Motivic homotopy theory I

12-10-2026

1 Classifying topological vector bundles

Recall: + X reasonable top. space (manifold, CW complex)

Then $\mathrm{Vect}_r^{top}(X) \simeq \mathrm{Vect}_r^{top}(X \times [0,1])$

↑ isomorphism classes of rank r complex vector bundles on X

Consequence: if $f, g : X \longrightarrow Y$ are homotopic maps and $E \longrightarrow Y$ is a vect. bundle,

then $f^*E \simeq g^*E$.

+ Let $Gr_{\mathbb{C},r}$ be the Grassmannian of r-planes in $\mathbb{C}^\infty$.

It comes with a universal rank r vb

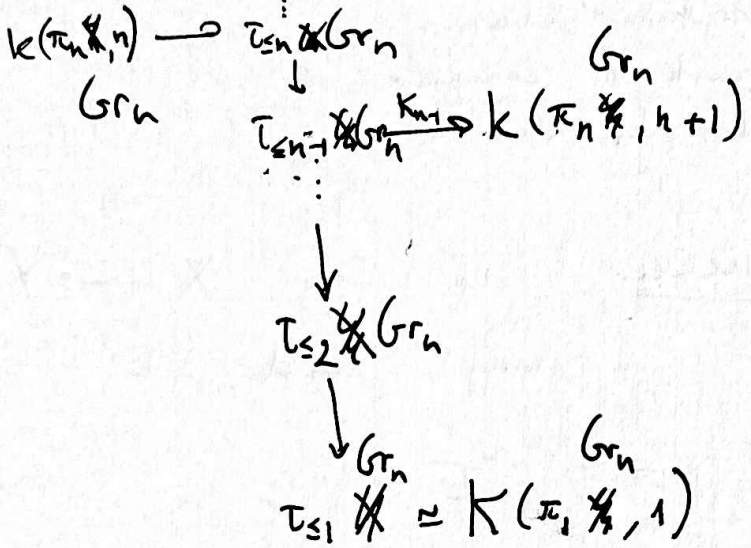
$E_r \longrightarrow Gr_r$, such that we have a bijection

$$[X, Gr_r] \xrightarrow{\sim} \mathrm{Vect}_r(X)$$
$$f \longmapsto f^*E_r.$$

+ **Theorem** (Bott)

$$\pi_k Gr_n \simeq \begin{cases} 0 & k \leq 2n,\ k \text{ odd} \\ \mathbb{Z} & k \leq 2n,\ k \text{ even} \\ \mathbb{Z}/n! & k = 2n+1 \end{cases}$$

+ ~~For any~~ There is a **Postnikov tower**



+ Suppose that we have constructed a map $f: X \to \tau_{\leq k} Gr_n$. There exists a lift of f to $\tau_{\leq k+1} Gr_n$ iff $f^*(k_k) \in H^{k+2}(X; \pi_{k+1} Gr_n)$ vanishes.

Ex: We have $\tau_{\leq 4} Gr_2 \cong K(\mathbb{Z},2) \times K(\mathbb{Z}_{4},4)$, since $\tau_{\leq 2} Gr_2 \simeq \tau_{\leq 3} Gr_2$ and $k_3 \in H^5(K(\mathbb{Z},2), \mathbb{Z})$

$K(\mathbb{Z},2)$ $\qquad$ $\| \; 0$

~~Corollary: If X is n-dimensional~~

- If X is 4-dimensional, $[X, \tau_{\leq 4} Gr_2] = [X, Gr_2]$

since the obstructions for lifting lie in $H^n(X)$ for $n > 4$.

Hence $Vect_2(X) \simeq H^2(X;\mathbb{Z}) \times H^4(X;\mathbb{Z})$

$$E \longmapsto (c_1(E),\ c_2(E))$$

2. ~~Motivic homotopy theory~~ Algebraic vector bundles.

~~Theorem~~ R commutative ring

$$\left\{\begin{array}{l}\text{algebraic vector}\\ \text{bundles on } \operatorname{Spec} R\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{f.g. projective}\\ R\text{-modules}\end{array}\right\}.$$

Theorem (Quillen, Suslin 1976)

k field. Every f.g. projective module over $k[x_1, \dots, x_n]$ is free.

In other words

$$Vect_r^{alg}(\operatorname{Spec} k) \xrightarrow{\sim} Vect_r^{alg}(\mathbb{A}^n_k)$$

Conjecture (Bass-Quillen) X regular Noetherian affine scheme of finite Krull dimension.

Then $$Vect_r^{alg}(X) \xrightarrow{\sim} Vect_r^{alg}(X \times \mathbb{A}^n)$$

Rk: Proved in many cases, for instance if X is of finite type over a field.

3. $\mathbb{A}^1$-invariant motivic spaces

Idea: Construct a homotopy theory of smooth algebraic varieties s.t. $\mathbb{A}^1$ becomes contractible.

