Topics in Koszul Duality, Michaelmas 2019, Oxford University

Lecture 2: Categorical Background

In this lecture, we will take a second look at Morita theory, and illustrate several categorical concepts along the way. This will set the stage for the Barr–Beck theorem, which will be discussed in the next lecture, and for its higher categorical generalisation.

Abstract categorical arguments of this kind can often be circumvented when dealing with "discrete" objects such as rings or modules, but become an indispensable tool in the study of "homotopical" objects like differential graded algebras or chain complexes.

As we have seen in the overview class last week, Koszul duality forces us into a differential graded setting because certain hom-spaces only become nontrivial after having been derived (e.g. remember that for $R = k[\epsilon]/\epsilon^2$, we had $\mathrm{Hom}_R(k,k) = k$ but $\mathbb{R}\,\mathrm{Hom}_R(k,k) \cong k[x]$).

2.1. **A reminder on Morita theory.**

The first result from last class will serve as a toy model; we recall it using Notation 1.7 from last class.

Proposition 2.1. Let $Q \in \mathrm{Mod}_R^\heartsuit$ be a left module over an associative ring R such that

1. Q is *finite projective*, i.e. a direct summand of $R^{\oplus n}$ for some n;
2. Q is a *generator*, which means that the functor $\mathrm{Hom}_R(Q,-)$ is faithful.

Then R and $S = \mathrm{End}_R(Q)^{op}$ are Morita equivalent, which is witnessed by inverse equivalences

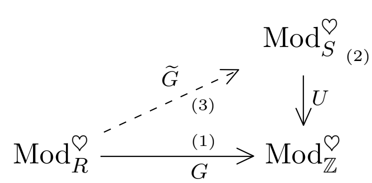
$$\widetilde{G} : \mathrm{Mod}_R^\heartsuit \to \mathrm{Mod}_S^\heartsuit, \quad M \mapsto \mathrm{Hom}_R(Q,M)$$
$$\widetilde{F} : \mathrm{Mod}_S^\heartsuit \to \mathrm{Mod}_R^\heartsuit, \quad N \mapsto Q \otimes_S N.$$

Proving Proposition 2.1 by hand is a reasonably straightforward, yet tedious, exercise. We will adopt a categorical approach which we have learned from Lurie [Lur] (for a related discussion of Serre's criterion for affineness, see [Mat]).

This approach might seem needlessly abstract, but has the advantage of generalising nicely to the ∞-category Mod_R of chain complexes of R-modules.

We start by observing that the functor $\widetilde{G}$ can be constructed in three steps:

1. Consider $M \mapsto \mathrm{Hom}_R(Q,M)$ as a functor $G : \mathrm{Mod}_R^\heartsuit \to \mathrm{Mod}_\mathbb{Z}^\heartsuit$ to abelian groups;
2. Define the associative ring $S = \mathrm{End}_R(Q)^{op}$;
3. Lift G to an enhanced functor $\widetilde{G} : \mathrm{Mod}_R^\heartsuit \to \mathrm{Mod}_S^\heartsuit$ by exhibiting a left S-module structure on every abelian group $\mathrm{Hom}_R(Q,M)$.



We will give a categorical reformulation of the assumptions of Proposition 2.1, eventually, only referencing the functor $G = \mathrm{Hom}_R(Q,-) : \mathrm{Mod}_R^\heartsuit \to \mathrm{Mod}_\mathbb{Z}^\heartsuit$ to abelian groups.

2.2. **Categorical notions.** First, we reformulate condition (1) of Proposition 2.1, asserting that $Q \in \mathrm{Mod}_R$ be finite projective, in terms of the functor $G : \mathrm{Mod}_R^{\heartsuit} \to \mathrm{Mod}_{\mathbb{Z}}^{\heartsuit}$. We treat the "finite" and the "projective" part separately, and start with the former.

Compactness. The categorical notion of compactness aims to capture the smallness of a given object X by asserting that it cannot be "spread out" arbitrarily.

For example, given a diagram $Y_0 \to Y_1 \to Y_2 \to \ldots$, any map from a small object X to the sequential colimit $\mathrm{colim}_i Y_i$ (which we might think of as an increasing union) should factor through one of the finite stages Y_i.

