Topics in Koszul Duality, Michaelmas 2019, Oxford University

Lecture 4: Higher Categorical Background

Last week, we discussed the Barr-Beck theorem, which specifies conditions under which an adjunction $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ induces an equivalence $\mathcal{D} \xrightarrow{\simeq} \mathrm{Alg}_T(\mathcal{C})$. Here, $T = GF$ is the monad associated with $F \dashv G$, and $\mathrm{Alg}_T(\mathcal{C})$ is the category of T-algebras $(X \in \mathcal{C}, T(X) \xrightarrow{\alpha} X)$.

As a toy application, we proved that if R is a ring and $Q \in \mathrm{Mod}_R^\heartsuit$ is a finite projective generator, then R and $S = \mathrm{End}_R(Q)^{op}$ have equivalent categories of (left) modules, via the functor $M \mapsto \mathrm{Map}_R(Q, M)$. In fact, any equivalence of module categories arises in this way.

It is natural to wonder if a similar statement holds true on the level of derived categories:

Question 1. Given two rings R and S, is there an equivalence between the derived categories $\mathrm{Mod}_R = \mathcal{D}(\mathrm{Mod}_R^\heartsuit)$ and $\mathrm{Mod}_S = \mathcal{D}(\mathrm{Mod}_S^\heartsuit)$?

To address this question, we will make use of Lurie's higher categorical generalisation of the classical Barr-Beck theorem. In this lecture, we will cover some necessary higher categorical background, and in particular define what monads are in this context.

Due to our time restrictions, we can only scratch the surface – for a comprehensive treatment, we recommend [Lur09] (for ∞-categories) and [Lur07] (for monoidal aspects).

4.1. **Simplicial sets.** While ∞-categories might look scary (possibly due to symbol "∞"), it can be helpful to remember that they are just simplicial sets satisfying a certain property.

To fix notation, we briefly recall the basic setup of simplicial sets.

Write Δ for the simplex category; its objects are the nonempty finite linearly ordered sets

$$[0] = \{0\} \ , \quad [1] = \{0 < 1\} \ , \quad [2] = \{0 < 1 < 2\} \ , \quad \ldots \ ,$$

and morphisms are order-preserving maps.

Definition 4.1 (Simplicial sets). A *simplicial set* is a functor $\Delta^{op} \to \mathrm{Set}$. Write **sSet** for the resulting (ordinary) category of simplicial sets.

Notation 4.2 (Simplices and horns). Fix integers $n \geq 0$ and $0 \leq i \leq n$.
(1) Write $\Delta^n = \mathrm{Map}_\Delta(-, [n]) : \Delta^{op} \to \mathrm{Set}$ for the simplicial set represented by $[n] \in \Delta$.
(2) Let Λ_i^n be the simplicial set sending $[k] \in \Delta$ to $\{f : [k] \to [n] \text{ s.t. } [n]\backslash\{i\} \not\subseteq f([k])\}$.
We refer to Δ^n as the simplicial n-simplex, and call Λ_i^n the ith horn of Δ^n.

Definition 4.3 (Mapping objects). Given $X, Y \in$ **sSet**, define $Y^X \in$ **sSet** by $(Y^X)_n = \mathrm{Map}_{\mathbf{sSet}}(\Delta^n \times X, Y)$; the simplicial structure maps are induced by the Yoneda embedding.

The category of simplicial sets is therefore enriched in **sSet**.

Simplicial sets are closely related to **Top**, the category of (compactly generated) spaces. To make this statement precise, we will need, for every $n \geq 0$, the topological n-simplex

$$\boldsymbol{\Delta}^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^n \mid x_0 + \ldots + x_n = 1, \ x_i \geq 0\}.$$

(1) Any $[n] \xrightarrow{f} [m]$ induces $\boldsymbol{\Delta}^n \xrightarrow{f_*} \boldsymbol{\Delta}^m$ with $f_*(s_0, \ldots, s_n) = (t_0, \ldots, t_m)$, $t_j = \sum_{f(i)=j} s_i$.

