Topics in Koszul Duality, Michaelmas 2019, Oxford University

LECTURE 1: OVERVIEW

The goal of these lectures is to give an introduction to Koszul duality, a simple, yet powerful, principle applicable in algebraic geometry and topology.

We illustrate this with three classical examples:

Homotopy groups of spheres. The theory of Koszul algebras leads to a small model for the E_2-page of the Adams spectral sequence – the so-called "Λ-algebra", cf. [BCK$^+$66][Pri70]. The Adams spectral sequence converges to the homotopy groups of spheres.

Coherent sheaves. The bounded derived category of coherent sheaves on $\mathbb{P}^n$ is equivalent to the stable graded module category of the exterior algebra $E[X_0, ..., X_n]$, cf. [BGG78][Ben17].

Deformation theory. Infinitesimal deformations of algebro-geometric objects in characteristic zero [Lur10][Pri10] and in characteristic p [BM19] are governed by Lie algebras. Over an algebraically closed field of characteristic 0, this can be used to prove the Bogomolov–Tian–Todorov theorem, which asserts that deformations of Calabi–Yau varieties are unobstructed, cf. [IM10]. In characteristic p, finding criteria for the unobstructedness of Calabi–Yau varieties is an active subject of research, cf. [AZ18, Conjecture 1.1].

In this first lecture, we will introduce some background, specify what we mean by "Koszul duality", and outline the scope of this class.

1.1. **Prologue: Morita Theory.** As a warm-up, we discuss a precursor of Koszul duality, which was developed by Morita in 1958 [Mor58]. He asked the following natural question:

Question 1. Given two associative rings R and S, when is there an equivalence between the categories of left modules Mod_R and Mod_S?

If such an equivalence exists, we call R and S *Morita equivalent.* Isomorphic rings are clearly Morita equivalent, but the converse need not be true:

Proposition 1.1 (Morita functors)**.** *Let $Q \in \mathrm{Mod}_R$ be a left module over a ring R such that*

(1) Q is finite projective, i.e. a direct summand of $R^{\oplus n}$ for some n;

(2) Q is a generator, which means that the functor $\mathrm{Hom}_R(Q, -)$ is faithful.

Then R and $S = \mathrm{End}_R(Q)^{op}$ are Morita equivalent, which is witnessed by inverse equivalences

$$G : \mathrm{Mod}_R \to \mathrm{Mod}_S, \quad M \mapsto \mathrm{Hom}_R(Q, M)$$

$$F : \mathrm{Mod}_S \to \mathrm{Mod}_R, \quad N \mapsto Q \otimes_S N.$$

We will prove this claim in the next lecture. In the meantime, we offer several exercises:

Exercise 1.2 (Examples of Morita equivalences)**.**

a) Prove directly that for any ring R and any $n > 0$, the ring R is Morita equivalent to $M_n(R)$.

b) Find a ring R and a finite projective generator $Q \in \mathrm{Mod}_R$ such that $S = \mathrm{End}_R(Q)^{op}$ is not a matrix algebra over R.

Exercise 1.3. *We now establish a converse to Proposition 1.1.*

a) Show that a functor $\mathrm{Mod}_R \to \mathrm{Mod}_S$ *is of the form* $M \mapsto B \otimes_R M$ *for some* (S,R)*-bimodule* B *if and only if it is right exact and preserves coproducts (this is known as the Eilenberg-Watts theorem).*

b) Deduce that all Morita equivalences are realised by the construction in Proposition 1.1 (and make this statement precise).

Exercise 1.4 (Morita duality)**.** *Given a ring* R *and a finite projective generator* $Q \in \mathrm{Mod}_R$*, Proposition 1.1 constructs a new ring* $S = \mathrm{End}_R(Q)^{op}$ *and an* S*-module* $P := \mathrm{Hom}_R(Q,R)$*. Show that* P *is a finite projective generator for* Mod_S*, and that applying the same procedure again brings us back to where we started, i.e.* $R \cong \mathrm{End}_S(P)^{op}$ *and* $Q = \mathrm{Map}_S(P,S)$*.*

Exercise 1.5 (Morita equivalence and centres)**.** *Let* R *be a ring.*

a) Show that the centre $Z(R)$ *of* R *is isomorphic to the ring of natural endomorphisms of the identity functor on* Mod_R*.*

b) Deduce that Morita equivalent rings have isomorphic centres. Hence commutative rings are Morita equivalent if and only if they are isomorphic.