In fact, we will also want to take slightly more general diagrams into account:

Definition 2.2 (Filtered categories)**.** A category I is *filtered* if it is nonempty and

a) any two objects x, y map into a third object z via morphisms $x \to z$, $y \to z$;
b) for all parallel morphisms $f, g : x \rightrightarrows y$ in $\mathcal{C}$, there exists $h : y \to z$ with $h \circ f = h \circ g$.

A *filtered colimit* in a category $\mathcal{C}$ is a colimit over a diagram $D : I \to \mathcal{C}$, where I is filtered.

Exercise 2.3. Establish the following facts:

(1) The category $\mathbb{N} = (\bullet \to \bullet \to \ldots)$ is filtered; hence sequential colimits are filtered;
(2) The product of filtered categories is filtered;
(3) The category $\bullet \;\; \bullet$ is *not* filtered, and neither is Δ^{op}, the opposite of the category of nonempty finite linearly ordered sets.

We can explicitly compute filtered colimits in the category of sets:

Exercise 2.4 (Filtered colimits of sets)**.** Given a diagram $D : I \to \mathrm{Set}$ with I a small filtered category, show that $\mathrm{colim}_{i \in I} D(i)$ is given by the set $\coprod_{i \in I} D(i)/_{\cong}$, where $\cong$ is the equivalence relation identifying $a \in D(i), b \in D(j)$ if there are arrows $f : i \to k$, $g : j \to k$ with $D(f)(a) = D(g)(b)$.

Exercise 2.5 (Limits of sets)**.** Given a diagram $D : I \to \mathrm{Set}$ from a small category I to sets, write down its limit.

We will often need the following important fact:

Proposition 2.6 (Filtered colimits commute with finite limits in Set)**.** Given a diagram $D : I \times J \to \mathrm{Set}$ with I a small filtered category and J a category with finitely many objects and morphisms, the following canonical arrow is an isomorphism:

$$\operatorname*{colim}_{i \in I} \left(\lim_{j \in J} D(i,j) \right) \xrightarrow{\cong} \lim_{j \in J} \left(\operatorname*{colim}_{i \in I} D(i,j) \right).$$

Proof. We leave this as an exercise; for a detailed proof, see [Bor94, Theorem 2.13.4]. □

The finiteness restriction in Proposition 2.6 is necessary:

Exercise 2.7. Show that filtered colimits generally do not commute with limits in Set.

Proposition 2.6 is not true in an arbitrary category:

Exercise 2.8. Show that in Set^{op}, filtered colimits need not commute with finite limits.

Filtered colimits and finite limits also commute in categories that are sufficiently similar to sets. To make this precise, we need several notions.

Notation 2.9. Given a category I, the *right cone* $I^{\triangleright}$ is obtained from I by adding a new object 1 and a unique morphism from every $i \in I$ to the new object 1.

Definition 2.10. Let I be a category. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ preserves and reflects colimits of shape I if $D^{\triangleright} : I^{\triangleright} \to \mathcal{C}$ is a colimit diagram if and only if this is true for $U \circ D^{\triangleright} : I^{\triangleright} \to \mathcal{D}$. A similar definition applies to limits.

Using that faithful functors reflect isomorphisms (which we establish in Proposition 2.26 below), we can deduce the following basic fact from Proposition 2.6:

Corollary 2.11. Let $U : \mathcal{C} \to \mathrm{Set}$ be a faithful functor which preserves and reflects finite limits and filtered colimits. Then finite limits commute with filtered colimits in $\mathcal{C}$.

Exercise 2.12. Show that for any ring R, the forgetful functor $U : \mathrm{Mod}_R^{\heartsuit} \to \mathrm{Set}$ satisfies the assumptions of Corollary 2.11. Hint: equip the colimit of sets $\mathrm{colim}_{i\in I}(U \circ D)(i)$ constructed in Exercise 2.5 with the structure of an R-module.

We can now give a categorical notion of smallness:

Definition 2.13. An object X in a locally small category $\mathcal{C}$ is called *compact* if the functor $\mathrm{Map}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathrm{Set}$ preserves filtered colimits.

Using Proposition 2.6, we can prove a closure property for compact objects:

Corollary 2.14. Finite colimits of compact objects in a category $\mathcal{C}$ are compact.

