Topics in Koszul Duality, Michaelmas 2019, Oxford University

Lecture 5: The Barr–Beck–Lurie Theorem

Last week, we discussed several basic notions in higher category theory. In particular, we introduced monoidal ∞-categories $\mathcal{C}^{\circledast} \to \mathrm{N}(\Delta^{op})$, defined the relative nerve construction $(J \to \mathbf{sSet}) \rightsquigarrow (\mathrm{N}_F(J) \to \mathrm{N}(J))$, and used it to equip ∞-categories of endofunctors $\mathcal{C} = \mathrm{End}(\mathcal{D})$ with monoidal structures. This allowed us to define monads on ∞-categories as algebra objects in endofunctors.

Today, we will explain Lurie's higher categorical variant of Barr–Beck's monadicity theorem from Lecture 3. This generalisation is a key tool in constructing equivalences between higher categories, and will be of great value in our later applications.

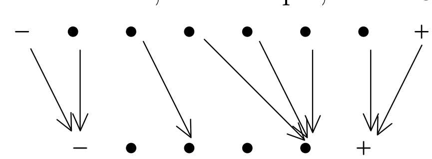
4.1. **Algebras over monads.** To state the monadicity theorem, we will need to define what we mean by algebras over a monad. We will use the setup of tensored ∞-categories. Let $\mathcal{C}^{\circledast} \to \mathrm{N}(\Delta^{op})$ be a monoidal ∞-category, written informally as $(\mathcal{C}, \otimes, 1)$.

Definition 4.1 (Tensored ∞-categories)**.** A $\mathcal{C}$*-tensored* ∞*-category* is given by a diagram of ∞-categories $\mathcal{M}^{\circledast} \xrightarrow{q} \mathcal{C}^{\circledast} \xrightarrow{p} \mathrm{N}(\Delta^{op})$ satisfying the following conditions:

a) $p \circ q : \mathcal{M}^{\circledast} \to \mathrm{N}(\Delta^{op})$ is a coCartesian fibration;
b) $q : \mathcal{M}^{\circledast} \to \mathcal{C}^{\circledast}$ is a categorical fibration sending $(p\circ q)$-coCartesian to p-coCartesian edges;
c) For all n, the inclusion $\{n\} \subset [n]$ induces an equivalence $\mathcal{M}^{\circledast}_{[n]} \xrightarrow{\simeq} \mathcal{C}^{\circledast}_{[n]} \times \mathcal{M}^{\circledast}_{\{n\}}$.

We say that the ∞-category $\mathcal{M} := \mathcal{M}^{\circledast}_{[0]}$ is equipped with a $\mathcal{C}$-tensored structure, written $\otimes$.

Informally, elements of $\mathcal{M}^{\circledast}_{[n]}$ correspond to tuples $(c_1, c_2, \dots, c_n, m)$ with $c_i \in \mathcal{C}$, $m \in \mathcal{M}$; we think of the c_i's as labels of the bullets and m as a label of the $+$. The $(p\circ q)$-coCartesian lifts tensor according to the arrows; for example, the coCartesian lift of the morphism

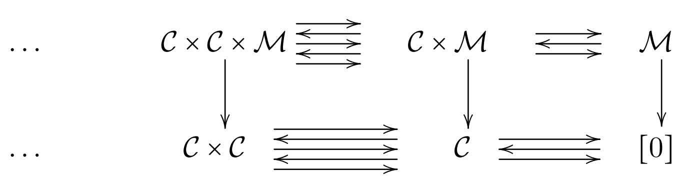


starting at a tuple $(c_1, c_2, c_3, c_4, c_5, c_6, m)$ ends at the tuple $(1, c_2, 1, c_3 \otimes c_4 \otimes c_5, c_6 \otimes m)$.

Example 4.2. Any ∞-category $\mathcal{M} = \mathcal{D}$ is naturally tensored over the monoidal ∞-category $\mathcal{C} = \mathrm{End}(\mathcal{D})$, where the tensoring evaluates functors on objects.

To formally construct this tensored structure, observe that the simplicial set $\mathcal{M} = \mathcal{D}$ is equipped with an action by the simplicial monoid $\mathcal{C} = \mathrm{End}(\mathcal{D})$.

