathrm{dAPerf}_A \simeq \mathrm{APerf}_A^{\mathrm{op}}$ is equivalent to the usual symmetric monoidal structure on almost perfect A -modules, i.e. $M^\vee \otimes_A N^\vee = (M \otimes_A N)^\vee$ for all $M, N \in \mathrm{APerf}_A$. Every map $f: A \rightarrow B$ between coherent animated rings induces a symmetric monoidal functor $f^*: \mathrm{QC}_A^\vee \rightarrow \mathrm{QC}_B^\vee$.*

Proof. Using part (3) of Lemma A.42, it suffices to verify that $\mathrm{QC}_A^\vee \subseteq \mathrm{Fun}(\mathrm{APerf}_{A, \geq 0}, \mathcal{S})$ is a monoidal left Bousfield localisation with respect to the Day convolution product. This follows from the fact that if F is a functor as in (3) and $M \in \mathrm{APerf}_{A, \geq 0}$, then $F(M \otimes_A -)$ satisfies the conditions from (3) as well. \square

Remark A.46. One readily verifies that $\iota: \mathrm{Mod}_A \rightarrow \mathrm{QC}_A^\vee$ is symmetric monoidal and that the tensoring from (26) is induced by this symmetric monoidal functor. It follows that the pro-coherent dual of an A -module M from Definition A.31 coincides with the dual of $\iota(M)$ with respect to the symmetric monoidal structure from Corollary A.45.

APPENDIX B. DERIVED ALGEBRAIC GEOMETRY RECOLLECTIONS

We recall the following terminology:

Definition B.1 (Prestacks). A *prestack* is an accessible functor $X: \mathrm{CAlg}^{\mathrm{an}} \rightarrow \mathcal{S}$ from animated rings to spaces. We will write PrStk for the ∞ -category of prestacks. Given a prestack S , we let

- (1) $\mathrm{PrStk}_{/S}$ be the ∞ -category of S -prestacks, that is, prestacks X endowed with a map $X \rightarrow S$.
- (2) $\mathrm{Aff}_{/S}$ be the ∞ -category of *affine derived schemes* with a map to S , i.e. all $\mathrm{Spec}(A) \rightarrow S$ where $\mathrm{Spec}(A)$ is the functor corepresented by an animated ring A .

The ∞ -category of S -prestacks can then also be identified with the ∞ -category of accessible functors $(\mathrm{Aff}_{/S})^{\mathrm{op}} \rightarrow \mathcal{S}$.

Example B.2. When $S = \mathrm{Spec}(R)$ is affine, $\mathrm{Aff}_{/S} \simeq \mathrm{CAlg}_R^{\mathrm{an}, \mathrm{op}}$ can be identified with the opposite of the ∞ -category of animated commutative R -algebras. Consequently, a prestack X over R is simply an accessible functor $X: \mathrm{CAlg}_R^{\mathrm{an}} \rightarrow \mathcal{S}$; we will write PrStk_R for the resulting ∞ -category of R -prestacks.

The purpose of this section is to briefly recall some of the results and constructions from [GR17a] in this setting.

B.1. Finiteness and deformation-theoretic properties of prestacks. We begin with a recollection of some of the conditions that one can impose on a prestack, following [GR17a] and [SAG, Section 17]. As in [GR17a, Volume 1, Chapter 2] and [SAG, Section 17.4], the following finiteness condition will play a central role in this text:

Definition B.3. A map of prestacks $X \rightarrow S$ is said to be

- *convergent* if for all $A \in \mathbf{CAlg}^{\text{an}}$, the square

$$\begin{array}{ccc} X(A) & \longrightarrow & \lim X(\tau_{\leq n} A) \\ \downarrow & & \downarrow \\ S(A) & \longrightarrow & \lim S(\tau_{\leq n} A) \end{array}$$

is cartesian.

- *locally almost finitely presented* if it is convergent and satisfies the following condition for each $n \geq 0$: for each filtered diagram $A_\alpha : I \rightarrow \mathbf{CAlg}_{\leq n}^{\text{an}}$ of n -truncated animated rings with colimit A , the square

$$\begin{array}{ccc} \text{colim } X(A_\alpha) & \longrightarrow & X(A) \\ \downarrow & & \downarrow \\ \text{colim } S(A_\alpha) & \longrightarrow & S(A) \end{array}$$

is cartesian. Equivalently, X is convergent and for any $\text{Spec } A \rightarrow S$ and each $n \geq 0$, the base change $X \times_S \text{Spec}(A)$ restricts to a functor $\mathbf{CAlg}_{A, \leq n}^{\text{an}} \rightarrow \mathcal{S}$ that preserves filtered colimits.

We will write $\text{PrStk}_S^{\text{laft}} \hookrightarrow \text{PrStk}_S^{\text{conv}} \hookrightarrow \text{PrStk}_S$ for the full subcategories spanned by the S -prestacks that are locally of finite presentation, resp. convergent.

Next, let us recall the conditions on a prestack that guarantee that it has a well-behaved infinitesimal structure, using the following algebraic terminology:

Definition B.4. A map of animated rings $A \rightarrow B$ is said to be a *nilpotent extension* if $\pi_0(A) \rightarrow \pi_0(B)$ is surjective and its kernel is a nilpotent ideal.

Lemma B.5 ([GR17a, Volume 2, Chapter 1, Proposition 5.5.3]). *Let $f : A \rightarrow B$ be a nilpotent extension. Then f can be decomposed as the limit of an almost eventually constant tower*

$$A = A_\infty \longrightarrow \dots \longrightarrow A_n \longrightarrow A_{n-1} \longrightarrow \dots \longrightarrow A_0 = B$$

where each $A_n \rightarrow A_{n-1}$ is a square zero extension by a $\pi_0(B)$ -module M_n and for each N .

Definition B.6 ([GR17a, Volume 2, Chapter 1, Definition 7.1.2]). An S -prestack X is said to *have deformation theory* (relative to S) if it is convergent (relative to S) and for each pullback diagram of animated rings on the left in which $B \rightarrow A$ (and hence $B' \rightarrow A'$) is a nilpotent extension, the square on the right is cartesian

$$\begin{array}{ccc} B' \longrightarrow B & & X(B') \longrightarrow X(B) \times_{X(A)} X(A') \\ \downarrow & & \downarrow \\ A' \longrightarrow A & & S(A') \longrightarrow S(B) \times_{S(A)} S(A'). \end{array}$$

Equivalently, for each $s : \text{Spec}(A) \rightarrow S$, the fibre $X_s : \mathbf{CAlg}_A^{\text{an}} \rightarrow \mathcal{S}$ has deformation theory, i.e. preserves limits of Postnikov towers and pullbacks along nilpotent extensions of animated A -algebras.