From Morita to Koszul. The basic setup for Koszul duality is a field k and an augmented associative k-algebra R. Note that this gives k the structure of an R-module.

Taking inspiration from Morita theory (cf. Proposition 1.1), we may ask:

Question 2. Is the functor $G : \mathrm{Mod}_R \to \mathrm{Mod}_{\mathrm{Hom}_R(k,k)^{op}}$, $M \mapsto \mathrm{Map}_R(k,M)$ an equivalence?

The answer is a resounding *"no"*, as is manifest from the following simple example:

Example 1.6. For $R = k[\epsilon]/\epsilon^2$, we have $\mathrm{Hom}_R(k,k)^{op} = k$, and the functor G sends $M \in \mathrm{Mod}_{k[\epsilon]/\epsilon^2}$ to $\ker(\epsilon : M \to M)$. Hence G is far from an equivalence.

However, a more sophisticated variant of this construction will give an interesting functor.

1.2. **A reminder on derived categories.** Refining the functor $(M \mapsto \mathrm{Map}_R(k,M))$ requires some basic homological algebra. If M and N are left modules over a ring R, then $\mathrm{Hom}_R(M,N)$ is only a fragment of a more refined construction called $\mathbb{R}\mathrm{Hom}_R(M,N)$, which is a *complex* of R-modules. It can be computed by first choosing a projective resolution $\ldots \to P_2 \to P_1 \to P_0 \to M$ of M and then setting

$$\mathbb{R}\mathrm{Hom}_R(M,N) \;:=\; (\ldots \to \;0\; \to \;\mathrm{Hom}_R(P_0,N)\; \to \;\mathrm{Hom}_R(P_1,N)\; \to \ldots)\ .$$

This complex depends on the chosen projective resolution $P_\bullet$, but different resolutions give quasi-isomorphic complexes. One can give a clean formulation of this phenomenon in the language of ∞-categories; we preemptively adjust our notation:

Notation 1.7. If R is a ring, we write $\mathrm{Mod}_R^\heartsuit$ for the category of ordinary left R-modules. We will soon define an ∞-category Mod_R whose objects are chain complexes of left R-modules. This is an enhancement of the classical triangulated category $D(R)$.

Notation 1.8. Given a chain complex $M \in \mathrm{Mod}_R$, write $\pi_*(M)$ for its homology groups. Concretely, if $M = (\ldots \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M_{-1} \xrightarrow{d_{-1}} \ldots)$, then $\pi_i(M) = \ker(d_i)/\mathrm{im}(d_{i+1})$.

Remark 1.9. Note that $\pi_*(\mathbb{R}\operatorname{Hom}_R(M,N))\cong\operatorname{Ext}_R^{-*}(M,N)$ recovers the usual Ext-groups.

We will also consider the following mild generalisation of the above setup:

Definition 1.10. A *differential graded ring* is a graded ring $R=\bigoplus_{n\in\mathbb{Z}} R_n$ with homomorphisms $d\colon R_n\to R_{n-1}$ such that $d^2=0$ and $d(xy)=d(x)y+(-1)^i x d(y)$ for all $x\in R_i, y\in R_j$.

Exercise 1.11. *What is a (left) differential graded module over a differential graded ring R?*

We can think of ordinary rings (like the field k) as differential graded rings concentrated in degree 0. Given an augmented differential graded k-algebra R, we introduce the following class of modules as a substitute for projective resolutions:

Definition 1.12. A differential graded R-module M is said to be *cofibrant* if it admits an exhaustive filtration $0=M_0\hookrightarrow M_1\hookrightarrow\ldots\hookrightarrow M$ such that each quotient M_{i+1}/M_i is isomorphic to a summand of a direct sum of shifts of R.

Proposition 1.13. *For any differential graded R-module M, there is a quasi-isomorphism $P\to M$ with P cofibrant; we will call P a projective resolution of M.*

Given another differential graded module N, we set $\mathbb{R}\operatorname{Hom}_R(M,N):=\operatorname{Hom}_R(P,N)$. We refer to [Sch04, Section 3.2] for a more detailed discussion of these graded constructions.