We obtain the diagram $\mathrm{N}(\Delta^{op}) \times \Delta^1 \to \mathbf{sSet}$ drawn below.



Exercise. Applying the relative nerve construction to this diagram gives rise to an $\mathcal{C} = \mathrm{End}(\mathcal{D})$-tensored structure on $\mathcal{M} = \mathcal{C}$.

Let $\mathcal{C}^{\circledast} \xrightarrow{p} \mathrm{N}(\Delta^{op})$ be a monoidal ∞-category and $\mathcal{M}^{\circledast} \xrightarrow{q} \mathcal{C}^{\circledast} \xrightarrow{p} \mathrm{N}(\Delta^{op})$ be a $\mathcal{C}$-tensored ∞-category. Fix an algebra object A in $\mathcal{C}$, parametrised by a section $s : \mathrm{N}(\Delta^{op}) \to \mathcal{C}^{\circledast}$ of p.

Definition 4.3 (Modules)**.** An A-module M in $\mathcal{M}$ consists of a section $s' : \mathrm{N}(\Delta^{op}) \to \mathcal{M}^{\circledast}$ with $q \circ s' = s$ and such that all morphisms drawn below are sent to $(p \circ q)$-coCartesian edges:

$$\begin{tikzcd}
- \arrow[rrd] & \cdots \arrow[rd] & \bullet \arrow[d] & \bullet \arrow[d] & \bullet \arrow[d] & \bullet \arrow[d] & \bullet \arrow[d] & \cdots & \bullet \arrow[d] & + \arrow[d] \\
 & & - & \bullet & \bullet & \bullet & \bullet & \cdots & \bullet & +
\end{tikzcd}$$

Informally, an A-module is an element $M \in \mathcal{M}$ with a multiplication map $A \otimes M \to M$ which is unital and associative up to coherent homotopy.

Definition 4.4 (Algebras over monads)**.** Given a monad T on an ∞-category $\mathcal{D}$, i.e. an algebra object in the monoidal ∞-category $\mathrm{End}(\mathcal{D})$, a T-algebra is simply a T-module object in the $\mathrm{End}(\mathcal{D})$-tensored ∞-category $\mathcal{D}$.

Remark 4.5. One could argue that T-algebras should be called T-modules instead, and this notational convention is indeed implemented in [Lur07]. However, we decided against this for higher consistency with the 1-categorical literature on monads.

4.2. **A reflection on adjunctions.** We wish to construct monads from adjunctions.

For ordinary categories, this was straightforward: given an an adjunction $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ with unit η and counit ϵ, the triple $(T = GF,\ G\epsilon_F : TT \to T,\ \eta : \mathrm{id}_{\mathcal{C}} \to T)$ evidently satisfies the axioms of a monad (cf. Definition 3.2 of Lecture 3).

The corresponding construction for ∞-categories is more complicated, as we must supply an infinite amount of coherence data to specify a monad.

Adjunctions of ∞-categories can be defined in several ways:

a) Most efficiently, we can define an adjunction as a functor $F : \mathcal{C} \to \mathcal{D}$ for which the corresponding coCartesian fibration $\mathcal{M} \to \mathrm{N}(\Delta^1)$ has the property of also being Cartesian.

b) Slightly less efficiently, we could also specify both functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$. However, this datum alone is overdetermined. To fix this, we must specify a unit natural transformation $u : \mathrm{id}_{\mathcal{C}} \to GF$ verifying that F and G are indeed adjoint, which means that $\mathrm{Map}_{\mathcal{D}}(FX, Y) \to \mathrm{Map}_{\mathcal{C}}(GFX, GY) \xrightarrow{u \circ -} \mathrm{Map}_{\mathcal{C}}(X, GY)$ is a weak equivalence for all X, Y.

c) Even less efficiently, we could also specify two functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ and two natural transformations $\eta : \mathrm{id}_{\mathcal{C}} \to GF$, $\epsilon : FG \to \mathrm{id}_{\mathcal{D}}$ satisfying the natural conditions for a unit and counit. Again, this quadruple alone would be overdetermined, which we can fix by also specifying a 2-simplex $\Delta^2 \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$:

$$\begin{tikzcd}
F \arrow[r, "F\eta"] \arrow[rd] & FGF \arrow[d, "\epsilon_F"] \\
 & F
\end{tikzcd}$$

d) ...

