Topics in Koszul Duality, Michaelmas 2019, Oxford University

Lecture 8: Koszul Duality for Commutative Algebras

In previous lectures, we studied Koszul duals of associative algebras and their modules. Indeed, recall that given an augmented differential graded algebra A over a field k, we set

$$\mathfrak{D}^{(1)}(A) = \mathbb{R}\operatorname{Hom}_A(k,k), \tag{1}$$

and given a chain complex of left A-modules M, we defined

$$M \in \operatorname{Mod}_A \quad \mapsto \quad \mathbb{R}\operatorname{Hom}_A(k,M) \in \operatorname{Mod}_{\mathfrak{D}^{(1)}(A)^{\mathrm{op}}}. \tag{2}$$

There is also a *contravariant* Koszul duality functor, which is given by

$$M \in \operatorname{Mod}_A \quad \mapsto \quad \mathbb{R}\operatorname{Hom}_A(M,k) \in \operatorname{Mod}_{\mathfrak{D}^{(1)}(A)}. \tag{3}$$

But as the basic building blocks of algebraic geometry are commutative algebras, it is important that we also address the following question:

Question 1. What is the right way of dualising an augmented *commutative* k-algebra R?

We could, of course, simply treat R as an augmented associative k-algebra and consider its Koszul dual $\mathfrak{D}^{(1)}(R)$, which is generally no longer commutative.

Exercise 8.1. Find an example of an augmented commutative k-algebra R such that $\mathfrak{D}^{(1)}(R)$ is not commutative.

However, the construction $R \mapsto \mathfrak{D}^{(1)}(R)$ is not optimal for commutative algebras. Today, we will use the commutative nature of R to define a much smaller Koszul dual $\mathfrak{D}(R)$, which will in fact carry the structure of a (generalised) Lie algebra.

8.1. From associative to commutative Koszul duality.

The key idea which allows us to take Koszul duals of augmented commutative algebras can be summarised as follows: we should *not* modify the Koszul duality functor for augmented associative algebras in (1); instead, we should generalise the contravariant Koszul duality functor for modules in (3).

At a first glance, this does not seem to make any sense, as the category $\mathrm{CR}_k^{\mathrm{aug}}$ of augmented commutative k-algebras is not equivalent to modules over any specific ring.

Taking a closer look, however, we realise that the category $\mathrm{CR}_k^{\mathrm{aug}}$ is controlled by an augmented monad $\operatorname{Sym}^* = \bigoplus_n (-)^{\otimes n}_{\Sigma_n}$ on $\operatorname{Mod}_k^{\heartsuit}$, i.e. an augmented associative algebra object in the monoidal category of endofunctors on $\operatorname{Mod}_k^{\heartsuit}$.

As Koszul duality is an inherently derived phenomenon, we will in fact need to enlarge $\mathrm{CR}_k^{\mathrm{aug}}$ to the larger ∞-category $\mathrm{SCR}_k^{\mathrm{aug}}$ of *augmented simplicial commutative k-algebras*

$$\ldots \substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow} R_1 \substack{\leftarrow\\ \rightarrow\\ \rightarrow} R_0.$$

To construct $\mathrm{SCR}_k^{\mathrm{aug}}$, we will need the important $\mathcal{P}_\Sigma$-contruction, which we will introduce in Section 8.4 below.

8.2. **Warmup: Contravariant Koszul duality for modules.** But first, let us unravel the contravariant Koszul duality functor

$$M \in \mathrm{Mod}_A \quad \mapsto \quad \mathbb{R}\,\mathrm{Hom}_A(M,k) \in \mathrm{Mod}_{\mathfrak{D}(A)}$$

for chain complexes of modules over an augmented k-algebra A in (3) above.