Lemma B.7. *Let X be a convergent S -prestack. Then X has deformation theory if and only if one of the following two equivalent conditions holds:*

(1) For each square in $(\text{Aff}/S)^{\text{op}}$ opposite to a pullback square of animated rings

$$\begin{array}{ccc} B' & \longrightarrow & A \\ \downarrow & & \downarrow (\text{id}, 0) \\ B & \longrightarrow & A \oplus I[1] \end{array}$$

with $I \in \text{QC}(A)_{\geq 0}$, its image under $X: (\text{Aff}/S)^{\text{op}} \rightarrow \mathcal{S}$ is a pullback diagram of spaces.

(1') The condition of (1) holds for all squares such that A, B and I are eventually coconnective.

Proof. The equivalence between X having deformation theory and condition (1) follows from Lemma B.5 by decomposing a general nilpotent extension as a tower of successive square-zero extensions. The equivalence between (1) and (1') follows from X being convergent. \square

Finally, let us introduce two versions of the formal completion of a prestack, based on the following algebraic constructions:

Definition B.8 (Reduction and pro-reduction). Let $A \in \text{CAlg}_R^{\text{an}}$ be an animated R -algebra.

- (1) The *reduction* A_{red} is the quotient of $\pi_0(A)$ by its nilradical.
- (2) The *pro-reduction* A_{prored} to be the ind-ring

$$A_{\text{prored}} = \text{“colim”}_I \pi_0(A)/I,$$

where the colimit is taken over all nilpotent ideals I of $\pi_0(A)$.

For $S = \text{Spec}(A)$ affine, we write $S_{\text{red}} = \text{Spec}(A_{\text{red}})$ for the underlying reduced scheme. We moreover define a pro-scheme S_{prored} over R as the formal limit

$$\text{Spec}(A_{\text{prored}}) = \text{“lim”}_I \text{Spec}(\pi_0(A)/I),$$

where I again ranges over all nilpotent ideals of $\pi_0(A)$. Given another R -prestack X , we set

$$X(A_{\text{prored}}) := \text{Map}_{\text{Pro}(\text{PrStk})}(S_{\text{prored}}, X) = \text{colim}_I X(\pi_0(A)/I).$$

Definition B.9. Let $X \rightarrow Y$ be a map of prestacks over R .

- (1) The *formal completion* Y_X^\wedge is the R -prestack sending an animated R -algebra A to

$$Y_X^\wedge(A) = X(A_{\text{red}}) \times_{Y(A_{\text{red}})} Y(A).$$

- (2) The *formal infinitesimal completion* of Y at X is the R -prestack sending A to

$$(X/Y)_{\text{inf}}(A) = X(A_{\text{prored}}) \times_{Y(A_{\text{prored}})} Y(A).$$

Remark B.10. When $X \rightarrow Y$ is left, then the canonical map $(X/Y)_{\text{inf}} \rightarrow Y_X^\wedge$ is an equivalence, as $X(A_{\text{red}}) \simeq X(A_{\text{prored}}) \times_{Y(A_{\text{prored}})} Y(A_{\text{red}})$.

Notation B.11. For $Y = *$ the terminal prestack, we write $X_{\text{inf}} = (X/*)_{\text{inf}}$. In characteristic 0, the prestack X_{inf} is often called the *de Rham stack* of X .

Definition B.12. We will say that a map of R -prestacks $f: X \rightarrow Y$ is a *nil-isomorphism* if for every $A \in \text{CAlg}_R^{\text{an}}$, the map

$$X(A_{\text{prored}}) \rightarrow Y(A_{\text{prored}})$$

is an equivalence.

Remark B.13. When $X \rightarrow Y$ is left, then it is a nil-isomorphism if and only if $X(A) \rightarrow Y(A)$ is an equivalence for all reduced commutative rings.

Lemma B.14. *Completion satisfies the following properties:*

- (1) *For any $X \rightarrow Y$, the map $Y_X^\wedge \rightarrow Y$ has deformation theory. If $X \rightarrow Y$ is left, then $X \rightarrow Y_X^\wedge$ and $Y_X^\wedge \rightarrow Y$ are left.*
- (2) *Let $X' \rightarrow X$ be an étale map between derived schemes and let $X \rightarrow Y$ be a nil-isomorphism. Then the square*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y_{X'}^\wedge & \longrightarrow & Y \end{array}$$

is cartesian.

Proof. The first part of (1) follows from the fact that for any nilpotent extension $A' \rightarrow A$ of animated rings, $Y_X^\wedge(A') \simeq Y_X^\wedge(A) \times_{Y(A)} Y(A')$. For the second part, it is enough to show that $Y_X^\wedge \rightarrow Y$ is left. This is the base change of the map $*_X^\wedge \rightarrow *_Y^\wedge$, which is left because for any filtered diagram of n -coconnective animated rings A_α with colimit A , $\text{colim } X(A_{\alpha, \text{red}}) \simeq \text{colim } Y(A_{\alpha, \text{red}}) \times_{Y(A_{\text{red}})} X(A_{\text{red}})$. For (2), the fact that $X' \rightarrow X$ is left implies that $Y_{X'}^\wedge \rightarrow Y$ is the base change of $X'_{\text{inf}} \rightarrow X_{\text{inf}}$ (Remark B.10). It now suffices to verify that the map $X' \rightarrow X'_{\text{inf}} \times_{X_{\text{inf}}} X$ is an equivalence. This follows from $X' \rightarrow X$ being étale, so that $X'(A) \simeq X(A) \times_{X(\pi_0 A/I)} X'(\pi_0 A/I)$ for any nilpotent ideal I . \square

B.2. Pro-coherent sheaves. The theory of pro-coherent sheaves on affine derived schemes from Section B.2 has an evident global analogue:

Definition B.15. For a prestack X , we define the stable presentable ∞ -category QC_X^\vee as

$$\text{QC}_X^\vee = \lim_{S \in \text{Aff}/X} \text{QC}_{\mathcal{O}(S)}^\vee$$

where the limit is taken with respect to the f^* -functoriality. Every map $f: Y \rightarrow X$ between prestacks induces an adjoint pair $f^*: \text{QC}_X^\vee \rightleftarrows \text{QC}_Y^\vee: f_*$.