Further evidence that this is a reasonable idea: Many important cohomology theories on smooth alg. varieties are $\mathbb{A}^1$-invariant

1) Étale cohomology
2) Algebraic K-theory
3) Chow groups.

Def: k field. Let Sm_k be the category of ~~qcqs~~ finitely presented smooth schemes over $\operatorname{spec} k$.

Step 1: add (homotopy) colimits

Def: Let $\mathcal{P}(Sm_k) := Fun(Sm_k^{op}, \mathcal{S})$
↑ ∞-category of spaces.

- Comes with a Yoneda embedding
$$y : Sm_k \hookrightarrow \mathcal{P}(Sm_k)$$
- $\mathcal{P}(Sm_k)$ has all colimits and $Fun^L(\mathcal{P}(Sm_k), C) \simeq Fun(Sm_k, C)$



Step 2 remember covers

Def A cartesian square

$$\begin{array}{ccc} V & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ U & \xrightarrow{i} & X \end{array} \quad \text{in } Sm_k$$

is an **elementary Nisnevich square** if

- i is an open immersion
- f is étale
- $f : (Y \setminus V)_{red} \longrightarrow (X - U)_{red}$ is an isomorphism.

Ex: f is an open immersion and $Y \cup U = X$.

Def: $F : Sm_k^{op} \longrightarrow Spc$ is a **Nisnevich sheaf** if it sends elementary Nisnevich squares to pullback squares and $F(\emptyset) \simeq *$.

Write $\mathcal{P}_{Nis}(Sm_k) \subseteq \mathcal{P}(Sm_k)$ for the full subcat. on the Nisnevich sheaves.

RK: $y : Sm_k \hookrightarrow \mathcal{P}(Sm_k)$ factors through $\mathcal{P}_{Nis}(Sm_k)$

Hence every elementary Nis. square becomes a pushout in $\mathcal{P}_{Nis}(Sm_k)$.

Step 3: $\mathbb{A}^1$-invariance.

Def: A functor $F : Sm_k^{op} \longrightarrow Spc$ is **$\mathbb{A}^1$-invariant** if $F(X) \xrightarrow{\sim} F(X \times \mathbb{A}^1)$ for all $X \in Sm_k$

Let $\mathcal{P}_{\mathbb{A}^1} \subseteq \mathcal{P}(Sm_k)$ be the full subcat on the $\mathbb{A}^1$-invariant functors.

Def (Morel-Voevodsky) The category of ($\mathbb{A}^1$-) motivic spaces /k Spc_k is $\mathcal{P}_{\mathbb{A}^1}(Sm_k) \cap \mathcal{P}_{Nis}(Sm_k)$.

There is a left adjoint to the inclusion

$$L_{mot} : \mathcal{P}(Sm_k) \longrightarrow Spc_k$$

so every $X \in Sm_k$ gives an object $X \in Spc_k$ by applying the composite

$$Sm_k \xrightarrow{y} \mathcal{P}(Sm_k) \xrightarrow{L_{mot}} Spc_k$$

We write $Spc_{k,*} = (Sm_k)_{Spec\,k/}$ for the category of pointed objects.

Adjunction $Spc_k \underset{\longleftarrow}{\xrightarrow{(-)_+}} (Spc_k)_*$

$$X_+ = X \sqcup *$$

4 Properties of Spc_k

$\mathbb{A}^1$-invariance

In Spc_k, $\mathbb{A}^1 \simeq *$, so that in fact

$$X \times \mathbb{A}^1 \simeq X \quad \text{for all } X \in Spc_k.$$

Realization functors

The functor $Sm_{\mathbb{C}} \longrightarrow Spc$

$$X \longmapsto X(\mathbb{C})$$

extends to a functor

$$Re_{\mathbb{C}} : Spc_{\mathbb{C}} \longrightarrow Spc$$

called the complex realization functor.

Spheres

~~Adjunction $Spc_k \rightleftarrows Spc_k$~~

Def: ~~[illegible]~~

The topological circle is $S^1 := \Sigma(\mathrm{spec}\, k)_+$

Def: The Tate circle is $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$.

We write $S^{p,q} := (S^1)^{\wedge p-q} \wedge \mathbb{G}_m^{\wedge q}$

Rk: $Re_{\mathbb{C}} S^{p,q} \simeq S^p$

Ex $\mathbb{P}^1 \cong S^{2,1}$

proof pushout

$$\begin{array}{ccc} \mathbb{A}^1 \setminus \{0\} & \longrightarrow & \mathbb{A}^1 \cong * \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \\ \| & & \\ * & & \end{array}$$

Hence, $\mathbb{P}^1 \simeq \Sigma \mathbb{A}^1 \setminus \{0\} \simeq S^1 \wedge \mathbb{G}_m$

Ex $\mathbb{A}^n \setminus \{0\} \simeq S^{2n-1,n}$

Homotopy sheaves $X, Y \in Spc_{k,*}$

Write $[X,Y]_{\mathbb{A}^1}$ for the set of homotopy classes of maps $X \to Y$ in Spc_k.