This functor can be constructed in four steps:

(1) Take the left derived tensor product to define a colimit-preserving functor

$$k \otimes_A^L (-) : \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_k;$$

its right adjoint is restriction of scalars along the augmentation $A \to k$;

(2) Postcompose with k-linear duality $(-)^\vee$ to obtain a limit-preserving functor

$$\begin{aligned} \mathrm{Mod}_A^{\mathrm{op}} &\longrightarrow \mathrm{Mod}_k \\ M \quad &\mapsto \quad (k \otimes_A^L M)^\vee \simeq \mathbb{R}\,\mathrm{Hom}_A(M,k); \end{aligned}$$

(3) Construct a differential graded k-algebra $\mathfrak{D}^{(1)}(A) = \mathbb{R}\,\mathrm{Hom}_A(k,k)$;

(4) Lift $\mathbb{R}\,\mathrm{Hom}_A(-,k) : \mathrm{Mod}_A^{\mathrm{op}} \to \mathrm{Mod}_k$ to a refined functor

$$\begin{array}{ccc} & & \mathrm{Mod}_{\mathfrak{D}^{(1)}(A)} \\ & \nearrow & \downarrow \\ \mathrm{Mod}_A^{\mathrm{op}} & \longrightarrow & \mathrm{Mod}_k. \end{array}$$

8.3. **From homological to homotopical algebra.** In step (1) above, we implicitly used homological algebra to replace the tensor product $k \otimes_A M$ by the more refined *derived* tensor product $k \otimes_A^L M$. Explicitly, $k \otimes_A^L M$ can be computed by first picking a projective resolution

$$\ldots \to P_2 \to P_1 \to P_0$$

of k and then setting $k \otimes_R^{\mathbb{L}} M := (\ldots \to M \otimes_R P_1 \to M \otimes_R P_0)$, thereby computing the value of the left derived functor of $k \otimes_A (-)$ on M.

To construct the desired commutative Koszul duality functor, we will need to derive the tangent space functor defined on the category of augmented commutative k-algebras. Classical homological algebra allows us to (left) derive additive functors $F : \mathcal{A} \to \mathcal{B}$ from an abelian category $\mathcal{A}$ with enough projectives to an abelian category $\mathcal{B}$. However:

Exercise 8.2. Show that the category of commutative k-algebras is not abelian.

To overcome this difficulty, we make use Quillen's theory of *homotopical algebra*, cf. [Qui06]. Its basic idea is straightforward: in homological algebra, we replaced modules by connective chain complexes (or equivalently simplicial modules) to left derive functors defined on modules; in homotopical algebra, we replace commutative algebras by *simplicial commutative algebras* in order to derive functors defined on algebras.

8.4. **Simplicial commutative rings.** For the rest of this lecture, we work over a commutative ring A. The ∞-category SCR_A of simplicial commutative A-algebras can be defined in two ways:

(1) We can either equip the category of simplicial objects in CR_A, the category of commutative A-algebras, with a suitable model structure and then pass to the underlying ∞-category;
(2) Or we can freely adjoin filtered colimits and geometric realisations to the category Poly_A of polynomial A-algebras $A[X_1, \dots, X_n]$.

Today, we will follow the second approach, as it will give as an excuse to learn an important ∞-categorical technique – the $\mathcal{P}_\Sigma$-construction.

In Lecture 6, we already encountered two ways of formally adjoining certain colimits:

(1) If $\mathcal{C}$ is a small ∞-category, the presheaf ∞-category $\mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ freely adds all colimits.
(2) If $\mathcal{C}$ admits finite limits, the ∞-category $\mathrm{Ind}(\mathcal{C}) = \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ of finite-limit-preserving presheaves freely adds filtered colimits to $\mathcal{C}$.

We will now introduce a third construction in this vein, which simultaneously adjoins filtered colimits and geometric realisations:

Definition 8.3. Given a small ∞-category $\mathcal{C}$ with finite coproducts, we let

$$\mathcal{P}_\Sigma(\mathcal{C}) \subset \mathcal{P}(\mathcal{C})$$

be the full subcategory spanned by all functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ which preserve finite products.