We will organise differential graded R-modules into an ∞-category denoted by Mod_R; when R is concentrated in degree zero, this will be consistent with Notation 1.7.

The bar construction. It is often useful to have explicit projective resolutions at hand:

Definition 1.14. Let k be a field and $R=k\oplus\overline{R}$ an augmented differential graded k-algebra with augmentation ideal $\overline{R}=\ker(R\to k)$. If M is a differential graded R-module, we define the *reduced bar construction* $\operatorname{Bar}(R,R,M)$ in several steps:

(1) First, we define a graded module $\Sigma\overline{R}$ by shifting the grading and set $(\Sigma\overline{R})_i:=\overline{R}_{i-1}$.
(2) Set $\operatorname{Bar}(R,R,M):=R\otimes_k T(\Sigma\overline{R})\otimes_k M$, with $T(\Sigma\overline{R})=\bigoplus_i(\Sigma\overline{R})^{\otimes i}$ the tensor algebra. If $r,r_1,\ldots,r_s\in R$, $m\in M$ are homogeneous elements, we write their tensor product as $r[r_1|\ldots|r_n]m\in R\otimes_k(s\overline{R})^{\otimes_k s}\otimes_k M$. It has degree $|r|+\sum|r_i|+|m|$.
(3) The differential ∂ on $\operatorname{Bar}(R,R,M)$ is given as a sum $\partial=\partial_h+\partial_v$, where
   a) the horizontal differential ∂_h sends $r[r_1|\ldots|r_n]m$ to
$$d(r)[r_1|\ldots|r_n]m-\sum_{i=1}^{n}(-1)^{\epsilon_i}r[r_1|\ldots|d(r_i)|\ldots|r_n]m+(-1)^{\epsilon_{n+1}}r[r_1|\ldots|r_n]d(m)$$
   b) the vertical differential ∂_v sends $r[r_1|\ldots|r_n]m$ to
$$(-1)^{|r|}rr_1[r_2|\ldots|r_n]m+\sum_{i=2}^{n}(-1)^{\epsilon_i}r[r_1|\ldots|r_{i-1}r_i|\ldots|r_n]m-(-1)^{\epsilon_n}r[r_1|\ldots|r_{n-1}]r_n m,$$
   Here $\epsilon_i=|r|+\sum_{j=1}^{i-1}|r_j|-i+1$ and the internal differentials of R, M are written as d.

The natural map $\operatorname{Bar}(R,R,M)\to M$ is in fact a projective resolution of M. Note that this construction simplifies whenever R is an ordinary ring.

1.3. **Koszul duality for associative algebras.** Replacing the functor $\mathrm{Hom}_R(k,-)$ by $\mathbb{R}\,\mathrm{Hom}_R(k,-)$ in our earlier attempt in Section 1.2, we arrive at the following definition:

Definition 1.15 (Koszul duality functors)**.**

a) The *Koszul dual* of an augmented differential graded k-algebra R is given by

$$\mathfrak{D}(R) := \mathbb{R}\,\mathrm{Hom}_R(k,k);$$

the Yoneda product makes $\mathfrak{D}(R)$ into an augmented differential graded k-algebra.

b) The *Koszul dual* of $M \in \mathrm{Mod}_R$ is given by the complex of $\mathfrak{D}(R)^{op}$-modules

$$\mathfrak{D}_R(M) := \mathbb{R}\,\mathrm{Hom}_R(k,M) \in \mathrm{Mod}_{\mathfrak{D}(R)^{op}}.$$

This definition requires some elaboration. While it is clear from ∞-categorical abstract nonsense that $\mathfrak{D}(R)$ is a differential graded algebra and that $\mathfrak{D}_R(M)$ is a differential graded $\mathfrak{D}(R)^{op}$-module, we will need explicit ways of computing these constructions.