Let us point out that in the presence of Serre duality, this is Serre dual to the ∞ -category $\text{QC}_X^!$ of ind-coherent sheaves on X defined using $f^!$ -functoriality [Gai11, 9.2.3]. The main features of QC_A^\vee carry over to pro-coherent sheaves on prestacks by naturality:

- (1) Since each $f^*: \text{QC}_A^\vee \rightarrow \text{QC}_B^\vee$ is right t -exact, for any prestack X the ∞ -category QC_X^\vee admits a right complete t -structure such that all f^* -functors are right t -exact. Since the functor $\iota: \text{Mod}_A \rightarrow \text{QC}_A^\vee$ intertwines the f^* -functors, there is a natural transformation $\iota: \text{QC}_X \rightarrow \text{QC}_X^\vee$ which induces an equivalence on connective objects.
- (2) Each QC_X^\vee is tensored and cotensored over QC_X and ι is compatible with the tensoring.
- (3) If M is an almost perfect A -module, then the natural map $f^*(M^\vee) \rightarrow (f^*M)^\vee$ is an equivalence. Using this, one sees that for any prestack X , pro-coherent duality induces an equivalence

$$(-)^\vee: \text{APerf}_X^{\text{op}} \xrightarrow{\sim} \text{dAPerf}_X$$

between the full subcategories of pro-coherent sheaves F that are (dually) almost perfect in the sense that f^*F is (dually) almost perfect for each $f: \text{Spec}(A) \rightarrow X$.

The ∞ -category QC_X^\vee is particularly well-behaved when X is locally coherent. In particular, we have the following generalisation of Lemma A.42 to derived schemes that are *locally coherent*, i.e. which admit a Zariski cover coherent affine open subschemes or equivalently, for which every affine open subscheme is coherent [Gla89, Theorem 2.4.2, Corollary 2.4.5].

Proposition B.16. *Let X be a qcqs derived scheme which is locally coherent and write \mathfrak{U} for the poset of affine opens of X . Then the natural map*

$$\mathrm{Coh}_X^{\mathrm{op}} = \lim_{U \in \mathfrak{U}} \mathrm{Coh}_U^{\mathrm{op}} \longrightarrow \lim_{U \in \mathfrak{U}} \mathrm{QC}_U^{\vee} = \mathrm{QC}_X^{\vee}$$

induces an equivalence $\mathrm{Ind}(\mathrm{Coh}_X^{\mathrm{op}}) \simeq \mathrm{QC}_X^{\vee}$.

Proof. By the usual reduction principle for qcqs schemes, it suffices to verify that for $U_0 \subseteq X \supseteq U_1$ an open cover such that the assertion holds on U_0 , U_1 and $U_{01} = U_0 \cap U_1$, the assertion holds also for X . Since QC^{\vee} satisfies Zariski descent by Corollary A.39, it suffices to verify that $\mathrm{Ind}(\mathrm{Coh}^{\mathrm{op}})$ also satisfies Zariski descent.

To see this, let us write $j^* : \mathrm{Ind}(\mathrm{Coh}_X^{\mathrm{op}}) \rightleftarrows \mathrm{Ind}(\mathrm{Coh}_U^{\mathrm{op}}) : j_*$ for the adjoint pair associated to an open inclusion $j : U \hookrightarrow X$. Since j_* preserves colimits, this is an adjoint pair in Pr^{L} . As such, it is the dual with respect to the Lurie tensor product of the adjoint pair $j^* : \mathrm{Ind}(\mathrm{Coh}_X) \rightleftarrows \mathrm{Ind}(\mathrm{Coh}_U) : j_*$ considered in [Gai11]. Using this duality, one sees the counit $j^*j_* \rightarrow \mathrm{id}$ is an equivalence [Gai11, Lemma 4.1.1] and that $\mathrm{Ind}(\mathrm{Coh}^{\mathrm{op}})$ satisfies base change for open inclusions [Gai11, Lemma 3.6.9]. It follows that

$$(j_0^*, j_{01}^*, j_1^*) : \mathrm{Ind}(\mathrm{Coh}_X^{\mathrm{op}}) \longrightarrow \mathrm{Ind}(\mathrm{Coh}_{U_0}^{\mathrm{op}}) \times_{\mathrm{Ind}(\mathrm{Coh}_{U_{01}}^{\mathrm{op}})} \mathrm{Ind}(\mathrm{Coh}_{U_1}^{\mathrm{op}})$$

is an equivalence as in the proof of [Gai11, Proposition 4.2.1]: this functor has a colimit-preserving right adjoint sending a compatible triple (F_0, F_1, F_{01}) to the fibre product $j_{0*}F_0 \times_{j_{01*}F_{01}} j_{1*}F_1$. Using base change and the fact that $j_i^*j_{i*} \simeq \mathrm{id}$, one sees that the counit is an equivalence. To verify that the unit $F \rightarrow j_{0*}j_0^*F \times_{j_{01*}j_{01}^*F} j_{1*}j_1^*F$ is an equivalence, it suffices to treat the case where $F \in \mathrm{Coh}(X)^{\mathrm{op}}$. In that case, the result follows from descent for coherent sheaves. \square

B.3. Tangent complex. In this section, we will discuss the tangent complex of a map of prestacks $X \rightarrow S$ (if it exists). The main goal will be to establish a form of functoriality of the tangent complex, which is used in Section 5.2.

Let us start by recalling the pointwise definition of the tangent and cotangent complex of an S -prestack. Let $\pi : X \rightarrow S$ be an S -prestack and $x : \mathrm{Spec}(A) \rightarrow X$ a point, and consider the functor

$$(28) \quad \mathrm{Mod}_{A, \geq 0} \rightarrow \mathcal{S}; \quad M \mapsto \left\{ \begin{array}{ccc} \mathrm{Spec}(A) & \xrightarrow{x} & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A \oplus M) & \longrightarrow & \mathrm{Spec}(A) \xrightarrow{\pi(x)} S \end{array} \right\}.$$

We then have the following:

Definition B.17. Let X be an S -prestack and let $x : \mathrm{Spec}(A) \rightarrow X$ be a point.

- (1) We will say that X admits a *cotangent complex* at x (relative to S) if there exists a (necessarily unique) eventually connective module $L_{X/S, x} \in \mathrm{Mod}_{A, > \infty}$ corepresenting the functor (28).
- (2) We will say that X admits a (pro-coherent) *tangent complex* at x (relative to S) if the functor (28) defines an object $T_{X/S, x} \in \mathrm{QC}_A^{\vee}$.