Recall from Definition 4.24 in Lecture 4 that an object is called *compact* if mapping out of it preserves filtered colimits. There is an analogous notion for geometric realisations:

Definition 8.4 (Projective object)**.** Let $\mathcal{C}$ be an ∞-category with geometric realisations. An object $X \in \mathcal{C}$ is called *projective* if the functor

$$\mathrm{Map}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathcal{S}$$

preserves geometric realisations.

The $\mathcal{P}_\Sigma$-construction has the following universal property (cf. [Lur09, Theorem 5.5.8.15]):

Proposition 8.5 (Universal property of $\mathcal{P}_\Sigma$)**.** Let $\mathcal{C}$ be a small ∞-category and let $\mathcal{D}$ be any ∞-category containing filtered colimits and geometric realisations.

Restriction along the Yoneda embedding induces an equivalence

$$\mathrm{Fun}_\Sigma(\mathcal{P}_\Sigma(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

between the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ of all functors $\mathcal{C} \longrightarrow \mathcal{D}$ and the full subcategory $\mathrm{Fun}_\Sigma(\mathcal{P}_\Sigma(\mathcal{C}), \mathcal{D}) \subset \mathrm{Fun}(\mathcal{P}_\Sigma(\mathcal{C}), \mathcal{D})$ spanned by those functors $\mathcal{P}_\Sigma(\mathcal{C}) \longrightarrow \mathcal{D}$ which preserve filtered colimits and geometric realisations.

Notation 8.6 (Nonabelian left derived functors)**.** Given a functor $f : \mathcal{C} \longrightarrow \mathcal{D}$ in the above situation, we call the corresponding functor $Lf : \mathcal{P}_\Sigma(\mathcal{C}) \longrightarrow \mathcal{D}$ the nonabelian left derived functor of f.

This universal property in Proposition 8.5 is very helpful in applications, as it allows us to easily construct functors out of categories of the form $\mathcal{P}_\Sigma(\mathcal{C})$. It is therefore important to identify when a given ∞-category takes this special form.

We state the following key criterion (cf. [Lur09, Proposition 5.5.8.22]):

Proposition 8.7. Let $f : \mathcal{C} \to \mathcal{D}$ be a functor from a small ∞-category $\mathcal{C}$ with finite coproducts to an ∞-category $\mathcal{D}$ with filtered colimits and geometric realisations.

Then $Lf : \mathcal{P}_\Sigma(\mathcal{C}) \longrightarrow \mathcal{D}$ is an equivalence if and only if the following conditions hold:

(1) The functor f is fully faithful;
(2) The essential image of f consists of compact projective objects of $\mathcal{D}$
(3) The essential image of f generates $\mathcal{D}$ under filtered colimits and geometric realisations.

Remark 8.8. If only conditions (1) and (2) hold, then Lf is automatically fully faithful.

Exercise 8.9. Write Vect_A^ω for the ordinary category of finite free (left) A-modules. Construct an equivalence between $\mathcal{P}_\Sigma(\mathrm{Vect}_A^\omega)$ and the full subcategory $\mathrm{Mod}_{A,\geq 0} \subset \mathrm{Mod}_A$ spanned by all connective chain complexes over A.

We will now use the $\mathcal{P}_\Sigma$-construction to define simplicial commutative algebras.

Definition 8.10. Write $\mathrm{Poly}_A \subset \mathrm{CR}_A$ be the full subcategory spanned by all commutative A-algebras of the form

$$A[x_1, \dots, x_n].$$

The ∞-category of *simplicial commutative A-algebras* is defined as

$$\mathrm{SCR}_A := \mathcal{P}_\Sigma(\mathrm{Poly}_A)$$

Exercise 8.11.

a) Define a model structure on the category $\mathbf{SCR}_A$ of simplicial objects in commutative A-algebras whose weak equivalences are weak equivalences of underlying simplicial sets and whose fibrations are levelwise surjections.
b) Prove that the underlying ∞-category of $\mathbf{SCR}_A$ is equivalent to $\mathrm{SCR}_A := \mathcal{P}_\Sigma(\mathrm{Poly}_A)$.