Continuing in this fashion, we obtain infinitely many definitions of what an adjunction is; one can prove that all these notions are equivalent up to a contractible space of choices.

Exercise 4.6. Given a functor $F:\mathcal{C}\to\mathcal{D}$ as in a) above, construct a functor $G:\mathcal{D}\to\mathcal{C}$ and a natural transformation $u:\mathrm{id}_{\mathcal{C}}\to GF$ satisfying the confitions specified in b).

For most applications, the most economical definition a) is entirely sufficient. However, the infinitely many coherences required for a monad force us to use the "*least*" efficient definition of adjunctions, which we will explain in the following sections.

4.3. **The $(\infty,2)$-category $\{\mathcal{C},\mathcal{D}\}$.** Given an ∞-category $\mathcal{C}$, we have constructed a monoidal structure $\mathrm{End}(\mathcal{C})^{\circledast}\to \mathrm{N}(\Delta^{op})$ on $\mathrm{End}(\mathcal{C})$ in Definition 4.36 of the preceeding lecture.

Shifting perspective, we may think of $\mathrm{End}(\mathcal{C})$ as a model for the full subcategory $\{\mathcal{C}\}$ of the $(\infty,2)$-category of $(\infty,1)$-categories and (not necessarily invertible) natural transformations.

To construct monads from adjunctions, we will need a similar description of the full subcategory $\{\mathcal{C},\mathcal{D}\}$ spanned by two ∞-categories $\mathcal{C},\mathcal{D}$. To this end, we proceed in three steps:

a) *Define a labelled version $\Delta^{op}_{\{\mathcal{C},\mathcal{D}\}}$ of Δ^{op}.* We have seen that Δ^{op} is modelled by diagrams

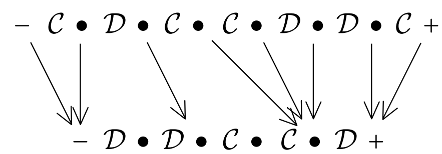
$$[n] = (\,-\quad\bullet\quad\bullet\quad\ldots\quad\bullet\quad+\,),$$

with morphisms corresponding to order-preserving maps sending $-$ to $-$ and $+$ to $+$.

The objects of $\Delta^{op}_{\{\mathcal{C},\mathcal{D}\}}$ are given by such diagrams with gaps labelled by either the symbol $\mathcal{C}$ or the symbol $\mathcal{D}$. For example, we have the following object:

$$(\,-\ \mathcal{C}\ \bullet\ \mathcal{C}\ \bullet\ \mathcal{D}\ \bullet\ \mathcal{D}\ \bullet\ \mathcal{C}\ \bullet\ \mathcal{D}\ +\,),$$

More formally, objects of $\Delta^{op}_{\{\mathcal{C},\mathcal{D}\}}$ are given by pairs $([n]\in\Delta^{op},\ c:[n]\to\{\mathcal{C},\mathcal{D}\})$. Morphisms in $\Delta^{op}_{\{\mathcal{C},\mathcal{D}\}}$ are order-preserving maps sending $-$ to $-$ and $+$ to $+$, which have the additional property that all gaps between two arrows carry the same label:

(1) 

b) *Define a functor $F:\Delta^{op}_{\{\mathcal{C},\mathcal{D}\}}\to\mathbf{sSet}$.* On objects, we define

$$(\,-\ \ c_0\ \bullet\ c_1\ \bullet\ \ldots\ \bullet\ c_{n-1}\ \bullet\ c_n\ +\,)\ \mapsto\ \mathrm{Fun}(c_0,c_1)\times\ldots\times\mathrm{Fun}(c_{n-1},c_n).$$