Proof. For any finite diagram $D : J \to \mathcal{C}$ which admits a colimit in $\mathcal{C}$, we have a natural isomorphism of functors $\mathrm{Map}_{\mathcal{C}}(\mathrm{colim}_{j\in J} D(j), -) \xrightarrow{\cong} \lim_{j\in J} \mathrm{Map}_{\mathcal{C}}(D(j), -)$. For any filtered diagram $D' : I \to \mathcal{C}$, compactness of all $D(j)$ and Proposition 2.6 implies:

$$\begin{aligned} &\mathrm{Map}_{\mathcal{C}}(\mathrm{colim}_{j\in J} D(j), \mathrm{colim}_{i\in I} D'(i)) &&\cong \lim_{j\in J} \mathrm{colim}_{i\in I} \mathrm{Map}_{\mathcal{C}}(D(j), D'(i)) \\ \cong\ &\mathrm{colim}_{i\in I} \lim_{j\in J} \mathrm{Map}_{\mathcal{C}}(D(j), D'(i)) &&\cong \mathrm{colim}_{i\in I} \mathrm{Map}_{\mathcal{C}}(\mathrm{colim}_{j\in J} D(j), D'(i)) \end{aligned}$$

□

Example 2.15 (Compact sets)**.** A set is compact if and only if it is finite.

For the "if" part, we first observe that the set $*$ with one object is compact. As finite sets are finite coproducts of points, Corollary 2.14 shows that they are compact.

To see the "only if" part, let S be an infinite set and consider the category I with objects $\{ x_T \mid T \subset S \text{ finite} \}$ and a unique morphism $x_T \to x_{T'}$ whenever T is contained in T'. An easy check shows that I is filtered, and that S is the colimit of the functor $D : I \to \mathrm{Set}$ given by $x_T \mapsto T$. If S were compact, then $\mathrm{Map}_{\mathrm{Set}}(S, S) \cong \mathrm{colim}_{i\in I} \mathrm{Map}_{\mathrm{Set}}(S, D(i))$ and we could factor the identity map $S \to S$ through a finite subset, which is absurd.

Exercise 2.16 (Compact topological spaces)**.** Compact objects in the category of topological spaces are finite sets with the discrete topology. We will revisit this example later.

Example 2.17 (Compact modules)**.** A (left) module M over a ring R is compact if and only if it is finitely presented. The proof is almost identical to Example 2.15.

First observe that R is compact because $\mathrm{Map}_R(R, M) \cong M$ and the forgetful functor $\mathrm{Mod}_R^\heartsuit \to \mathrm{Set}$ preserves filtered colimits by Exercise 2.12. Since any finitely presented R-module is an iterated finite colimit of copies of R, the "if" part follows.

For the converse direction, we need that any R-module is a filtered colimit of finitely presented modules; we leave this as an exercise. If M is compact, then we can factor the identity map on M through a finitely presented submodule. This shows that M is a summand of a finitely presented module, and hence finitely presented itself.

We have completed the first step towards the desired reformulation of Proposition 2.1:

Corollary 2.18. A module $Q \in \mathrm{Mod}_R$ is finitely presented if and only if the functor $G = \mathrm{Map}_{\mathrm{Mod}_R}(Q, -) : \mathrm{Mod}_R^\heartsuit \to \mathrm{Mod}_\mathbb{Z}^\heartsuit$ preserves filtered colimits.

Remark 2.19. Since the forgetful functor $\mathrm{Mod}_\mathbb{Z}^\heartsuit \to \mathrm{Set}$ preserves and reflects filtered colimits, this is an instance of Definition 2.13.

Projectivity. We now give a reformulation of the condition that a module $Q \in \mathrm{Mod}_R$ be projective, with an eye towards later higher-categorical generalisations.

First, we recall a well-known result in homological algebra:

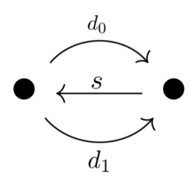
Proposition 2.20. Given a module $Q \in \mathrm{Mod}_R^\heartsuit$, the following are equivalent:

a) Q is a summand of a free module;
b) The functor $\mathrm{Map}_R(Q, -)$ preserves surjections;
c) The functor $\mathrm{Map}_R(Q, -)$ preserves short exact sequences;
d) The functor $\mathrm{Map}_R(Q, -)$ preserves cokernels.

If these conditions hold, we call the module Q *projective*.