Exercise 8.12. Let k be a field and write $\mathbf{cdga}_k$ for the category of commutative differential graded k-algebras.

(1) Show that if $\mathrm{char}(k) = 0$, the category $\mathbf{cdga}_k$ admits a model structure whose weak equivalences are given by quasi-isomorphisms and whose fibrations are given by levelwise surjections. Writing cdga_k for the underlying ∞-category of $\mathbf{cdga}_k$, show that SCR_k is equivalent to full subcategory $\mathrm{cdga}_{k,\geq 0} \subset \mathrm{cdga}_k$ spanned by all connective objects.
(2) Show that if $\mathrm{char}(k) = p$, the category $\mathbf{cdga}_k$ *does not* admit a model structure whose weak equivalences are given by quasi-isomorphisms and whose fibrations are given by levelwise surjections.

Let $A \in \mathrm{CR}$ again be a general commutative ring. We need two variants of Definition 8.10:

Variant 8.13. Let $\mathrm{Poly}_A^{\mathrm{aug}}$ be the category of augmented commutative A-algebras of the form

$$A[x_1,\dots,x_n] \to A$$

with morphisms given by those maps of A-algebras which commute with the augmentation. The ∞-category of *augmented simplicial commutative A-algebras* (or *animated* nonunital A-algebras) is then defined as $\mathrm{SCR}_A^{\mathrm{aug}} := \mathcal{P}_\Sigma(\mathrm{Poly}_A^{\mathrm{aug}})$.

Variant 8.14. Let $\mathrm{Poly}_A^{\mathrm{nu}}$ be the category of nonunital commutative A-algebras of the form

$$\mathrm{I}\,A[x_1,\dots,x_n] = \ker(A[x_1,\dots,x_n] \to A).$$

The ∞-category of *nonunital simplicial commutative A-algebras* (or *animated* nonunital A-algebras) is defined as $\mathrm{SCR}_A^{\mathrm{nu}} := \mathcal{P}_\Sigma(\mathrm{Poly}_A^{\mathrm{nu}})$.

Exercise 8.15.

(1) Show that the augmentation ideal functor $\mathrm{I} : \mathrm{SCR}_A^{\mathrm{aug}} \to \mathrm{SCR}_A^{\mathrm{nu}}$ is an equivalence.
(2) Define the forgetful functor $\mathrm{forget}^{\mathrm{nu}} : \mathrm{SCR}_A^{\mathrm{nu}} \to \mathrm{Mod}_{A,\geq 0}$ and show that it is part of a monadic adjunction $\mathrm{free}^{\mathrm{nu}} \dashv \mathrm{forget}^{\mathrm{nu}}$. Write $\mathbb{L}\,\mathrm{Sym}_A^{\mathrm{nu}}$ for the corresponding monad.
(3) Show that the underlying functor of $\mathbb{L}\,\mathrm{Sym}_A^{\mathrm{nu}}$ is given by $\bigoplus_{n>0} \mathbb{L}\,\mathrm{Sym}_A^n$, where $\mathbb{L}\,\mathrm{Sym}_A^n$ is the left derived functor of the n^{th} symmetric power functor $M \mapsto M_{\Sigma_n}^{\otimes n}$.

8.5. **The cotangent fibre.** The formalism of nonabelian left derived functors will allow us to construct the desired Koszul duality functor for commutative rings.

To begin with, recall that in Step 1) of Section 8.2, the augmentation $A \to k$ gave rise to the restriction-of-scalars functor $\mathrm{Mod}_k \to \mathrm{Mod}_A$, whose left adjoint $k \otimes_A^L (-)$ was the main ingredient for contravariant Koszul duality for modules.