On morphisms, this functor is defined by composing functors and inserting the identities as dictated from the arrows. For example, the morphism (1) above sends an element $(\mathcal{C}\xrightarrow{F_0}\mathcal{D}\xrightarrow{F_1}\mathcal{C}\xrightarrow{F_2}\mathcal{C}\xrightarrow{F_3}\mathcal{D}\xrightarrow{F_4}\mathcal{D}\xrightarrow{F_5}\mathcal{C})$ to the element $(\mathcal{D}\xrightarrow{\mathrm{id}_{\mathcal{D}}}\mathcal{D}\xrightarrow{F_1}\mathcal{C}\xrightarrow{\mathrm{id}_{\mathcal{C}}}\mathcal{C}\xrightarrow{F_4\circ F_3\circ F_2}\mathcal{D})$.

c) *Unstraighten.* We apply the relative nerve construction introduced in Definition 4.28 of last class to obtain a coCartesian fibration $p:\mathrm{End}(\mathcal{C},\mathcal{D})^{\circledast}\to\mathrm{N}(\Delta^{op}_{\{\mathcal{C},\mathcal{D}\}})$.

This construction will allow us access all functors between $\mathcal{C}$ and $\mathcal{D}$, and all natural transformation between such functors, in an effective way.

4.4. **Adjunction data.** We can now keep track of all higher coherence data of adjunctions. We need the following auxiliary definition:

Definition 4.7. A morphism $([n], c) \to ([m], d)$ is said to be $\mathcal{C}$*-inert* if any $\mathcal{C}$-label in the domain $([n], c)$ sits in one of the following five configurations:

$$
\begin{tikzcd}[column sep=tiny]
- \arrow[d] & \mathcal{C} & \bullet \arrow[dll] & \dots \\
- & \dots & &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
\dots & \bullet \arrow[d] & \mathcal{C} & \bullet \arrow[dll] & \dots \\
\dots & - & \dots & &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
\dots & \bullet \arrow[d] & \mathcal{C} & \bullet \arrow[d] & \dots \\
\dots & \bullet & \mathcal{C} & \bullet & \dots
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
\dots & \bullet \arrow[drr] & \mathcal{C} & \bullet \arrow[d] & \dots \\
& & \dots & + &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
\dots & \bullet \arrow[drr] & \mathcal{C} & + \arrow[d] \\
& & \dots & +
\end{tikzcd}
$$

Definition 4.8 (Adjunction data)**.** An *adjunction datum* for a pair of ∞-categories $(\mathcal{C}, \mathcal{D})$ consists of a section $s : \mathrm{N}(\Delta^{op}_{\{\mathcal{C},\mathcal{D}\}}) \to \mathrm{End}(\mathcal{C}, \mathcal{D})^{\circledast}$ of $p : \mathrm{End}(\mathcal{C}, \mathcal{D})^{\circledast} \to \mathrm{N}(\Delta^{op}_{\{\mathcal{C},\mathcal{D}\}})$ sending $\mathcal{C}$-inert morphisms to p-coCartesian edges. Write $\mathrm{Adj}(\mathcal{C}, \mathcal{D})$ for the full subcategory of $\mathrm{Fun}_{\mathrm{N}(\Delta^{op}_{\{\mathcal{C},\mathcal{D}\}})}(\mathrm{N}(\Delta^{op}_{\{\mathcal{C},\mathcal{D}\}}), \mathrm{End}(\mathcal{C}, \mathcal{D})^{\circledast})$ spanned by such sections.

We now unravel the information contained in an adjunction datum $\mathrm{N}(\Delta^{op}_{\{\mathcal{C},\mathcal{D}\}}) \xrightarrow{s} \mathrm{End}(\mathcal{C}, \mathcal{D})^{\circledast}$.