We can build spaces from simplicial sets:

Definition 4.4 (Geometric realisation)**.** The *geometric realisation* of a simplicial set X is given by $|X| = \operatorname{colim}_{\Delta^n \to X}(\mathbf{\Delta}^n)$; this colimit is computed in the ordinary category **Top**.

We call a simplicial set X weakly contractible if $|X|$ has this property.

Exercise 4.5.
a) Reformulate Definition 4.4 both as a left Kan extension and as a coend.
b) Give an explicit formula for $|X|$ as a quotient of a coproduct by an equivalence relation.
c) Describe the spaces $|\Delta^n|$ and $|\Lambda^n_i|$ (cf. Notation 4.2).

We can also go into the reverse direction and attach simplicial sets to spaces:

Definition 4.6 (Singular chains)**.** Given $X \in \mathbf{Top}$, the simplicial set $\operatorname{Sing}(X)$ satisfies

$$\operatorname{Sing}(X)_n = \operatorname{Map}_{\mathbf{Top}}(\mathbf{\Delta}^n, X),$$

Given a map $[n] \xrightarrow{f} [m]$ in Δ, the corresponding structure map $\operatorname{Sing}(X)_m \to \operatorname{Sing}(X)_n$ is obtained by precomposing with the map $f^* : \mathbf{\Delta}^n \xrightarrow{f_*} \mathbf{\Delta}^m$ from (1) above.

Exercise 4.7. Show that the singular chains functor Sing is right adjoint to the geometric realisation functor $|-|$ from Definition 4.4.

Simplicial sets arising as the singular chains of a topological space have a special property:

Definition 4.8 (Kan complexes)**.** A simplicial set X is called a *Kan complex* if it satisfies the right lifting property for all horns. Concretely, this means that for all n and any $0 \le i \le n$, every map $f_0 : \Lambda^n_i \to X$ extends to a map $\overline{f} : \Delta^n \to X$ from the n-simplex:

$$
\begin{array}{ccc}
\Lambda^n_i & \xrightarrow{\ f\ } & X \\
\big\downarrow & \nearrow & \\
\Delta^n & \overline{f} &
\end{array}
$$

We can also attach simplicial sets to ordinary categories. For this, we identify the linearly ordered set $[n]$ with the category $(0 \to 1 \to \ldots \to n)$ and define:

Definition 4.9. The *nerve* of a category $\mathcal{C}$ is the simplicial set $\mathrm{N}(\mathcal{C})$ with $\mathrm{N}(\mathcal{C})_n = \operatorname{Fun}_{\mathrm{Cat}}([n], \mathcal{C})$; The structure maps are induced by pullback along maps $[n] \to [m]$ in Δ^{op}.

Remark 4.10. The observant reader might object that if $\mathcal{C}$ is not small, then $\mathrm{N}(\mathcal{C})$ is too large to be a set. This technical difficulty can be handled rigorously using Grothendieck universes; we refer to [Lur09, Section 1.2.15] for a discussion. In these expository lectures, we will confidently sweep size issues of this kind under the rug.

Simplicial sets which arise as nerves of ordinary categories share a special property:

Exercise 4.11. Show that a simplicial set $X \in \mathbf{sSet}$ is the nerve $\mathrm{N}(\mathcal{C})$ of a category $\mathcal{C}$ if and only if for all $0 < i < n$ and each map $f : \Lambda^n_i \to X$, there is a *unique* extension to Δ^n.

4.2. **Higher categories.** To define ∞-categories, also known as quasi-categories or weak Kan complexes, we relax the uniqueness assertion in Exercise 4.11:

Definition 4.12 (Boardman-Vogt)**.** An ∞-category is a simplicial set $\mathcal{D} \in \mathbf{sSet}$ such that for all $0 < i < n$ and each map $f : \Lambda_i^n \to \mathcal{D}$, there is a (not necessarily unique) extension

$$
\begin{array}{ccc}
\Lambda_i^n & \xrightarrow{\;f\;} & \mathcal{D} \\
\cap\; \downarrow & \nearrow & \\
\Delta^n & \overline{f} &
\end{array}
$$

A functor between ∞-categories $\mathcal{C}$ and $\mathcal{D}$ is simply a map of simplicial sets.

The 0-simplices of an ∞-category $\mathcal{C}$ are its *objects*; the 1-simplices are the *morphisms*.