In the commutative setting, we note that, given some $A \in \mathrm{CR}$, the monad

$$\mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}} = \mathbb{L}\,\mathrm{Sym}_A^{\mathrm{nu}} = \bigoplus_{n>0} \mathbb{L}\,\mathrm{Sym}_A^n$$

parametrising nonunital simplicial commutative A-algebras is naturally augmented over the identity monad $\mathbf{1} = \mathbb{L}\,\mathrm{Sym}_A^1$, and the restriction along $\mathbb{L}\,\mathrm{Sym}_A^{\mathrm{nu}} \to \mathbf{1}$ defines a functor $\mathrm{sqz}^{\mathrm{nu}} : \mathrm{Mod}_{A,\geq 0} \to \mathrm{SCR}_A^{\mathrm{nu}}$.

Exercise 8.16.

a) Use the equivalence $\mathrm{Mod}_{A,\geq 0} \simeq \mathcal{P}_\Sigma(\mathrm{Vect}_A^\omega)$ to construct the (nonunital) trivial square-zero algebra functor $\mathrm{sqz}^{\mathrm{nu}}$ rigorously.
b) Show that $\mathrm{sqz}^{\mathrm{nu}}$ admits a left adjoint $\mathrm{cot}^{\mathrm{nu}}$ and construct an equivalence $\mathrm{cot}^{\mathrm{nu}} \circ \mathrm{free}^{\mathrm{nu}} \simeq \mathrm{id}$, where $\mathrm{free}^{\mathrm{nu}} : \mathrm{Mod}_{A,\geq 0} \to \mathrm{SCR}_A^{\mathrm{nu}}$ is left adjoint to the forgetful functor $\mathrm{forget}^{\mathrm{nu}}$.

Definition 8.17 (Cotangent fibre)**.** The *cotangent fibre* of an augmented simplicial commutative A-algebra $R \in \mathrm{SCR}_A^{\mathrm{aug}}$ is given by

$$\cot(R) := \cot^{\mathrm{nu}}(\mathrm{I}\,R),$$

where I is the augmentation ideal functor from Exercise 8.15.

The most general technique for computing cotangent fibres uses the so-called *bar construction.* Indeed, given an augmented simplicial commutative A-algebra $R \in \mathrm{SCR}_A^{\mathrm{aug}}$ with augmentation ideal $\mathrm{I}\,R \in \mathrm{SCR}_A^{\mathrm{nu}}$, we consider the following simplicial object in $\mathrm{SCR}_A^{\mathrm{nu}}$:

$$\mathrm{Bar}_\bullet(\mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}, \mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}, \mathrm{I}\,R) = \Big(\dots \; \substack{\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow} \; \mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}(\mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}(\mathrm{I}\,R)) \; \substack{\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow} \; \mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}(\mathrm{I}\,R) \Big)$$

Exercise 8.18. Use the structure map $\mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}(\mathrm{I}\,R) \to \mathrm{I}\,R$, the monadic multiplication $\mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}} \circ \mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}} \to \mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}$, and the monadic unit $\mathrm{id} \to \mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}$ to define all morphisms in the above simplicial diagram.

Using the structure map of $\mathrm{I}\,R$, we can extend $\mathrm{Bar}_\bullet(\mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}, \mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}, \mathrm{I}\,R)$ to an *augmented* simplicial object in the ∞-category $\mathrm{SCR}_A^{\mathrm{nu}}$.

Proposition 8.19. The induced map

$$|\,\mathrm{Bar}_\bullet(\mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}, \mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}, \mathrm{I}\,R)| \coloneqq \mathrm{colim}_{\Delta^{\mathrm{op}}} \mathrm{Bar}_\bullet(\mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}, \mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}, \mathrm{I}\,R) \xrightarrow{\simeq} \mathrm{I}\,R$$

is an equivalence.