If X is an S -prestack with deformation theory, then each map of rings $f : A \rightarrow B$ induces a natural equivalence

$$X(A \oplus M) \times_{X(A)} \{x\} \simeq X(B \oplus f_*M) \times_{X(B)} \{f^*x\}$$

for each connective A -module M . This implies that $L_{X/S, f^*x} \simeq f^*L_{X/S, x}$ and $T_{X/S, f^*x} \simeq f^*T_{X/S, x}$ whenever X admits a (co)tangent complex at the point x . Consequently, if X has deformation theory and admits a (co)tangent complex at each of its points, then these (co)tangent complexes glue to global objects on X :

Definition B.18. Let X be an S -prestack with deformation theory.

- (1) If X admits a cotangent complex at each point, then the *cotangent complex* $L_{X/S} \in \mathrm{QC}_X$ is the unique quasi-coherent sheaf such that $L_{X/S, x} \simeq x^*L_{X/S}$ for each point of X .
- (2) If X admits a tangent complex at each point, then the *tangent complex* $T_{X/S} \in \mathrm{QC}_X^\vee$ is the unique pro-coherent sheaf such that $T_{X/S, x} \simeq x^*T_{X/S}$ for each point of X .

Example B.19. Suppose that X is a left S -prestack with deformation theory. Then X admits a relative tangent complex at each point.

Having a tangent complex at each point requires a finiteness condition on $X \rightarrow S$ similar to Example B.19 (but at the linear level). However, there are many prestacks for which one can make sense of their tangent complex, without requiring the existence of a tangent complex at each point. For example, for every affine derived R -scheme X , we can define the tangent complex $T_{X/R} = L_{X/R}^\vee$ to be the pro-coherent dual of its cotangent complex. In order to define the tangent complex in such cases, and to make the functoriality of the tangent complex more explicit, we will need a somewhat technical argument.

Construction B.20. Let $\mathrm{Mod}_{\geq 0, *}$ \rightarrow Aff be the cocartesian fibration classifying the functor sending $f: \mathrm{Spec}(A) \rightarrow \mathrm{Spec}(B)$ to the direct image $f_*: \mathrm{Mod}_{A, \geq 0} \rightarrow \mathrm{Mod}_{B, \geq 0}$ and let us consider the following pullback

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & \mathrm{Fun}([1], \mathrm{PrStk}) \\ \downarrow & & \downarrow \mathrm{ev}_0 \\ \mathrm{Mod}_{\geq 0, *} & \longrightarrow & \mathrm{Aff} \longrightarrow \mathrm{PrStk} \end{array}$$

An object of \mathcal{M} corresponds a map of prestacks $x: \mathrm{Spec}(A) \rightarrow X$ together with a connective A -module M . The functor $\mathcal{M} \rightarrow \mathrm{PrStk}$ sending $(\mathrm{Spec}(A) \rightarrow X, M) \mapsto X$ is a cocartesian fibration and each fibre \mathcal{M}_X comes equipped with a cocartesian fibration $\mathcal{M}_X \rightarrow \mathrm{Aff}/_X$ parametrising the ∞ -category of connective modules at each point of X .

Let $\mathrm{Fun}_{/\mathrm{PrStk}}(\mathcal{M}, \mathcal{S}) \rightarrow \mathrm{PrStk}$ be the relative functor ∞ -category over PrStk ; by [HTT] this is a cartesian fibration, classified by the functor $\mathrm{PrStk}^{\mathrm{op}} \rightarrow \mathrm{Cat}_\infty$ sending a prestack X to $\mathrm{Fun}(\mathcal{M}_X, \mathcal{S})$ and sending $f: X \rightarrow Y$ to restriction along $\mathcal{M}_X \rightarrow \mathcal{M}_Y$. Let us write

$$\mathcal{D} \hookrightarrow \mathrm{Fun}_{/\mathrm{PrStk}}(\mathcal{M}, \mathcal{S})$$

for the full subcategory of tuples (X, F) consisting of a prestack X and a functor $F: \mathcal{M}_X \rightarrow \mathcal{S}$ satisfying the following two properties:

- (1) F sends cocartesian arrows with respect to $\mathcal{M}_X \rightarrow \mathrm{Aff}/_X$ to equivalences.
- (2) For each $x: \mathrm{Spec}(A) \rightarrow X$, the restriction of F to the fibre

$$F_x: \mathcal{M}_{X, x} \simeq \mathrm{Mod}_{A, \geq 0} \rightarrow \mathcal{S}$$

is accessible and preserves the terminal object, pullbacks along π_0 -surjections and limits of almost eventually constant towers.

Let us write $\mathcal{D}^{\text{aft}} \subseteq \mathcal{D}$ and $\mathcal{D}^{\text{corep}} \subseteq \mathcal{D}$ for the subcategories of tuples (X, F) where, in addition to condition (2), each restriction F_x is almost finitely presented, respectively corepresentable by an eventually connective A -module.

One readily verifies that for any map of prestacks $f: X \rightarrow Y$, restriction along $\mathcal{M}_X \rightarrow \mathcal{M}_Y$ preserves these conditions, so that for any cartesian arrow $(X, G) \rightarrow (Y, F)$ with $(Y, F) \in \mathcal{D}$, (X, G) is in \mathcal{D} as well (and similarly in the other cases). Consequently, we obtain a diagram of cartesian fibrations (and maps preserving cartesian arrows)

$$\begin{array}{ccccc} \mathcal{D}^{\text{aft}} & \longleftrightarrow & \mathcal{D} & \longleftrightarrow & \mathcal{D}^{\text{corep}} \\ & \searrow & \downarrow & \swarrow & \\ & & \text{PrStk} & & \end{array}$$

Proposition B.21. *Each of the above three cartesian fibrations classifies a functor $\text{PrStk}^{\text{op}} \rightarrow \text{Cat}_{\infty}$ preserving limits. Furthermore, the cartesian fibration $\mathcal{D}^{\text{aft}} \rightarrow \text{PrStk}$ is classified by the functor $X \mapsto \text{QC}_X^{\vee}$ and the cartesian fibration $\mathcal{D}^{\text{corep}} \rightarrow \text{PrStk}$ is classified by the functor $X \mapsto (\text{QC}_X^-)^{\text{op}}$.*