In an ordinary category, we can compose morphisms $x \xrightarrow{f} y$, $y \xrightarrow{g} z$ and obtain a third morphism $x \xrightarrow{g \circ f} z$. This is reflected in the fact that any $\Lambda_1^2 \to \mathrm{N}(\mathcal{C})$ admits a *unique* filler.

In an ∞-category $\mathcal{D}$, the composite of morphisms $x \xrightarrow{f} y$ and $y \xrightarrow{g} z$ is no longer defined uniquely. Instead, there could be many 2-simplices $\Delta^2 \to \mathcal{D}$ of the form

$$
\begin{array}{ccc}
x & \xrightarrow{\;f\;} & y \\
 & & \downarrow g \\
 & & z
\end{array}
$$

For any two such fillers $\Delta^2 \to \mathcal{D}$ with $\{0,2\}$-edges $h_1, h_2 \in \mathcal{D}_1$, which we think of as two composites of f and g, we obtain a morphism $\Lambda_1^3 \to \mathcal{D}$ depicted below:

By the inner horn filling condition in Definition 4.12, we can again extend this to a map $\Delta^3 \to \mathcal{D}$, which we think of as an identification between h_1 and h_2. There could of course be many such 3-simplices, but any two can be "identified" by a 4-simplex, and so on.

We give an explicit model for the space of maps between two objects in an ∞-category:

Definition 4.13 (Mapping space)**.** Given objects x, y in an ∞-category $\mathcal{C}$, we define the *space of right morphisms* $\mathrm{Hom}_{\mathcal{C}}^R(x, y)$ as the simplicial set with

$$(\mathrm{Hom}_{\mathcal{C}}^R(x, y))_n = \{z : \Delta^{n+1} \to \mathcal{C} \mid z|_{\Delta^{0,\ldots,n}} = \mathrm{id}_x, \ z(n+1) = y\}$$

Exercise 4.14. Define the simplicial structure maps in Definition 4.13 and prove that $\mathrm{Hom}_{\mathcal{C}}^R(x, y)$ is a Kan complex for all x, y.

Definition 4.15 (Initial objects)**.** An object x in an ∞-category $\mathcal{C}$ is said to be *initial* if for all $y \in \mathcal{C}$, the simplicial set $\mathrm{Hom}_{\mathcal{C}}^R(x, y)$ is weakly contractible.

We will now specify three important examples of ∞-categories:

Example 4.16.
a) For any ordinary category $\mathcal{C}$, the nerve $\mathrm{N}(\mathcal{C})$ is an ∞-category.
b) The ∞-category of (compactly generated Hausdorff) spaces is defined in several steps. Let **Kan** $\subset$ **sSet** be the full subcategory spanned by all Kan complexes (cf. Definition 4.8). By Definition 4.3, **Kan** is in fact a simplicial category (i.e. enriched in simplicial sets). For each n, we define a simplicial category $\mathfrak{C}[\Delta^n]$ with objects $0, 1, \ldots, n$, and where $\mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i, j)$ is given by the nerve of the poset $\{S \mid \{i, j\} \subset S \subset \{i, i+1, i+2, \ldots, j-1, j\}\}$. We then define the ∞-category $\mathcal{S}$ of spaces using Cordier's simplicial nerve, i.e. set

$$\mathcal{S}_n = \mathrm{Map}_{\mathrm{sCat}}(\mathfrak{C}[\Delta^n], \mathbf{Kan}),$$

where sCat is the category of simplicial categories. We leave it as an exercise to define the simplicial structure maps, and to verify that $\mathcal{S}$ satisfies the inner horn filling axiom.
c) The ∞-category Cat_∞ of ∞-categories is defined by a very similar procedure.

We start with $\mathbf{Cat}_\infty^\Delta$, the simplicial category whose objects are (small) ∞-categories and where $\mathrm{Map}_{\mathrm{Cat}_\infty^\Delta}(\mathcal{C}, \mathcal{D})$ is the largest Kan complex contained in $\mathcal{D}^{\mathcal{C}}$ (cf. Definition 4.3). We then define the ∞-category Cat_∞ of small ∞-categories by

$$(\mathrm{Cat}_\infty)_n = \mathrm{Map}_{\mathrm{sCat}}(\mathfrak{C}[\Delta^n], \mathbf{Cat}_\infty^\Delta).$$

Again, we leave the definition of the simplicial structure maps as an exercise.

