Topology Advanced Class, Michaelmas 2018, Oxford University

AN INTRODUCTION TO CHROMATIC HOMOTOPY THEORY

(1) **Overview.**

(2) **Complex Oriented Cohomology Theories.**
Review the geometric definition of complex K-theory via vector bundles ([Ati89], [Hat03]) and complex cobordism via manifolds ([Sto15]). Describe their values on a point.
Explain the Eilenberg-Steenrod axioms and Brown's representability theorem ([Bro62]), thereby introduing the notion of a spectrum. Give a direct construction of the ring spectra KU and MU in terms of classifying spaces of unitary groups and their Thom spaces.
Introduce the notion of a complex oriented cohomology theory following [Hop99, Sections 1].

(3) **Formal Group Laws and Complex Cobordism.**
Explain why MU is the universal complex oriented cohomology theory ([Lur10, Lecture 6]). Describe how these theories give rise to formal group laws and introduce the Lazard ring L ([Rav03, Theorem A2.1.8., Theorem A2.1.10]). State Quillen's Theorem $L \cong MU_*$ ([Ada74, Part II], [Hop99, Sections $1-4$], [Lur10, Lectures 2-11]).
Refine Quillen's Theorem as follows. First show that the pair of functors $(FGL(-), SI(-))$ is representable by two rings (L, LB) ([Rav03, Proposition A2.1.15]). Then observe that (L, LB) is a *Hopf algebroid* ([Rav03, Definition A1.1.1]), a notion you should introduce. Construct the Hopf algebroid (MU_*, MU_*MU) and state the Landweber-Novikov isomorphism $(L, LB) \cong (MU_*, MU_*MU)$ ([Rav03, Theorem 4.1.11]).

(4) **The Brown-Peterson Spectrum.**
Over $\mathbb{Q}$, every formal group law is strictly isomorphic to $\hat{\mathbb{G}}_a$ ([Rav03, Theorem A2.1.6]). Cartier's Theorem ([Rav03, Definition A2.1.18]) asserts that over $\mathbb{Z}_{(p)}$, every formal group law is strictly isomorphic to a *p-typical* one ([Rav03, Definition A2.1.17]). Define the idempotent $\phi : L\otimes\mathbb{Z}_{(p)} \to L\otimes\mathbb{Z}_{(p)}$ and use it to construct the universal p-typical formal group law on the ring V ([Rav03, A2.1.25]). Introduce Hazewinkel generators for V ([Rav03, A2.2.1]). Lift ϕ to the Quillen idempotent on $MU_{(p)}$, and use this to define the complex oriented ring spectrum BP ([Rav03, p.107]). Enhance V to a Hopf algebroid (V, VT) ([Rav03, p.347]) and state the "p-typical Landweber-Novikov isomorphism" $(V, VT) \cong (BP_*, BP_*BP)$ ([Rav03, Theorem 4.1.19]). Describe the structure maps of this Hopf algebroid in terms of ℓ_i's as in [Rav03, A2.1.27] and explain the "mod p formula" [Rav76, Theorem 1] for the right unit η_R.

(5) **The Adams-Novikov Spectral Sequence and Invariant Ideals.**
Construct the Adams Spectral Sequence for a generalised homology theory E ([Rav03, Chapter 2]) using the theory of comodules over Hopf algebroids ([Rav03, Appendix A1]). For $E = BP$, the E_2-page of the resulting Adams-Novikov spectral sequence is given by $\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*)$. Present the figure depicted on ([Rav03, p.13]).
Prove Landweber-Morava's Invariant Prime Ideal Theorem ([Rav03, Theorem 4.3.2]). Introduce Johnson-Wilson theory $E(n)$ and Morava K-theory $K(n)$ by following [Rav03, p.111-112]. If there is time, mention the Landweber Exact Functor Theorem and the Conner-Floyd isomorphism $MU^*(X) \underset{MU^*}{\otimes} KU^* \cong KU^*(X)$ ([CF66]) .

(6) **Smith-Toda complexes and Greek Letter Elements.**
Construct Greek letter elements $\alpha_t^{(n)}$ in the E_2-page of the ANSS ([MRW77, Section 3.B], [Rav03, Section 5.1]). *Caveat*: the ℓ_i's in [Rav03] are called m_i's in [MRW77]).
Construct the Adams self-map on Moore spaces ([Ada, Theorem 1.7, Section 12], or spell out [Rav03, p.18]). Deduce that for p odd, each element in the α-family represents a permanent cycle in the ANSS ([MRW77, p.478], no need to prove nonvanishing of the α-family here). Discuss the image of J ([Rav03, Section 5.3]). State existence results for higher Smith-Toda complexes (cf. [Nav10, Introduction], [Rav03, Theorem 5.5.2], [Smi70], [Tod71]).

(7) **The Chromatic Spectral Sequence.**
Set up the chromatic spectral sequence ([MRW77, Section 3.A], [Rav03, Proposition 5.1.8]). Describe the Hopf algebra $K(n)_*K(n)$ ([MRW77, (3.14)]) and discuss the Change-of-Rings Theorem $\mathrm{Ext}^*_{BP_*BP}(BP_*, M) \cong \mathrm{Ext}^*_{K(n)_*K(n)}(K(n)_*, K(n)_* \otimes_{BP_*} M)$ ([MR77, Theorem 2.10], [MRW77, Theorem 3.15]).
Prove nontriviality of the α- and β-family ([MRW77, Theorem 2.2.a)] and [MRW77, Theorem 2.6] both follow from [MRW77, Corollary 4.8]) and state the nontriviality of the γ-family ([MRW77, Theorem 2.7]).

(8) **The Chromatic Tower.**
Review Bousfield localisation ([Rav84, Chapter 1-2], [Bou75][Bou79], [Lur10, Lecture 20]). Define the notion of a type n complex.
Discuss $L_n(BP)$ ([Rav84, Theorem 6.2]) and state Ravenel's Localization Conjecture [Rav84, Conjecture 5.8]. Outline a proof (see [Rav16, Theorem 7.5.2]) assuming Mitchell's Theorem on existence of type n complexes [Mit85]. If there is time, you could mention the Smash Product Theorem [Rav16, Theorem 7.5.6]).
Set up the chromatic tower ([Rav16, Definition 7.5.3]) and state the Chromatic Convergence Theorem ([Rav16, Theorem 7.5.7]). If there is time, describe the chromatic fracture square $L_n(X) \simeq L_{n-1}(X) \times^h_{L_{n-1}(L_{K(n)}(X))} L_{K(n)}(X)$.

(9) **Nilpotence and Periodicity.**
State the Nilpotence Theorem ([DHS88], [HS98]) and use it to deduce Nishida's Nilpotence Theorem ([Lur10, Lecture 25]) and the Thick Subcategory Theorem ([Lur10, Lecture 26]). State the Periodicity Theorem ([HS98]) and deduce it from the Thick Subcategory Theorem assuming that complexes of type n with v_n-self maps exist ([Lur10, Lecture 27]).
Discuss how the Periodicity Theorem allows us to generate periodic families in the homotopy groups of spheres ([BL10, Introduction]). If there is time, you could discuss v_n-periodic homotopy groups ([AM99, Appendix A]) and the Bousfield-Kuhn functor.

References

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