For simplicity, assume that R is an ordinary augmented k-algebra of finite type. Using the bar construction $\mathrm{Bar}(R,R,k) \to k$ in Definition 1.14, we can compute

$$\mathfrak{D}(R) = \mathbb{R}\,\mathrm{Hom}_R(k,k) \simeq \mathrm{Hom}_R(\mathrm{Bar}(R,R,k),k) \cong \mathrm{Hom}_k(\mathrm{Bar}(k,R,k),k),$$

which is the chain complex

$$\mathfrak{D}(R) = (\ldots \to 0 \xrightarrow{\delta} k \xrightarrow{\delta} \mathrm{Map}_k(\overline{R},k) \xrightarrow{\delta} \mathrm{Map}_k(\overline{R},k)^{\otimes 2} \xrightarrow{\delta} \mathrm{Map}_k(\overline{R},k)^{\otimes 3} \xrightarrow{\delta} \ldots).$$

Given $\phi_1,\ldots,\phi_n : \overline{R} \to k$, we will write the corresponding element in $\mathrm{Map}_k(\overline{R}^{\otimes n},k) \cong \mathrm{Map}_k(\overline{R},k)^{\otimes n}$ as $[\phi_1|\ldots|\phi_n]$. The product on $\mathfrak{D}(R)$ is then given by juxtaposition:

$$[\phi_1|\ldots|\phi_n]\cdot[\psi_1|\ldots|\psi_m] = [\phi_1|\ldots|\phi_n|\psi_1|\ldots|\psi_m]$$

We refer to [McC01, Section 9.2] for a more detailed treatment of this circle of ideas.

Exercise 1.16. *Given a differential graded R-module M, explicitly construct the $\mathfrak{D}(R)^{op}$–module structure on the chain complex $\mathfrak{D}_R(M) = \mathbb{R}\,\mathrm{Hom}_R(k,M)$.*

Remark 1.17. Strictly speaking, $\mathbb{R}\,\mathrm{Hom}_R(k,k)$ and $\mathbb{R}\,\mathrm{Hom}_R(k,M)$ again depend on the chosen projective resolution for the R-module k, but different resolutions will lead to quasi-isomorphic results.

Example 1.18 (The exterior algebra)**.** Consider $R = k[\epsilon]/\epsilon^2$ as an augmented k-algebra, where the element ϵ maps to 0 in k. It admits the following projective resolution:

$$P_\bullet := (\ldots \to k[\epsilon]/\epsilon^2 \xrightarrow{1\mapsto\epsilon} k[\epsilon]/\epsilon^2 \xrightarrow{1\mapsto\epsilon} k[\epsilon]/\epsilon^2 \to 0 \to \ldots)$$

We deduce an equivalence of complexes

$$\mathbb{R}\,\mathrm{Hom}_R(k,k) \simeq \mathrm{Map}_{k[\epsilon]/\epsilon^2}(P_\bullet,k) \simeq (\ldots \to 0 \to k \xrightarrow{0} k \xrightarrow{0} k \to \ldots),$$

Exercise 1.19.

a) Use the reduced bar construction to show that $\mathbb{R}\,\mathrm{Hom}_R(k,k) \simeq k[x]$ is a tensor algebra on a class in degree -1.

b) Let R be an augmented commutative k-algebra. Is $\mathfrak{D}(R)$ necessarily commutative?

Considering $R = k[x]$ as an augmented differential graded k-algebra with vanishing differential, we observe that $\mathbb{R}\operatorname{Hom}_{\mathbb{R}\operatorname{Hom}_{k[\epsilon]/\epsilon^2}(k,k)}(k,k) \cong \mathbb{R}\operatorname{Hom}_{k[x]}(k,k) \cong k[\epsilon]/\epsilon^2$. We say that the exterior algebra $k[\epsilon]/\epsilon^2$ and the symmetric algebra $k[x]$ *lie in Koszul duality.*

In fact, this biduality for algebras is a quite general phenomenon:

Proposition 1.20. *Let R be an augmented differential graded k-algebra such that*

(1) R is coconnective, i.e. $\pi_0(R) \cong k$ via the unit and $\pi_i(R) = 0$ for $i > 0$.
(2) R is of finite type, i.e. $\pi_i(R)$ is a finite-dimensional k-vector space for all i.

Then there is a canonical quasi-isomorphism $R \to \mathfrak{D}(\mathfrak{D}(R)) = \mathbb{R}\operatorname{End}_{\mathbb{R}\operatorname{Hom}_R(k,k)}(k)$.