Note that in Cat_∞, we have not captured noninvertible natural transformations.

4.3. **Colimits.** To define colimits in ∞-categories, we will make use of the following notion:

Definition 4.17 (Join)**.** Given $X, Y \in \mathbf{sSet}$, we define a new simplicial set $X \star Y$ with

$$(X \star Y)_n = \coprod_{n=a+b} X_a \times Y_b$$

Exercise 4.18.
a) Complete Definition 4.17 by describing the structure maps of $X \star Y$.
b) Verify that $\Delta^n \star \Delta^m = \Delta^{n+m-1}$.

Let now $F : I \to \mathcal{C}$ be a functor of ∞-categories. A colimit of D will be an initial cone. More precisely, consider the ∞-category of cones $\mathcal{C}_{F/}$ with $(\mathcal{C}_{F/})_n = \{\overline{F} : I \star \Delta^n \to \mathcal{C} \mid \overline{F}|_I = F\}$.

Example 4.19. If $I = (\bullet \leftarrow \bullet \rightarrow \bullet)$ and $F : I \rightarrow \mathcal{C}$ picks out a diagram $(b \leftarrow a \rightarrow c)$, then $\mathcal{C}_{F/}$ is the ∞-category of all diagrams

$$
\begin{array}{ccc}
a & \longrightarrow & b \\
\downarrow & & \vdots \\
\downarrow & & \vdots \\
c & \dashrightarrow & d
\end{array}
$$

Definition 4.20 (Colimits). Given a diagram $F : I \rightarrow \mathcal{C}$, a colimit of F is an initial object (cf. Definition 4.15) in the ∞-category $\mathcal{C}_{F/}$.

A result of Joyal shows that if a colimit exists, then it is unique up to a contractible space of choices (cf. [Lur09, Proposition 1.2.12.9]). Limits are defined in a dual fashion.

We will often want to talk about filtered colimits in a higher categorical setting. To this end, we generalise the notion of a filtered category from ordinary to higher categories. Given $n \geq 0$, write $\partial\Delta^n$ for the simplicial subset of Λ^{n+1}_{n+1} spanned by all simplices not containing the vertex $n + 1$.

Definition 4.21. An ∞-category I is said to be *filtered* if for all integers $n \geq 0$, any map $f : \partial\Delta^n \rightarrow I$, extends to Λ^{n+1}_{n+1}:

$$
\begin{array}{ccc}
\partial\Delta^n & \xrightarrow{f} & \mathcal{D} \\
\big\uparrow & \nearrow & \\
\big\downarrow & \overline{f} & \\
\Lambda^{n+1}_{n+1} & &
\end{array}
\;.
$$

The case $n = 0$ shoes that I is nonempty. For $n = 1$, we conclude that for any diagram $\partial\Delta^1 \rightarrow I$ picking out two objects x, y, we can find an object z and morphisms $x \rightarrow z$, $y \rightarrow z$.

Exercise 4.22. Show that if I is the nerve of an ordinary category, then Definition 4.21 recovers Definition 2.2 from Lecture 2.

In the ordinary category Set of sets, filtered colimits commuted with finite limits. The higher categorical analogue of this fact is given by the following result:

Proposition 4.23. Filtered colimits and finite limits commute in the ∞-category $\mathcal{S}$ of spaces.

We refer to [Lur09, Proposition 5.3.3.3] for a proof of Proposition 4.23. This result illustrates the general paradigm that $\mathcal{S}$ plays the same role for ∞-categories as Set plays for ordinary categories. We generalise Definition 2.13 from Lecture 2 to this setting:

Definition 4.24 (Compact objects). An object X in an ∞-category $\mathcal{C}$ is called *compact* if the functor $\mathrm{Map}_{\mathcal{C}}(X, -) : \mathcal{C} \rightarrow \mathcal{S}$ preserves filtered colimits.