Proof. Let X be a prestack and consider the cocartesian fibration $\mathcal{M}_X \rightarrow \text{Aff}/_X$ that classifies the diagram of categories $\text{Aff}/_X \rightarrow \text{Cat}_{\infty}$ sending a map of affines $f: \text{Spec}(A) \rightarrow \text{Spec}(B)$ over X to $f_*: \text{Mod}_{A, \geq 0} \rightarrow \text{Mod}_{B, \geq 0}$. The localisation of \mathcal{M}_X at the cocartesian arrows classifies the colimit of this diagram of categories. Condition (1) therefore implies that \mathcal{D}_X is a full subcategory of the limit

$$\lim_{\text{Spec}(A) \in \text{Aff}/_X} \text{Fun}(\text{Mod}_{A, \geq 0}, \mathcal{S})$$

where each morphism $f: \text{Spec}(A) \rightarrow \text{Spec}(B)$ in $\text{Aff}/_X$ is sent to restriction along f_* . Using this, Condition (2) then identifies

$$\mathcal{D}_X \simeq \lim_{\text{Spec}(A) \in \text{Aff}/_X} \text{Fun}_{\text{rex, acc}}(\text{Mod}_{A, \geq 0}^+, \mathcal{S})$$

with the limit of the ∞ -categories of accessible functors that are right exact (or equivalently, that preserve the terminal object and pullbacks along π_0 -surjections). This identification is natural in X , so that $X \mapsto \mathcal{D}_X$ evidently preserves limits. The categories $\mathcal{D}_X^{\text{aft}}$ and $\mathcal{D}_X^{\text{corep}}$ are the limits over $\text{Aff}/_X$ of the diagram of full subcategories of $\text{Fun}_{\text{rex, acc}}(\text{Mod}_{A, \geq 0}^+, \mathcal{S})$ spanned by almost finitely presented and corepresentable functors. It follows that there are natural equivalences $\mathcal{D}^{\text{aft}} \simeq \text{QC}_X^{\vee}$ and $\mathcal{D}^{\text{corep}} \simeq (\text{QC}_X^-)^{\text{op}}$. \square

Observation B.22. *Given a prestack X , the inclusion*

$$\text{QC}_X^{\vee} = \lim_{\text{Spec}(A) \rightarrow X} \text{QC}_A^{\vee} \hookrightarrow \lim_{\text{Spec}(A) \rightarrow X} \text{Fun}_{\text{rex, acc}}(\text{Mod}_{A, \geq 0}^+, \mathcal{S}) \simeq \mathcal{D}_X$$

preserves colimits. Since QC_X^{\vee} is presentable, it follows that there is a right adjoint $\mathcal{D}_X \rightarrow \text{QC}_X^{\vee}$. Given $F \in \mathcal{D}_X$, we will refer to its image under this right adjoint F^{aft} as the almost finitely presented approximation to F . Let us point out that the right adjoints $\mathcal{D}_X \rightarrow \text{QC}_X^{\vee}$ do not assemble into a right adjoint $\mathcal{D} \rightarrow \mathcal{D}^{\text{aft}}$.

Let us now use this construction to give a functorial description of the tangent complex.

Construction B.23. Let us write $\text{Fun}(\Delta^1, \text{PrStk})^{\text{def}} \subseteq \text{Fun}(\Delta^1, \text{PrStk})$ for the full subcategory of maps $X \rightarrow S$ that have deformation theory. We define a functor

$$\Phi: \mathcal{M} \times_{\text{PrStk}} \text{Fun}(\Delta^1, \text{PrStk})^{\text{def}} \rightarrow \mathcal{S};$$

as follows. The domain of Φ is the fibre product of the cocartesian fibration $\mathcal{M} \rightarrow \text{PrStk}$ from Construction B.20 and the functor $\text{ev}_0: \text{Fun}(\Delta^1, \text{PrStk})^{\text{def}} \rightarrow \text{PrStk}$ taking the domain. We then define Φ by

$$\Phi(x: \text{Spec}(A) \rightarrow X, M, X \rightarrow S) = \left\{ \begin{array}{ccc} \text{Spec}(A) & \xrightarrow{x} & X \\ \downarrow & \nearrow & \downarrow \\ \text{Spec}(A \oplus M) & \longrightarrow & \text{Spec}(A) \xrightarrow{\pi(x)} S \end{array} \right\}.$$

The functor Φ is adjoint to

$$\begin{array}{ccc} \text{Fun}(\Delta^1, \text{PrStk})^{\text{def}} & \xrightarrow{\quad} & \text{Fun}/\text{PrStk}(\mathcal{M}, S) \\ & \searrow \text{ev}_0 & \swarrow \\ & \text{PrStk} & \end{array}$$

sending $X \rightarrow S$ to the restriction $\Phi(-, X \rightarrow S): \mathcal{M}_X \rightarrow S$. The fact that $X \rightarrow S$ has deformation theory implies that $\Phi(-, X \rightarrow S)$ satisfies conditions (1) and (2) from Construction B.20. Consequently, we obtain a functor that we will denote by

$$(29) \quad \begin{array}{ccc} \text{Fun}(\Delta^1, \text{PrStk})^{\text{def}} & \xrightarrow{T^{\text{pre}}} & \mathcal{D} \\ & \searrow \text{ev}_0 & \swarrow \\ & \text{PrStk} & \end{array}$$

If $X \rightarrow S$ is left, then $T^{\text{pre}}(X/S)$ is contained in \mathcal{D}^{aft} . Under the equivalence $\mathcal{D}_X^{\text{aft}} \simeq \text{QC}^\vee(X)$, this corresponds to the tangent complex $T_{X/S}$ from Definition B.18. If $X \rightarrow S$ admits cotangent complexes at all points of X , then $T^{\text{pre}}(X/S) \in \mathcal{D}^{\text{corep}}$ and corresponds under the equivalence $\mathcal{D}_X^{\text{corep}} \simeq (\text{QC}_X^-)^{\text{op}}$ to the cotangent complex $L_{X/S}$ from Definition B.18.

From the formula for Φ , one readily sees:

Lemma B.24. *The functor T^{pre} preserves pullbacks.*

Since $\mathcal{D}^{\text{corep}} \subseteq \mathcal{D}$ is closed under pullbacks, this reproduces for example the well-known behaviour of the cotangent complex with respect to base change and composition.

Let us now use the functor T^{pre} to define the tangent complex of a prestack with deformation theory:

Definition B.25. Let X be an S -prestack with deformation theory. We define the (pro-coherent) *tangent complex of X over S* to be the almost finitely presented approximation $T_{X/S} = T_{X/S}^{\text{pre, aft}} \in \text{QC}_X^\vee$ from Observation B.22.

Example B.26. If X is a *left* S -prestack with deformation theory, this coincides with the pointwise definition of tangent complex from Definition B.18. Indeed, since $X \rightarrow S$ is left, $T_{X/S}^{\text{pre}}$ is already contained in $\mathcal{D}_X^{\text{aft}} \subseteq \mathcal{D}_X$.