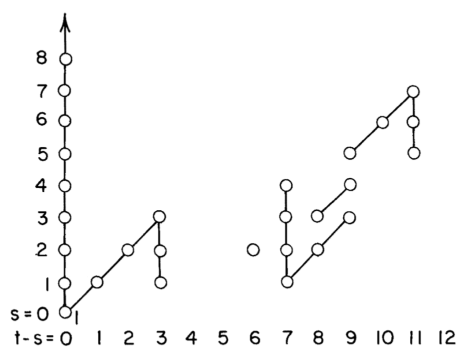
Outlook: Koszul algebras. Given an augmented k-algebra R, computing the Koszul dual $\mathbb{R}\operatorname{Hom}_R(k,k)$ and its homology algebra $\operatorname{Ext}_R^{-*}(k,k) = \pi_*(\mathbb{R}\operatorname{Hom}_R(k,k))$ can be an extremely challenging and rewarding problem, which is illustrated by the following result:

Theorem 1.21 (Adams spectral sequence)**.** *Write $\mathcal{A}$ for the Steenrod algebra at $p = 2$, i.e. the algebra of stable cohomology operations acting on $H^*(X, \mathbb{F}_2)$ for any space X.*

Then there is a spectral sequence of signature

$$E_s^{s,t} = \operatorname{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \quad \Rightarrow \quad \pi_*^s(S)_2^\wedge$$

We depict the E_2-term of this spectral sequence (in Adams convention), cf. [Rav78]:



There is a distinguished class of algebras R for which $\mathfrak{D}(R)$ is particularly simple:

Definition 1.22. Let R be an augmented k-algebra equipped with a grading $R = \bigoplus_m R[m]$ with $R[0] = k$. Note that $\operatorname{Ext}_R^*(k,k) \cong \bigoplus_m \operatorname{Ext}_R^*(k,k)[m]$ inherits an additional grading.

Then R is a *Koszul algebra* if $\operatorname{Ext}_R^i(k,k)[m] = 0$ unless $i = m$.

We will later explain the work of Priddy [Pri70], which shows that if R is a Koszul algebra, then $\mathfrak{D}(R) = \mathbb{R}\operatorname{Hom}_R(k,k) \simeq \operatorname{Ext}_R(k,k)$ is formal and easy to compute.

Exercise 1.23. *Find a grading on $k[\epsilon]/\epsilon^2$ which makes it a Koszul algebra.*

Unfortunately, the Steenrod algebra $\mathcal{A}$ is not a Koszul algebra in the above sense – otherwise stable homotopy theory would be significantly easier. However, it is closely related to a Koszul algebra; this will allow us to exhibit a small differential graded algebra model for $\mathbb{R}\operatorname{Hom}_{\mathcal{A}}(k,k)$ – the Λ-*algebra.*

1.4. **Koszul duality for modules over associative algebras.** Let R be an augmented differential graded k-algebra. Recall that the Koszul duality functor on R-modules, a derived substitute for the Morita functor in Proposition 1.1, is defined as:

$$\mathfrak{D}_R : \mathrm{Mod}_R \to \mathrm{Mod}_{\mathfrak{D}(R)^{op}}, \qquad M \mapsto \mathfrak{D}_R(M) = \mathbb{R}\mathrm{Hom}_R(k, M)$$

One might hope that by this functor is an equivalence. However, this is again not true, but this time for much more subtle reasons:

Exercise 1.24. *Show that the functor* $\mathbb{R}\mathrm{Hom}_{k[\epsilon]/\epsilon^2}(k, -) : \mathrm{Mod}_{k[\epsilon]/\epsilon^2} \to \mathrm{Mod}_{k[x]}$ *is not an equivalence by checking that* $k[x] \in \mathrm{Mod}_{k[x]}$ *is compact, but* $k \in \mathrm{Mod}_{k[\epsilon]/\epsilon^2}$ *is not.*

Here, we encounter an ∞-categorical property, which will discuss more carefully later:

Definition 1.25. An object X in an ∞-category $\mathcal{C}$ is said to be *compact* if the functor $\mathrm{Map}_{\mathcal{C}}(X, -)$ preserves filtered homotopy colimits.