Let $\underline{\pi}_k^{\mathbb{A}^1}(X)$ be the Nisnevich sheafification of $U \longmapsto [U_+ \wedge S^k, X]_{\mathbb{A}^1}$ $U \in Sm_k$.

This is a sheaf of sets for $k=0$

——— groups for $k=1$

——— abelian groups for $k \geq 2$

Thm (Whitehead)

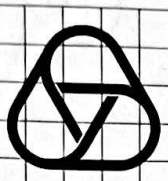
Suppose $X \xrightarrow{f} Y \in Spc_k$ such that

$$f : \underline{\pi}_0^{\mathbb{A}^1}(X) \xrightarrow{\sim} \underline{\pi}_0^{\mathbb{A}^1}(Y)$$

and for all $x \in \pi_0^{\mathbb{A}^1}(X)$

$$f : \underline{\pi}_n^{\mathbb{A}^1}(X,x) \xrightarrow{\sim} \underline{\pi}_n^{\mathbb{A}^1}(Y, f(x)) \quad \forall\, n \geq 1$$

Then f is an equivalence.



Ex $\underline{\pi}_i^{\mathbb{A}^1}(S^n_t) = \underline{\pi_i(S^n)}$ constant sheaf.

Theorem (Morel) k perfect field, ~~char $k \neq 2$~~

$$\underline{\pi}_i^{\mathbb{A}^1}(S^{2n-1,n}) = \underline{\pi}_i^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0) = \begin{cases} 0 & \text{if } i < n-1 \\ \underline{K}_n^{MW} & \text{if } i = n-1 \end{cases}$$

$\underline{K}_n^{MW}$: Unramified Milnor-Witt K-theory sheaf.

+ Fiber sequences give rise to long exact sequences on homotopy sheaves, there are Eilenberg-Mac Lane objects and Postnikov towers.

+ Let $Gr_{r,N}$ be the grassmannian in Sm_k and write $Gr_r = \operatorname{colim}_{N\to\infty} Gr_{r,N}$ in Spc_k.

Theorem (Asok-Hoyois-Wendt)

Let X be a smooth affine k-scheme and $r \geq 0$.

Then

$$Vect_r^{alg}(X) \simeq [X, Gr_r]_{\mathbb{A}^1}.$$

Rk: $Vect_r^{alg}$ is not $\mathbb{A}^1$-invariant for non-affine schemes.

Thm Prop $\pi_0^{\mathbb{A}^1} Gr_2 = *$

$\pi_1 Gr_2 \cong \mathbb{G}_m$

$\pi_2 Gr_2 \cong K_2^{MW}$

"proof" $\pi_0^{\mathbb{A}^1}$: vector bundles are locally trivial.

$\pi_1^{\mathbb{A}^1}$: $\pi_1 Gr_2 = \pi_1 Gr_\infty \cong K_1 \cong \mathbb{G}_m$

($=$: stable range; $\cong$: alg K-theory)

$\pi_2^{\mathbb{A}^1}$: fiber sequence $\mathbb{A}^2 \setminus \{0\} \to Gr_1 \to Gr_2$

$Gr_1 = K(\mathbb{G}_m, 1)$

use LES and apply Morel's result.

+ For X a smooth affine surface, have

$$Vect_2^{alg}(X) = [X, Gr_2]_{\mathbb{A}^1} \cong [\Sigma X, \tau_{\leq 2} Gr_2]_{\mathbb{A}^1}$$

Exact sequence:

$$[X, \Omega\tau_{\leq 2}Gr_2] \xrightarrow{\text{surj.}} [X, K(\mathbb{G}_m, 0)]_{\mathbb{A}^1} \to [X, K^{\mathbb{G}_m}(K_2^{MW}, 2)]_{\mathbb{A}^1} \to [X, \tau_{\leq 2} Gr_2]$$

$[X, K(\mathbb{G}_m, 0)]_{\mathbb{A}^1} = \mathbb{G}_m(X)$

$[X, K^{\mathbb{G}_m}(K_2^{MW}, 2)]_{\mathbb{A}^1} \to [X, \tau_{\leq 2} Gr_2]$: injective

$[X, \tau_{\leq 2} Gr_2] \to [X, \tau_{\leq 1} Gr_2]$

$[X, \tau_{\leq 1} Gr_2] = Pic(X)$



k quadratically closed.

- $[X, K^{Gm}(K_2^{MW}, 2)]_{\mathbb{A}^1} \cong CH^2(X)$

- $Pic(X) \cong CH^1(X)$.

Proposition There is a bijection

$$\begin{aligned} Vect_2^{alg}(X) &\longrightarrow CH^1(X) \times CH^2(X) \\ E &\longmapsto (c_1(E), c_2(E)) \end{aligned}$$