- First, define two functors $F := s\,(\ -\ \mathcal{C}\ \bullet\ \mathcal{D}\ +\ )$ and $G := s\,(\ -\ \mathcal{D}\ \bullet\ \mathcal{C}\ +\ )$; these will serve as left and right adjoint, respectively.
- Next, define two endofunctors $T := s\,(\ -\ \mathcal{C}\ \bullet\ \mathcal{C}\ +\ )$ and $\iota := s\,(\ -\ \mathcal{D}\ \bullet\ \mathcal{D}\ +\ )$; while T will be the monad induced by the adjunction, ι will just be a version of $\mathrm{id}_{\mathcal{D}}$.
- We obtain a natural transformation $\mathrm{id}_{\mathcal{C}} \to T$ from the following morphism:

$$
\begin{tikzcd}[column sep=small]
& - \arrow[dl] & \mathcal{C} & + \arrow[dr] & \\
- & \mathcal{C} & \bullet & \mathcal{C} & +
\end{tikzcd}
$$

- We obtain a natural equivalence $\mathrm{id}_{\mathcal{C}} \xrightarrow{\simeq} \iota$ from the following $\mathcal{C}$-inert morphism:

$$
\begin{tikzcd}[column sep=small]
& - \arrow[dl] & \mathcal{D} & + \arrow[dr] & \\
- & \mathcal{D} & \bullet & \mathcal{D} & +
\end{tikzcd}
$$

- The triangle $s\,(\ -\ \mathcal{C}\ \bullet\ \mathcal{D}\ \bullet\ \mathcal{C}\ +\ )$ gives two functors $F' : \mathcal{C} \to \mathcal{D}$ and $G' : \mathcal{D} \to \mathcal{C}$.

  Exercise. Use $\mathcal{C}$-inert morphisms to produce equivalences $F' \simeq F$, $G' \simeq G$, $G'F' \simeq T$.

- The triangle $s\,(\ -\ \mathcal{D}\ \bullet\ \mathcal{C}\ \bullet\ \mathcal{D}\ +\ )$ gives functors $G'' : \mathcal{D} \to \mathcal{C}$ and $F'' : \mathcal{C} \to \mathcal{D}$.

  Exercise. Use $\mathcal{C}$-inert morphisms to produce equivalences $F'' \simeq F$, $G'' \simeq G$, and find a non-$\mathcal{C}$-inert morphism inducing a natural transformation $\epsilon : F''G'' \to \mathrm{id}_{\mathcal{D}}$; this will be the counit.

We have produced functors F, G and natural transformations $\eta : \mathrm{id}_{\mathcal{C}} \to GF$, $\epsilon : FG \to \mathrm{id}_{\mathcal{D}}$. A more elaborate argument, which appears as [Lur07, Lemma 3.2.9], then shows that these satisfy the axioms of an adjunction between the homotopy categories $h\mathcal{C}$ and $h\mathcal{D}$.

4.5. **The Barr–Beck–Lurie theorem.** Fix two ∞-categories $\mathcal{C}$ and $\mathcal{D}$, and consider the following maximally efficient definition of adjunctions:

Definition 4.9. A functor $F : \mathcal{C} \to \mathcal{D}$ *is a left adjoint* if the corresponding coCartesian fibration over Δ^1 is also Cartesian. Write $\mathrm{Fun}'(\mathcal{C}, \mathcal{D}) \subset \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ for the subcategory whose objects are left adjoints and whose morphisms are natural equivalences.

It is not hard to show that any adjunction datum $s \in \mathrm{Adj}(\mathcal{C}, \mathcal{D})$ determines a left adjoint $s(-\mathcal{C} \bullet \mathcal{D}+)$, and we can ask how much information is lost in this process. The following hard theorem of Lurie (cf. [Lur07, Theorem 3.2.10]) shows that the infinitely many higher coherences present in an adjunction datum can be added in an essentially unique way:

Theorem 4.10 (Adjunction data from adjunctions)**.** Evaluation gives a trivial Kan fibration

$$\begin{aligned} \mathrm{Adj}(\mathcal{C}, \mathcal{D}) &\to \mathrm{Fun}'(\mathcal{C}, \mathcal{D}) \\ s &\mapsto s(-\mathcal{C} \bullet \mathcal{D}+) \end{aligned}$$

Now let $F : \mathcal{C} \to \mathcal{D}$ be a left adjoint. Using Theorem 4.10 above, we pick a preimage $s \in \mathrm{Adj}(\mathcal{C}, \mathcal{D})$ from a contractible space. Write $G = s(-\mathcal{D} \bullet \mathcal{C}+)$ for the corresponding right adjoint. Restricting s to the full subcategory $\mathrm{N}(\Delta^{op}) \simeq \mathrm{N}(\Delta^{op}_{\mathcal{C}})$ of all diagrams $(-\ \mathcal{C} \bullet \ldots \bullet \mathcal{C}\ +)$ labelled only by $\mathcal{C}$ gives rise to an algebra object T in $\mathrm{End}(\mathcal{C})$.