Example B.27. Suppose that X is a (not necessarily left) S -prestack with deformation theory and a cotangent complex at each point. Then there is an equivalence $T_{X/S} = L_{X/S}^\vee$ in QC_X^\vee , where $L_{X/S}^\vee$ is the pro-coherent linear dual of $L_{X/S} \in (\text{QC}_X^-)^{\text{op}}$ (see Example A.32). Since the right adjoint $\text{QC}_X^\vee \rightarrow \text{QC}_X$ preserves the linear dual of a quasi-coherent sheaf, one can think of $T_{X/S} \in \text{QC}_X^\vee$ as

a pro-coherent refinement of the *coherator* of the tangent sheaf on X , in the sense of [Sta19, Tag 08D6].

The tangent complex $T_{X/S}$ is much less well-behaved when $X \rightarrow S$ is not laft. For example, given a map of S -prestacks with deformation theory $X \rightarrow Y$, there is no direct map from $T_{X/S}$ to $f^*T_{Y/S}$. For instance, if X and Y admit a cotangent complex at each point, there is only a zig-zag of maps $L_{X/S}^\vee \rightarrow (f^*L_{Y/S})^\vee \leftarrow f^*(L_{Y/S}^\vee)$ because base change does not commute with duality without finiteness assumptions. Let us nonetheless record the following properties:

Proposition B.28. *Let $f: X \rightarrow Y$ be a map of S -prestacks with deformation theory. Then the following hold.*

- (1) *There is a fibre sequence $T_{X/Y} \rightarrow T_{X/S} \rightarrow (f^*T_{Y/S}^{\text{pre}})^{\text{aft}}$.*
- (2) *There is a zig-zag of maps $T_{X/S} \rightarrow (f^*T_{Y/S}^{\text{pre}})^{\text{aft}} \leftarrow f^*T_{Y/S}$ in QC_X^\vee .*
- (3) *If f is a formally étale, then there is a map $f^*T_{Y/S} \rightarrow T_{X/S}$.*
- (4) *$T_{X/S} \simeq T_{X/S_\Delta}$.*

If X and Y are furthermore laft S -prestacks with deformation theory, then:

- (5) *There is natural map $T_{X/S} \rightarrow f^*T_{Y/S}$ that fits into a fibre sequence $T_{X/Y} \rightarrow T_{X/S} \rightarrow f^*T_{Y/S}$ in QC_X^\vee .*
- (6) *Given a morphism $S' \rightarrow S$, there is a natural equivalence $T_{X \times_S S'/S'} = p^*T_{X/S}$ where $p: X \times_S S' \rightarrow X$ is the evident projection.*

Proof. The first three assertions follow from the natural fibre sequence $T_{X/Y}^{\text{pre}} \rightarrow T_{X/S}^{\text{pre}} \rightarrow f^*T_{Y/S}^{\text{pre}}$ in \mathcal{D}_X by taking almost finitely presented approximations, using that $T_{X/Y}^{\text{pre}} \simeq 0$ if $X \rightarrow Y$ is formally étale. Likewise, (4) follows from the fact that $T_{X/S}^{\text{pre}} \simeq T_{X/S_\Delta}^{\text{pre}}$. Assertion (5) follows from the fact that $T_{Y/S} = T_{Y/S}^{\text{pre}}$ and (6) follows from the equivalence $T_{X \times_S S'/S'}^{\text{pre}} = p^*T_{X/S}^{\text{pre}}$, which follows from Lemma B.24. \square

The somewhat problematic behaviour of $T_{X/S}$ when $X \rightarrow S$ is not laft will not be a problem for us, because we are mainly interested in the ∞ -category $(\text{QC}_X^\vee)_{/T_{X/S}}$ of pro-coherent sheaves over $T_{X/S}$. This ∞ -category is equivalent to the ∞ -category $(\text{QC}_X^\vee)_{/T_{X/S}^{\text{pre}}}$ of pro-coherent sheaves equipped with a map $F \rightarrow T_{X/S}^{\text{pre}}$ in \mathcal{D}_X , and $T_{X/S}^{\text{pre}}$ is very well-behaved.

For instance, each laft map $X \rightarrow Y$ of S -prestacks with deformation theory gives rise to a natural object $T_{X/Y} \in (\text{QC}_X^\vee)_{/T_{X/S}}$, or equivalently, in $(\text{QC}_X^\vee)_{/T_{X/S}^{\text{pre}}}$. Let us conclude with a functorial description of this construction:

Construction B.29. Let us fix a base prestack S and write $\text{PrStk}_S^{\text{def,laft-map}}$ for the subcategory of PrStk_S whose objects are prestacks with deformation theory over S and whose morphisms are the laft morphisms between such S -prestacks. Let us write

$$\mathcal{E} = \mathcal{D} \times_{\text{PrStk}} \text{PrStk}_S^{\text{def,laft-map}}, \quad \text{QC}^\vee = \mathcal{D}^{\text{aft}} \times_{\text{PrStk}} \text{PrStk}_S^{\text{def,laft-map}}$$

for the evident restrictions. The cartesian fibration $\mathcal{E} \rightarrow \text{PrStk}_S^{\text{def,laft-map}}$ then admits a section T^{pre} sending each $X \mapsto T_{X/S}^{\text{pre}}$. Taking fibrewise over-categories then produces a diagram that we

will denote by

$$\begin{array}{ccc} \mathrm{QC}_{/T}^{\vee} := \mathrm{QC}^{\vee} \times_{\mathcal{E}} \mathcal{E}_{/T^{\mathrm{pre}}} & \xleftarrow{\quad} & \mathcal{E}_{/T^{\mathrm{pre}}} \\ & \searrow & \swarrow \\ & \mathrm{PrStk}_S^{\mathrm{def,laft-map}} & \end{array}$$

At the level of the fibres over a prestack X , this is given by

$$(\mathrm{QC}_X^{\vee})_{/T_{X/S}} \simeq (\mathrm{QC}_X^{\vee}) \times_{\mathcal{D}_X} (\mathcal{D}_X)_{/T_{X/S}^{\mathrm{pre}}} \xleftarrow{\quad} (\mathcal{D}_X)_{/T_{X/S}^{\mathrm{pre}}}.$$

Lemma B.30. *The projection $\mathrm{QC}_{/T}^{\vee} \rightarrow \mathrm{PrStk}_S^{\mathrm{def,laft-map}}$ is a cartesian fibration.*