We then have the following generalisation of Corollary 2.14 from Lecture 2, which follows from Proposition 4.23:

Corollary 4.25. Finite colimits of compact objects in an ∞-category $\mathcal{C}$ are compact.

4.4. **CoCartesian fibrations.** To examine ∞-categories in families, we need the following

Definition 4.26 (coCartesian lifts)**.** Given a map of simplicial sets $p : \mathcal{C} \to S$ and an edge $f : x \to y$ in S, an edge $\widetilde{f} : \widetilde{x} \to \widetilde{y}$ in $\mathcal{C}$ is said to be a *p-coCartesian lift* of f if

a) The edge $\widetilde{f}$ lifts f, which means that $p(\widetilde{f}) = f$.

b) The map $\mathcal{C}_{\widetilde{f}/} \to \mathcal{C}_{\widetilde{x}/} \times_{S_{x/}} S_{f/}$ is a trivial Kan fibration of simplicial sets.

Condition b) says that in the diagram below, specifying the upper triangle, an element of $\mathcal{C}_{\widetilde{f}/}$, is equivalent to compatibly specifying $(\widetilde{x} \to \widetilde{z}) \in \mathcal{C}_{\widetilde{x}/}$ and the lower triangle, an element of $S_{f/}$.

(2)

$$
\begin{array}{ccc}
\widetilde{x} \xrightarrow{\ \widetilde{f}\ } \widetilde{y} & & \\
\downarrow \qquad\ \downarrow & \searrow \widetilde{z} & \\
x \xrightarrow{\ f\ } y & \downarrow & \\
& \searrow z &
\end{array}
$$

Definition 4.27 (CoCartesian fibration)**.** A map $\mathcal{C} \xrightarrow{p} S$ in **sSet** is a *coCartesian fibration* if

(1) p is an *inner fibration*, i.e. it satisfies the right lifting property for all inner horns:

$$
\begin{array}{ccc}
\Lambda^n_i & \longrightarrow & \mathcal{C} \\
\downarrow & \nearrow & \downarrow \\
\Delta^n & \longrightarrow & S
\end{array}
$$

(2) Given $x \xrightarrow{f} y$ in S and $\widetilde{x} \in \mathcal{C}$ with $p(\widetilde{x}) = x$, there is a p-coCartesian lift $\widetilde{x} \xrightarrow{\widetilde{f}} \widetilde{y}$ of f.

As a heuristic, it might be helpful to think of coCartesian fibrations as bundles with flat connection; in this picture, coCartesian lifts correspond to paths along the connection.

4.5. **Unstraightening.** One can show that coCartesian fibrations over S are equivalent to functors from S into the ∞-category Cat_∞ introduced in Example 4.16. The proof of this result is challenging, and we refer to [Lur09, Section 3.2] for a comprehensive treatment.

We will content ourselves with constructing coCartesian fibrations for *certain* functors to Cat_∞. More precisely, let J be an ordinary category and $F : J \to \mathbf{sSet}$ a functor.

Definition 4.28 (Relative nerve)**.** The *relative nerve* $\mathrm{N}_F(J)$ is the simplicial set over $\mathrm{N}(J)$ with

$$\mathrm{N}_F(J)_0 = \{(j_0 \in \mathrm{N}(J)_0 \ , \ x_0 \in F(j_0)\}$$

$$\mathrm{N}_F(J)_1 = \{(j_0 \to j_1) \in \mathrm{N}(J)_1 \ , \ \begin{smallmatrix} x_0 \in F(j_0) \\ x_1 \in F(j_1) \end{smallmatrix} \ , \ F(j_0 \to j_1)(x_0) \to x_1 \}$$

$$
\mathrm{N}_F(J)_2 = \left\{
\begin{array}{l}
\begin{array}{ccc}
j_0 & \longrightarrow & j_1 \\
& \searrow & \downarrow \\
& & j_2
\end{array}
\ , \
\begin{smallmatrix} x_0 \in F(j_0) \\ x_1 \in F(j_1) \\ x_2 \in F(j_2) \end{smallmatrix}
\ , \
\begin{smallmatrix} F(j_0 \to j_1)(x_0) \to x_1 \\ F(j_0 \to j_2)(x_0) \to x_2 \\ F(j_1 \to j_2)(x_1) \to x_2 \end{smallmatrix}
\ ,
\begin{array}{ccc}
F(j_0 \to j_2)(x_0) & \Longrightarrow & F(j_1 \to j_2)(x_1) \\
& \searrow & \downarrow \\
& & x_2
\end{array}
\right\}
$$