Proof. By Exercise 8.15 (3), the monad $\mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}$ preserves geometric realisations. By [Lur07, Corollary 2.3.7], this implies that the forgetful functor $\mathrm{forget}^{\mathrm{nu}}$ detects geometric realisations, so it suffices to show that $|\,\mathrm{Bar}_\bullet(\mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}, \mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}, \mathrm{I}\,R)| \to \mathrm{I}\,R$ is a colimit diagram in chain complexes. But in Mod_A, this augmented simplicial diagram admits an extra degeneracy, which implies that it is a colimit diagram. □

Corollary 8.20. Given an augmented simplicial commutative ring $R \in \mathrm{SCR}_A^{\mathrm{aug}}$, there is an equivalence

$$\cot(R) \simeq |\,\mathrm{Bar}_\bullet(\mathrm{id}, \mathbb{L}\,\mathrm{Sym}^{\mathrm{nu}}, \mathrm{I}\,R)|$$

Proof. As cot is a left adjoint and therefore preserves colimits, this follows immediately from Exercise 8.16 and Proposition 8.19. □

8.6. **Koszul duals of commutative algebras.** Using the constructions introduced above, we can generalise step (1) and (2) from Section 8.2 to the setting of nonunital simplicial commutative k-algebras as follows:

(1') Take the colimit-preserving cotangent fibre functor

$$\cot : \mathrm{SCR}_k^{\mathrm{aug}} \longrightarrow \mathrm{Mod}_{k,\geq 0}.$$

(2') Postcompose with k-linear duality $(-)^\vee$ to obtain a limit-preserving functor

$$(\mathrm{SCR}_k^{\mathrm{nu}})^{\mathrm{op}} \longrightarrow \mathrm{Mod}_k,\ R \mapsto \cot(R)^\vee$$

Generalising step (3) and (4) to the commutative setting is technically more challenging; these steps are needed to describe the full structure acting on the chain complex $\cot(R)^\vee$, which is needed to recover R in good cases (e.g. for R complete local Noetherian).

We will not have time to cover the details in this class, but shall give a brief outline.

In step (3) above, we defined the Koszul dual of an augmented associative algebra A as $\mathfrak{D}^{(1)}(A) = \mathbb{R}\operatorname{Hom}_A(k,k)$. In the commutative setting, we want to construct a well-behaved Koszul dual monad of the augmented monad $\mathbb{L}\operatorname{Sym}_k^{\mathrm{nu}}$, which we want to think of as "$\mathfrak{D}^{(1)}(\mathbb{L}\operatorname{Sym}_k^{\mathrm{nu}})$".

Abstract nonsense gives rise to a natural candidate T^{naive} via the tangent fibre adjunction, but this is *not* the right monad as it fails to preserve sifted colimits. To circumvent these obstacles, we replace T^{naive} by a more well-behaved monad Lie_k^{π} in [BM19], which is obtained by left Kan extending a certain restriction of T^{naive}.

Algebras for Lie_k^{π} are called *partition Lie algebras*, and there is a natural functor

$$\mathfrak{D} : \mathrm{SCR}_k^{\mathrm{aug}} \to \mathrm{Alg}_{\mathrm{Lie}_k^{\pi}}$$

sending $R \in \mathrm{SCR}_k^{\mathrm{aug}}$ to a partition Lie algebra $\mathfrak{D}(R)$ with underlying chain complex $\cot(R)^{\vee}$.

If $\mathrm{char}(k) = 0$, then Lie_k^{π} is equivalent to the (shifted) differential graded Lie algebra monad; in other words, partition Lie algebras are the same thing as (shifted) differential graded Lie algebras. In characteristic p, however, partition Lie algebras do not reduce to any classically known structure.

The complex $\cot(R)^{\vee}$ together with its partition Lie algebra structure contains a lot of information. For example, we can use the Barr–Beck–Lurie theorem to show:

Theorem 8.21 ([BM19])**.** The functor $\mathfrak{D}$ restricts to an equivalence between complete local Noetherian $R \in \mathrm{SCR}_k^{\mathrm{aug}}$ and partition Lie algebras $\mathfrak{g}$ whose underlying chain complex is coconnective and satisfies $\dim(\pi_i(\mathfrak{g})) < \infty$ for all i.

General partition Lie algebras are Koszul dual to so–called *formal moduli problems*, which encode the derived infinitesimal deformations of algebro-geometric objects over k. This makes partition Lie algebras a helpful tool in deformation theory.

References

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