Proof. Let us first consider the cartesian fibration $\pi: \mathcal{E} \rightarrow \mathrm{PrStk}_S^{\mathrm{def,laft-map}}$. This classifies a functor sending X to $\mathcal{E}_X = \lim_{\mathrm{Spec}(A) \rightarrow X} \mathrm{Fun}_{\mathrm{ex,acc}}(\mathrm{Mod}_A^+, \mathrm{Sp})$ and $f: X \rightarrow Y$ to the evident functor between limits

$$f^*: \mathcal{E}_Y = \lim_{\mathrm{Spec}(A) \rightarrow Y} \mathrm{Fun}_{\mathrm{ex,acc}}(\mathrm{Mod}_A^+, \mathrm{Sp}) \rightarrow \lim_{\mathrm{Spec}(A) \rightarrow X} \mathrm{Fun}_{\mathrm{ex,acc}}(\mathrm{Mod}_A^+, \mathrm{Sp}) = \mathcal{E}_X.$$

Note that each \mathcal{E}_X is a stable ∞ -category and that f^* is exact. This implies that \mathcal{E} admits finite limits and that π admits a fully faithful right adjoint (taking the terminal object fibrewise). The projection $\mathcal{E}_{/T^{\mathrm{pre}}} \rightarrow \mathrm{PrStk}_S^{\mathrm{def,laft-map}}$ is then a cartesian fibration as well: indeed, given a map $f: X \rightarrow Y$ and $F \rightarrow T_{Y/S}^{\mathrm{pre}}$ in \mathcal{E}_Y , the cartesian lift of f in $\mathcal{E}_{/T^{\mathrm{pre}}}$ corresponds to the following pullback square in \mathcal{E} :

$$\begin{array}{ccc} T_{X/S}^{\mathrm{pre}} \times_{T_{Y/X}^{\mathrm{pre}}} F & \xrightarrow{\tilde{f}} & F \\ \downarrow & & \downarrow \\ T_{X/S}^{\mathrm{pre}} & \longrightarrow & T_{Y/X}^{\mathrm{pre}}. \end{array}$$

More explicitly, this is given by the pullback $T_{X/S}^{\mathrm{pre}} \times_{f^* T_{Y/X}^{\mathrm{pre}}} f^* F$ in the fibre \mathcal{E}_X .

Finally, it suffices that the full subcategory $\mathrm{QC}_{/T}^{\vee} \subseteq \mathcal{E}_{/T^{\mathrm{pre}}}$ is stable under these cartesian arrows. For this, suppose that $F \in \mathrm{QC}_Y^{\vee} \subseteq \mathcal{E}_Y$ and consider the fibre sequence in \mathcal{E}_X

$$T_{X/Y}^{\mathrm{pre}} \rightarrow T_{X/S}^{\mathrm{pre}} \times_{f^* T_{Y/X}^{\mathrm{pre}}} f^* F \rightarrow f^* F.$$

The base change functor f^* sends QC_Y^{\vee} to QC_X^{\vee} , so $f^* F \in \mathrm{QC}_X^{\vee}$. Since $f: X \rightarrow Y$ was assumed to be laft, $T_{X/Y}^{\mathrm{pre}} = T_{X/Y} \in \mathrm{QC}_X^{\vee}$ as well so that the middle term is contained in QC_X^{\vee} , as desired. \square

Remark B.31. Informally, Lemma B.30 asserts that there is a functor sending each S -prestack X with deformation theory to the ∞ -category $(\mathrm{QC}_X^{\vee})_{/T_{X/S}}$, and each map $f: X' \rightarrow X$ to a functor

$$f^{\sharp}: (\mathrm{QC}_X^{\vee})_{/T_{X/S}} \longrightarrow (\mathrm{QC}_{X'}^{\vee})_{/T_{X'/S}}; \quad (F \rightarrow T_{X/S}) \longmapsto T_{X'/S} \times_{f^*(T_{X/S}^{\mathrm{pre}})_{\mathrm{aft}}} F.$$

In particular, when f is formally étale, then $f^{\sharp}(F \rightarrow T_{X/S}) = f^* F$ by Proposition B.28.

If $f: X \rightarrow Y$ is a map of S -prestacks with deformation theory and a cotangent complex, Example B.27 identifies the resulting base change functor with $f^{\sharp}(F) = T_{X/S} \times_{(f^* L_{Y/S})^{\vee}} f^* F$, using the natural map $f^*(T_{Y/S}) = f^*(L_{Y/S}^{\vee}) \rightarrow (f^* L_{Y/S})^{\vee}$.

Proposition B.32. *There is a map of cartesian fibrations preserving cartesian arrows*

$$\begin{array}{ccc} \mathrm{Fun}(\Delta^1, \mathrm{PrStk}_S^{\mathrm{def}, \mathrm{laft}\text{-map}}) & \xrightarrow{T} & \mathrm{QC}_{/T}^\vee \\ & \searrow \mathrm{evo} & \swarrow \\ & \mathrm{PrStk}_S^{\mathrm{def}, \mathrm{laft}\text{-map}} & \end{array}$$

sending each $X \rightarrow Y$ to $T_{X/Y} \rightarrow T_{X/S}$. Furthermore, the induced functors between fibres preserve limits.

In other words, for every square of laft maps between S -prestacks with deformation theory

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

there is a natural map $T_{X'/Y'} \rightarrow f^\#(T_{X/Y})$ where $f^\#$ is as in Remark B.31. The fact that T preserves cartesian arrows corresponds to this map being an equivalence if $Y' = Y$. We are mostly interested for in the situation of (locally coherent) derived schemes X (over a base S): the above then provides a functor sending $X \rightarrow Y$ a map $T_{X/Y} \rightarrow L_{X/Y}^\vee$ in QC_X^\vee , functorial with respect to laft maps of schemes $X' \rightarrow X$.

Proof. The desired functor arises formally from the functor T^{pre} in (29), using that $T_{X/Y} = T_{X/Y}^{\mathrm{pre}}$ if $X \rightarrow Y$ is laft (Example B.26). To see that it preserves cartesian arrows, note that T^{pre} sends

$$\begin{array}{ccc} X' \xrightarrow{f} X & & T_{X'/Y}^{\mathrm{pre}} \longrightarrow T_{X/Y}^{\mathrm{pre}} \\ \downarrow & \longmapsto & \downarrow \\ Y \xrightarrow{\mathrm{id}} Y & & T_{X'/S}^{\mathrm{pre}} \longrightarrow T_{X/S}^{\mathrm{pre}} \end{array}$$

Here the left square defines a cartesian arrow covering f in the domain. To see that its image is cartesian, it suffices to verify that the right square is cartesian in \mathcal{E} ; this follows from the fact that the functor T^{pre} from (29) preserves pullbacks by Lemma B.24. \square

We conclude by recording the following property of the induced functors $T_{X/-}$ between the fibres.