Nonetheless, the Koszul functor in Exercise 1.24 restricts to an equivalence on certain subcategories. We use the following terminology, which we will define in full detail later:

Definition 1.26. A subcategory of a stable ∞-category $\mathcal{C}$ is called *thick* if it contains the zero object, and is closed under the formation of fibres, cofibres, and retracts. Given an object $X \in \mathcal{C}$, we write $\mathrm{Thick}_{\mathcal{C}}(X)$ for the thick subcategory generated by X.

Example 1.24 (continued)**.** We will later prove that $\mathfrak{D}_{k[\epsilon]/\epsilon^2}$ restricts to an equivalence

$$\{M \in \mathrm{Mod}_{k[\epsilon]/\epsilon^2} \mid \dim_k(\pi_*(M)) < \infty\} = \mathrm{Thick}_{\mathrm{Mod}_{k[\epsilon]/\epsilon^2}}(k) \simeq \mathrm{Thick}_{\mathrm{Mod}_{k[x]}}(k[x]) = \mathrm{Perf}_{k[x]},$$

where $\mathrm{Perf}_{k[x]} \subset \mathrm{Mod}_{k[x]}$ consists of all *perfect complexes* (here, a complex is said to be perfect if it is quasi-isomorphic to a bounded complex of finitely generated projectives).

Passing to the underlying triangulated categories, we deduce the well-known equivalence $D^b(k[\epsilon]/\epsilon^2) \cong D^b_{dg}(k[x])$ between bounded complexes of finitely generated $k[\epsilon]/\epsilon^2$-modules and differential graded $k[x]$-modules whose underlying $k[x]$-module is finitely generated.

A similar duality will in fact hold whenever R is sufficiently "small" (cf. [BGS96, Theorem 1.2.6], [Lur11, Proposition 3.5.2]). In fact, there is a Koszul-equivalence on general modules over general augmented associative k-algebras R (cf. [Lur11, Theorem 3.5.1]); to formulate it, one needs the concept of "Ind-coherent sheaves on formal moduli problems".

We conclude our introduction to Koszul duality on modules with a geometric application of duality for our favourite ring $R = k[\epsilon]/\epsilon^2$:

Theorem 1.25 (The Bernstein–Gel'fand–Gel'fand correspondence)**.** *There is an equivalence*

$$D^b\,\mathrm{Coh}(\mathbb{P}^n) \cong \mathrm{stMod}_{gr}(E),$$

where $E = E[X_0, \dots X_n]$ *is the free exterior algebra on* $n+1$ *generators. The stable graded module category* $\mathrm{stMod}_{gr}(E)$ *has objects given by graded* E*-modules, and we have*

$$\mathrm{Map}_{\mathrm{stMod}_{gr}(E)}(M, N) = \mathrm{Map}_{\mathrm{grMod}_E}(M, N)/\mathrm{P}\,\mathrm{Map}_{\mathrm{grMod}_E}(M, N).$$

Here $\mathrm{P}\,\mathrm{Map}_{\mathrm{grMod}_E}(M, N)$ *is the subspace of maps which factor through a projective* E*-module.*

1.5. **Outlook: Koszul duality for operads and monads.** We have seen that sending an augmented associative A over a field k to $\mathbb{R}\operatorname{Hom}_A(k,k) = \operatorname{Map}_k(\operatorname{Bar}(k,A,k),k)$ induces a subtle duality theory on associative algebras and their modules.

It is natural to ask if similar Koszul duality patterns exist for other kinds of algebras, e.g. augmented commutative k-algebras, Poisson algebras over k, or Lie algebras over k. We outline a rough and non-technical answer to this question, which we will elaborate on later.

The ingenuous idea (originating in work of Barr–Beck [BB69], André [And13], Quillen [Qui70], and Ginzburg–Kapranov [GK95]) is that we should *not* seek to directly replace {associative algebras} by {commutative algebras / Poisson algebras / Lie algebras / …} in the Koszul duality constructions in Definition 1.15. Instead, we should try to realise {commutative algebras / Poisson algebras / Lie algebras / … } as categories of modules over corresponding associative algebras { $\mathcal{O}_{\mathrm{Comm}}$ / $\mathcal{O}_{\mathrm{Poiss}}$ / $\mathcal{O}_{\mathrm{Lie}}$ / … }.