**Exercise 4.29.** a) Write down $N_F(J)_n$ for all n and check that it is a simplicial set.

b) Show that if $F(j)$ is an ∞-category for all j, then $\mathrm{N}_F(J) \xrightarrow{p} \mathrm{N}(J)$ is a coCartesian fibration.

4.6. **Monoidal ∞-categories.** We are finally in a position to define monoidal ∞-categories.

But first, we observe that the category Δ^{op} from Section 4.1 admits an alternative description. Indeed, the objects of Δ^{op} can be written as

$$[0] = (- \ +), \quad [1] = (- \ \bullet \ +), \quad [2] = (- \ \bullet \ \bullet \ +), \quad [3] = (- \ \bullet \ \bullet \ \bullet \ +), \quad \ldots$$

Morphisms from $[n]$ to $[m]$ are maps which preserve the order and send $-$ to $-$ and $+$ to $+$:
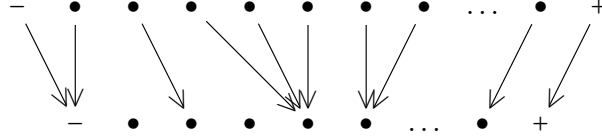
Exercise 4.30. Show that the category defined in this way indeed agrees with the opposite of the usual simplex category Δ.

Informally, we think of the bullets as placeholders of potential elements in a monoidal category. The symbols $+$ and $-$ will act as "trashcans"; arrows will parametrise multiplications.

We give a name to the morphisms which "throw away" all but one element:

Definition 4.31. Given $n \geq 0$ and $1 \leq i \leq n$, we write $\rho_i^n : [n] \to [1]$ for the morphism
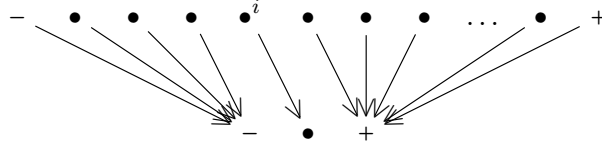
This motivates the following definition:

Definition 4.32 (Monoidal ∞-categories)**.** A monoidal ∞-category is a coCartesian fibration $p : \mathcal{C}^{\otimes} \to \mathrm{N}(\Delta^{op})$ such that for all n, the following morphism is an equivalence:

$$\mathcal{C}_{[n]}^{\otimes} \xrightarrow{\ \Pi_{i=1}^{n}(\rho_i^n)_! \ } \prod_{i=1}^{n} \mathcal{C}_{[1]}^{\otimes} \qquad \text{(Segal condition)}$$

Here $\mathcal{C}_{[n]}^{\otimes}$ denotes the fibre of p over $[n]$, and $\mathcal{C}_{[n]}^{\otimes} \xrightarrow{(\rho_i^n)_!} \mathcal{C}_{[1]}^{\otimes}$ is the functor associated with ρ_i^n.

Informally, we simply say that $\mathcal{C} \simeq \mathcal{C}_{[1]}^{\otimes}$ is equipped with a monoidal structure.

Remark 4.33. The functor $(\rho_i^n)_!$ sends $x \in \mathcal{C}_{[n]}^{\otimes}$ to the endpoint of a coCartesian lift of ρ_i^n starting at $\tilde{x}$. For a complete definition, we refer to [Lur09, Section 2.2.1].

The monoidal product $\circ$ is determined, up to equivalence, by the following composite:

$$\mathcal{C}_{[1]}^{\otimes} \times \mathcal{C}_{[1]}^{\otimes} \xleftarrow{\ \simeq \ } \mathcal{C}_{[2]}^{\otimes} \xrightarrow{\ m_! \ } \mathcal{C}_{[1]}^{\otimes},$$

where $m : [2] \to [1]$ is the morphism in Δ^{op} represented by the diagram
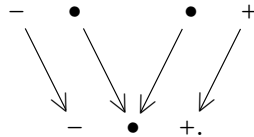
Exercise 4.34. Define the monoidal unit **1** of a monoidal ∞-category $\mathcal{C}^{\oplus} \to \mathrm{N}(\Delta^{op})$.