Proposition B.33. *Let X be an S -prestack with deformation theory and consider the composite*

$$\Psi: (\mathrm{PrStk}_S^{\mathrm{def}, \mathrm{laft}\text{-map}})_{X/-} \xrightarrow{T_{X/-}} (\mathrm{QC}_X^\vee)_{/T_{X/S}} \xrightarrow{\mathrm{cofib}} \mathrm{QC}_X^\vee$$

where the first functor is induced by the functor T from Proposition B.32 and the second functor sends $F \rightarrow T_{X/S}$ to its cofibre. This functor admits a left adjoint, sending each $F \in \mathrm{QC}_X^\vee$ to a map $X \rightarrow X_F$ that is furthermore a nil-isomorphism. In particular, the functor $T_{X/-}$ preserves limits.

Proof. Unravelling the definitions, $\Psi(f: X \rightarrow Y) = (f^* T_{Y/S}^{\mathrm{pre}})^{\mathrm{aft}}$. To describe its left adjoint, recall from Construction B.20 that each $F \in \mathrm{QC}_X^\vee$ defines a functor $F: \mathcal{M}_X \rightarrow \mathcal{S}$ on the ∞ -category of tuples $(x: \mathrm{Spec}(A) \rightarrow X, M \in \mathrm{Mod}_{A, \geq 0}^+)$. Let $\mathcal{F} \rightarrow \mathcal{M}_X$ be the (cocartesian) unstraightening of this functor. We then define

$$X_F = \mathrm{colim}_{(A, M) \in \mathcal{F}^{\mathrm{op}}} \mathrm{Spec}(A \oplus M)$$

as the colimit in convergent S -prestacks. There are natural maps $X \rightarrow X_F \rightarrow X$. The first map is the natural map from the colimit indexed by the subcategory of \mathcal{F}^{op} on those (A, M) where $M \simeq 0$ is the zero module. The second map is the colimit over the objects of \mathcal{F} , corresponding to tuples

$$(x: \text{Spec}(A) \rightarrow X, M \in \text{Mod}_{A, \geq 0}^+, y \in F(M)),$$

of the maps $\text{Spec}(A \oplus M) \rightarrow \text{Spec}(A) \xrightarrow{x} X$. Given a map $f: X \rightarrow Y$ to an S -prestack with deformation theory, a map $X_F \rightarrow Y$ of S -prestacks under X corresponds to a natural transformation $F \rightarrow f^*T_{Y/S}^{\text{pre}}$ between functors $\mathcal{M}_X \rightarrow \mathcal{S}$, i.e. a map $F \rightarrow (f^*T_{Y/S}^{\text{pre}})^{\text{aft}} = \Psi(f: X \rightarrow Y)$ in QC_X^\vee .

Hence, it remains to show that $X \rightarrow X_F$ is indeed a left nil-isomorphism and that X_F has deformation theory. For this it suffices to check that for each $x: \text{Spec}(A) \rightarrow X$ with A eventually coconnective, the base change x^*X_F is a left A -prestack with deformation theory that is nil-isomorphic to $\text{Spec}(A)$. This base change can be identified as follows. Let \mathcal{F}_x denote the unstraightening of the restriction $x^*F: \text{Mod}_{A, \geq 0}^+ \rightarrow \mathcal{S}$ of F to the fibre of $(\mathcal{M}_X) \times_{\text{Aff}_X} \{x\}$. Then

$$(30) \quad x^*X_F = \text{colim}_{M \in \mathcal{F}_x^{\text{op}}} \text{Spec}(A \oplus M)$$

where the colimit is taken in convergent A -prestacks. This is clearly nil-isomorphic to $\text{Spec}(A)$, as every $\text{Spec}(A \oplus M)$ is nil-isomorphic to $\text{Spec}(A)$. The category $\mathcal{F}_x^{\text{op}}$ is filtered, so that x^*X_F has deformation theory, being a filtered colimit of A -prestacks with deformation theory.

To see that x^*X_F is left, observe that for any eventually coconnective animated A -algebra B , one can either compute the value $x^*X_F(B)$ using the colimit (30), or as follows. Let us write $\text{Sqz}_{/B} = \text{Mod}_{A, \geq 0}^+ \times_{\text{CAlg}_A^{\text{an}}} \text{CAlg}_{A/B}^{\text{an}}$ for the ∞ -category of A -modules $M \in \text{Mod}_{A, \geq 0}^+$ equipped with a map of animated A -algebras $A \oplus M \rightarrow B$. Then

$$x^*X_F(B) = \text{colim}_{M \in \text{Sqz}_{/B}} x^*F(M).$$

If B is n -coconnective, it suffices to take the colimit over the cofinal subcategory $\text{Sqz}_{\leq n/B} \subseteq \text{Sqz}_{/B}$ of n -coconnective A -modules M equipped with a map $A \oplus M \rightarrow B$. Note that $\text{Sqz}_{\leq n/B}$ is compactly generated with compact objects given by

$$(M, A \oplus M \rightarrow B),$$

where M is the n -truncation of a perfect connective A -module; let us refer to such n -coconnective modules as n -compact. Since x^*F preserves colimits of filtered diagrams of n -coconnective A -modules, one can thus reduce the above colimit to the colimit over the full subcategory $\text{Sqz}_{\leq n/B}^\omega \subseteq \text{Sqz}_{\leq n/B}$ of n -compact A -modules M with a map $A \oplus M \rightarrow B$. Equivalently, this means that

$$x^*X_F(B) = \text{colim}_{M \in \mathcal{F}_{x, \leq n}^{\omega, \text{op}}} \text{Map}_{\text{CAlg}_A^{\text{an}}}(A \oplus M, B) \simeq \text{colim}_{M \in \mathcal{F}_{x, \leq n}^{\omega, \text{op}}} \text{Map}_{\tau_{\leq n} \text{CAlg}_A^{\text{an}}}(\tau_{\leq n} A \oplus M, B)$$

where $\mathcal{F}_{x, \leq n}^\omega \subseteq \mathcal{F}_x$ is the full subcategory of tuples $(M, y \in x^*F(M))$ where M is n -compact.

We conclude that the restriction of x^*X_F to $\tau_{\leq n} \text{CAlg}_A^{\text{an}}$ is equivalent to a colimit of representables $\text{Spec}(\tau_{\leq n} A \oplus M)$ where M is an n -compact A -module. By Lemma 2.20, each square zero extension $\tau_{\leq n} A \oplus M$ by an n -compact A -module defines a compact object in $\tau_{\leq n} \text{CAlg}_A^{\text{an}}$. It follows that x^*X_F restricts to a functor on $\tau_{\leq n} \text{CAlg}_A^{\text{an}}$ preserving filtered colimits, so that x^*X_F is indeed left. \square

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