Just like in the 1-categorical case discussed in Lecture 3, we can construct a diagram

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\widetilde{G}"] \arrow[dr, "G"'] & \mathrm{Alg}_T(\mathcal{C}) \arrow[d, "U"] \\ & \mathcal{C} \end{tikzcd}$$

For a formal construction of the functor $\widetilde{G}$, we refer to [Lur07, Section 3.3].

We are finally ready to state the ∞-categorical monadicity theorem:

Theorem 4.11 (Barr–Beck–Lurie, crude version)**.** Assume that

(1) $\mathcal{D}$ admits and G preserves geometric realisations, i.e. $\mathrm{N}(\Delta^{op})$-shaped colimits;
(2) G is conservative, i.e. if $G(f)$ is an equivalence in $\mathcal{C}$, then so is f in $\mathcal{D}$.

Then the functor $\widetilde{G} : \mathcal{D} \xrightarrow{\simeq} \mathrm{Alg}_T(\mathcal{C})$ is an equivalence of ∞-categories.

In this higher categorical result, geometric realisations play an analogous role to the reflexive coequalisers appearing in the ordinary crude Barr–Beck theorem.

To give a sharp criterion, we need a higher categorical generalisation of split coequalisers. To this end, we introduce the following enlargement of the simplex category:

Definition 4.12. The category $\Delta_{-\infty}$ has objects the finite linearly ordered sets

$$[-1] = \{\ \}\ ,\ [0] = \{0\}\ ,\quad [1] = \{0 < 1\}\ ,\quad [2] = \{0 < 1 < 2\}\ ,\quad \dots\ .$$

Morphisms $[n] \to [m]$ are given by order-preserving maps $[n] \cup \{-\infty\} \to [m] \cup \{-\infty\}$ which send $-\infty$ to $-\infty$; here $-\infty$ is defined as the least element.

Exercise 4.13.

a) Exhibit the simplex category Δ and the augmented simplex category Δ_+ as subcategories of $\Delta_{-\infty}$.
b) Show that any $\Delta_{-\infty}$-indexed diagram in an ordinary category gives a split coequaliser.

Definition 4.14 (Split simplicial objects)**.**

a) A simplicial object $X : \mathrm{N}(\Delta^{op}) \to \mathcal{C}$ in an ∞-category $\mathcal{C}$ is *split* if it extends to $\mathrm{N}(\Delta^{op}_{-\infty})$.
b) Given a functor $G : \mathcal{D} \to \mathcal{C}$, a simplicial object $X : \mathrm{N}(\Delta^{op}) \to \mathcal{D}$ is said to be *G-split* if the simplicial object $G \circ X : \mathrm{N}(\Delta^{op}) \to \mathcal{C}$ is split.

Remark 4.15. If X is a split simplicial diagram, then the restriction of X to Δ_+ is a colimit diagram; in other words, $X([-1])$ is the geometric realisation of $X|_{\mathrm{N}(\Delta^{op})}$.

Theorem 4.16 (Barr–Beck–Lurie, precise version)**.** Given a left adjoint $F : \mathcal{C} \to \mathcal{D}$ as above, the induced functor $\mathcal{D} \to \mathrm{Alg}_T(\mathcal{C})$ is an equivalence if and only if the following conditions hold:

(1) $\mathcal{D}$ admits and G preserves colimits of G-split simplicial diagrams in $\mathcal{D}$;
(2) G is conservative.

In practical applications, (1) is usually much harder to check than (2). In the next weeks, we will give several concrete applications of this result.

References

[Lur07] Jacob Lurie, *Derived algebraic geometry II: Noncommutative algebra*, Preprint from the author's web page (2007).