Notation 4.35. We will often say "let $(\mathcal{C}, \circ, \mathbf{1})$ be a monoidal ∞-category" instead of "let $\mathcal{C}^{\oplus} \to \mathrm{N}(\Delta^{op})$ be a monoidal ∞-category with $\mathcal{C}^{\oplus}_{[1]} \simeq \mathcal{C}$, multiplication $\circ$, and unit **1**".

Using the relative nerve from Definition 4.28, we can construct:

Definition 4.36 (Endomorphism ∞-categories)**.** Given an ∞-category $\mathcal{D}$, we equip

$$\mathcal{C} = \mathrm{End}(\mathcal{D}) \coloneqq \mathcal{D}^{\mathcal{D}}$$

(cf. Definition 4.3) with the structure of a monoidal ∞-category as follows.

First, use that $\mathcal{C}$ is a simplicial monoid (under composition) to construct a diagram

$$\ldots \qquad \mathcal{C} \times \mathcal{C} \quad \underset{\overrightarrow{}}{\overset{\overrightarrow{}}{\rightleftarrows}} \quad \mathcal{C} \quad \overset{\longrightarrow}{\underset{\longleftarrow}{\rightleftarrows}} [0]$$

Second, apply the relative nerve (cf. Definition 4.28) to obtain a coCartesian fibration

$$\mathrm{End}(\mathcal{D})^{\oplus} \to \mathrm{N}(\Delta^{op}).$$

Exercise 4.37. Check that $\mathrm{End}(\mathcal{D})^{\oplus} \to \mathrm{N}(\Delta^{op})$ is a monoidal ∞-category.

4.7. **Algebra objects.** To generalise the notion of a monad to a higher categorical context, we first need to define what we mean by an algebra A in a monoidal ∞-category $(\mathcal{C}, \circ, \mathbf{1})$.

We certainly want to specify a multiplication map $A \circ A \to A$, which, by diagram (2), is equivalent to lifting the morphism $m : [2] \to [1]$ in Δ^{op} drawn below along $p : \mathcal{C}^{\oplus} \to \mathrm{N}(\Delta^{op})$.

We can also specify higher compositions (e.g. $A \circ A \circ A \xrightarrow{m \circ \mathrm{id}} A \circ A$) as lifts of corresponding maps in Δ^{op}. One might hope that algebra objects are simply sections of $p : \mathcal{C}^{\oplus} \to \mathrm{N}(\Delta^{op})$.

This is almost true, but we need to make sure that certain dull morphisms have dull lifts:

Definition 4.38 (Inert morphism)**.** A morphism $f : [n] \to [m]$ in Δ^{op} is *inert* if every bullet $\bullet$ in $[m]$ has a unique preimage in $[n]$:

Definition 4.39 (Algebras)**.** An *algebra* in a monoidal ∞-category $p : \mathcal{C}^{\oplus} \to \mathrm{N}(\Delta^{op})$ is a section $s : \mathrm{N}(\Delta^{op}) \to \mathcal{C}^{\oplus}$ of p sending inert morphisms to p-coCartesian morphisms.

Exercise 4.40. Show that if $s : \mathrm{N}(\Delta^{op}) \to \mathcal{C}^{\oplus}$ specifies an algebra, then $s([2])$ corresponds to the pair $(s([1]), s([1]))$ under the equivalence $\mathcal{C}^{\oplus}_{[2]} \simeq \mathcal{C} \times \mathcal{C}$.

Finally, we can generalise Definition 3.2 from last lecture to the setting of ∞-categories:

Definition 4.41 (Monads)**.** A monad on an ∞-category $\mathcal{C}$ is an algebra object in $\mathrm{End}(\mathcal{C})$.

References

[Lur07] Jacob Lurie, *Derived algebraic geometry II: Noncommutative algebra*, Preprint from the author's web page (2007).

[Lur09] _____, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR 2522659