At first sight, this does not seem to make any sense. After all, there do not exist associative algebras $\mathcal{O}_{\mathrm{Comm}}$ / $\mathcal{O}_{\mathrm{Poiss}}$ / $\mathcal{O}_{\mathrm{Lie}}$ / … whose categories of modules have the desired form. However, one can generalise the meaning of the term "associative algebra" by replacing $\{k-\text{modules}\}$ by a more general monoidal category.

When the base field k has characteristic zero, "generalised associative algebras" will then be *operads*. Roughly speaking, an operad $\mathcal{O}$ over k consists of a collection $\{\mathcal{O}(n)\}_{n\geq 1}$ of Σ_n-representations; we think of these k-modules as n-ary operations. There are also composition maps $\mathcal{O}(n)\otimes\mathcal{O}(k_1)\otimes\ldots\otimes\mathcal{O}(k_n)\to\mathcal{O}(k_1+\ldots+k_n)$ satisfying various axioms.

For every operad $\mathcal{O}$, there will be a corresponding ∞-category of $\mathcal{O}$-algebras in Mod_k, which play the role of modules over the "generalised algebra" $\mathcal{O}$. Roughly speaking, an $\mathcal{O}$-algebra is a chain complex $M\in\mathrm{Mod}_k$ together with maps $\mathcal{O}(n)\otimes M^{\otimes n}\to M$ satisfying several axioms. We can construct operads $\mathcal{O}_{\mathrm{Comm}}$ / $\mathcal{O}_{\mathrm{Poiss}}$ / $\mathcal{O}_{\mathrm{Lie}}$ whose categories of algebras are augmented differential graded commutative k-algebras, differential graded Poisson algebras over k, and differential graded Lie algebras over k, respectively.

We then use the heuristic replacements

{Associative rings}	$\rightsquigarrow$	{Operads}
{Modules over a specific associative rings A}	$\rightsquigarrow$	{Algebras over a specific operad $\mathcal{O}$}

to generalise the Koszul duality functors in Definition 1.15; given an augmented operad $\mathcal{O}$, we will define its Koszul dual $\mathfrak{D}(\mathcal{O})$ as some kind of dualised bar construction.

Several substantial results in mathematics are due to the following simple fact:

Example 1.26. The Koszul dual of the operad $\mathcal{O}_{\mathrm{Comm}}$ is (a shift of) the Lie operad $\mathcal{O}_{\mathrm{Lie}}$.

As an example, we will use the induced Koszul duality functor between commutative algebras and Lie algebras to establish the following general paradigm (which originated in the work of Drinfeld [Dri], Deligne [Del], and Feigin, and was formulated as a precise correspondence by Lurie [Lur10] and Pridham [Pri10]):

'The deformation functor of any algebro-geometric object over a field k of characteristic zero is controlled by a differential graded Lie algebra.'

Example 1.27. The (derived) deformation functor of a smooth and proper variety Z over $\mathbb{C}$ can be recovered from the Dolbeault complex with values in the tangent bundle T_Z

$$C^*(Z, T_Z) = (\mathcal{A}^{0,0}(T_Z) \to \mathcal{A}^{0,1}(T_Z) \to \ldots),$$

together with the Lie bracket induced by the commutator bracket of vector fields.

As a concrete application of the above paradigm, we will discuss the following theorem (following the strategy of Iacono-Manetti [IM10]):

Theorem 1.28 (Bogomolov–Tian–Todorov theorem)**.** *Let Z be a smooth projective variety over an algebraically closed field k of characteristic zero. Moreover, assume that the canonical bundle K_Z of Z is trivial.*

Then Z has unobstructed deformations, which means that any deformation of Z over $\mathrm{Spec}(k[x]/x^2)$ *extends to* $\mathrm{Spec}(k[x]/x^n)$ *for all $n \geq 2$.*

When k is a field of characteristic p, infinitesimal deformations of algebro-geometric objects are no longer governed by differential graded Lie algebras, or algebras over any operad. Instead, we must use monads, which are also a form of generalised associative algebras.

This leads to the theory of "partition Lie algebras" (cf. [BM19]), which govern formal deformations over any field k (in characteristic 0, they recover differential graded Lie algebras).

If time permits, we will discuss partition Lie algebras and outline what is known and conjectured about versions of Theorem 1.28 in characteristic p.

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