We will reformulate the "cokernel" condition $d)$ using the following notion:

Definition 2.21. A *reflexive pair* in a category $\mathcal{C}$ is a diagram consisting of two arrows $d_0, d_1 : X_1 \rightrightarrows X_0$ and a common section $s : X_0 \to X_1$ satisfying $f \circ s = g \circ s = \mathrm{id}_{X_0}$. In other words, it is a $\Delta^{op}_{\leq 1}$-indexed diagram; we will return to this perspective in the next lectures.



A *reflexive coequaliser* is the colimit of a reflexive pair. Note that this agrees with the coequaliser of the arrows d_0 and d_1.

We also record the following notion:

Definition 2.22. A functor $F : \mathrm{Mod}_R^\heartsuit \to \mathrm{Mod}_\mathbb{Z}^\heartsuit$ is called *additive* if for all M, N, the functor $\mathrm{Map}_R(M, N) \to \mathrm{Map}_\mathbb{Z}(FM, FN)$ is a homomorphism of abelian groups.

Condition *d*) in Proposition 2.20 can be reformulated in terms of reflexive coequalisers:

Proposition 2.23. An additive functor $F : \mathrm{Mod}_R^\heartsuit \to \mathrm{Mod}_\mathbb{Z}^\heartsuit$ preserves cokernels if and only if it preserves reflexive coequalisers.

Proof. Assume that F preserves cokernels. The coequaliser of a reflexive pair $A \underset{g}{\overset{f}{\rightleftarrows}} B$ is the cokernel of $A \xrightarrow{f-g} B$. As F is additive, this shows that it preserves reflexive coequalisers. Conversely, assume that F preserves reflexive coequalisers. The cokernel of $A \xrightarrow{f} B$ agrees with the coequaliser of the reflexive pair $A \oplus B \underset{\mathrm{id}_B}{\overset{f+\mathrm{id}_B}{\rightleftarrows}} B$, which implies the claim. □

Corollary 2.24. A module $Q \in \mathrm{Mod}_R$ is projective if and only if the functor $\mathrm{Map}_R(Q, -)$ preserves reflexive coequalisers.

Exercise 2.25.

a) Prove that the forgetful functor $\mathrm{Mod}_\mathbb{Z}^\heartsuit \to \mathrm{Set}$ preserves and reflects reflexive coequalisers.
b) Show that this becomes false once we drop the word "reflexive".

2.3. **Conservativity.** Finally, we reformulate condition (2) of Proposition 2.1 by making the following simple observation:

Proposition 2.26. Any faithful functor $G : \mathrm{Mod}_R^\heartsuit \to \mathrm{Mod}_\mathbb{Z}^\heartsuit$ is conservative. Conversely, any conservative functor which preserves coequalisers is faithful.

Here G is called conservative if $f : X \to Y$ in $\mathrm{Mod}_R^\heartsuit$ is an isomorphism iff $G(f)$ is one.

Proof. First assume that G is faithful. If $G(f)$ is an isomorphism, then it is both an epi- and a monomorphism. Since G is faithful, this implies that f is both an epi- and a monomorphism, which shows that f is an isomorphism since $\mathrm{Mod}_R^\heartsuit$ is an abelian category.

Conversely, assume that G is a conservative functor which preserves coequalisers. Note that arrows $f, g : A \to B$ are equal if and only if in the coequaliser diagram $A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{h} C$, the map h is an isomorphism; this condition is preserved and reflected by the functor G. □

Combining Corollary 2.18, Corollary 2.24, and Proposition 2.26 with two straightforward observations *a*), *b*), we conclude that Proposition 2.1 is an instance of the following setup:

Setup 2.27. Let $G : \mathrm{Mod}_R^\heartsuit \to \mathrm{Mod}_\mathbb{Z}^\heartsuit$ be a functor satisfying the following conditions:

a) G admits a left adjoint;
b) G preserves finite coproducts;
c) G preserves filtered colimits;
d) G preserves reflexive coequalisers;
e) G is conservative.

In the next class, we will give a proof of Proposition 2.1 starting from these purely categorical assumptions, and establish the Barr-Beck theorem along the way.

References

[Bor94] Francis Borceux, *Handbook of categorical algebra: volume 1, basic category theory*, vol. 1, Cambridge University Press, 1994.

[Lur] Jacob Lurie, *Higher algebra*, Preprint from the author's web page.

[Mat] Akhil Mathew, *Serre's criterion for affineness*, Climbing Mount Bourbaki webblog.