Oxford Motivic Homotopy Lecture 3 Michaelmass Term 2026

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October 30, 2026

1 Intro

Plan :

(i) A little bit of motivation

(ii) Constructing MGL (Algebraic-Geometry version of MU)

(iii) Introduce t-structures on a category

(iv) Understand the "0-part" of MGL

Motivation [6] :
Let's say that it is established that we are interested in objects of $\mathcal{H}(S)$. This is nearly equivalent to being interested in invariants of $\mathcal{H}(S)$. But how do we choose "good invariants"? Stable homotopy teaches us to take those represented by motivic spectra (objects of $S\mathcal{H}(S)$). Some examples :

(i) Motivic cohomology $\leftrightarrow H\mathbb{Z}$.

(ii) Weibel's homotopy K-theory $\leftrightarrow$ BGL

(iii) Algebraic cobordism (the universal oriented cohomology theory) $\leftrightarrow$ MGL.

Quick AI disclaimer : Used to get some reminders in algebraic geometry (and to fix some Latex).
Quick Acknowledgment : Julie Banwart for answering some of my Motivic questions. Thank you to the organizers of the MT26 motivic homotopy theory advanced class.
Quick disclaimer : These are notes originally intended to help me present, use at your own risk, all mistakes are my own.

2 Constructing MGL

This story will very closely // the construction of MU in algebraic topology.

2.1 Motivic Thom spectra

Start with a vector bundle $p : E \to X$ over a smooth scheme of finite type.

Definition 2.1.1. (Section 27.6 in [5]) This is (anti-)equivalently a quasi-coherent module over X or an affine map, such that the pushforward of the structure sheaf of E can be identified with a polynomial algebra over the structure sheaf of X.

There is a zero section $s_p; X \to E$, which is a closed immersion. In particular, $E - s_p(E)$ is a clearly defined open subscheme of E

Remark 2.1.2. Viewing vector bundles as quasi-coherent sheaves, the zero section is given by the unique map to the zero sheaf.

Thinking by analogy with classical homotopy for a second, $E \to X$ can be thought of as the associated "disk"-bundle to E and $E - s_p(X)$ can be thought of as the associated "sphere"-bundle. This motivates.

Definition 2.1.3. (Section 6.3 in [6]) Given a vector bundle $E \to X$ over a smooth scheme of finite type, we denote by $\mathrm{Th}(E)$ the homotopy cofiber in the category of motivic spaces of the inclusion $E - s_p(X) \to E$.

We will be interested in this construction for the universal bundle, which requires a algebro-geometric version of the Grassmanian.

2.2 The Grassmanian

Definition 2.2.1. (27.2 in [5]) We denote by $Gr(k,n) : \mathrm{Aff}^{\mathrm{op}} \to \mathrm{Set}$ the functor which sends a ring R to the set of surjections up to isomorphism $q : R^{\oplus n} \to Q$ where Q is locally free of rank $n-k$. On maps $R \to S$, this functor can be seen as $- \otimes_R S$.

For this to actually exist in the motivic category, we need this to be represented by a scheme.

Lemma 2.2.2. *(27.22.1 in [5]) Let $0 < k < n$ be integers, then the functor $Gr(k,n)$ is representable by a scheme.*

Proof. For convenience we write $F = Gr(n,k) : Aff^{\mathrm{op}} \to \mathrm{Set}$, we need to show that F can be covered by affine sub-functors, such that the maps from these affines are open immersions and that F is a Zariski sheaf. We do not explicitly check the latter condition, which follows from gluing of sheaves.
Let $I = \binom{[n]}{[n-k]}$. Let R be some ring and $\{e_j\}_{j=1}^n$ be a canonical basis of R^n and $\{f_j\}_{j=1}^{n-k}$ the canonical basis of R^{n-k}.
Now for $A \in I$ we let $s_A : R^{n-k} \to R^n$ be the inclusion corresponding to $A \subset [n]$ under the free R-module functor. We let $F_A(R) \subset R$ be the set of maps $R^n \to Q$ with Q locally free of rank $(n-k)$ which become surjective upon precomposition by s_A. F_A is defined on maps in the same way as F and so it is clear that it defines a subfunctor.
We now prove representability of the F_A. We work with $A = [n-k]$ specifically, and all other cases are perfectly analogous. Notice that for $q : R^n \to Q$, the surjectivity of $q \circ s_{[n-k]}$ allows us to identify Q with R^{n-k} as a surjective map between locally free modules of same finite rank is an isomorphism. Under this isomorphism we know $q : R^n \to R^{n-k}$ maps $\{e_i\}_{i=1}^{n-k}$ to $\{f_i\}_{i=1}^{n-k}$, whereas the image of the remaining basis elements can be sent to any element of R^{n-k}. Since there are k remaining basis elements, we see that $F_{[n-k]}(R)$ is $R^{k(n-k)}$, and so $F_{[n-k]} = \mathbb{A}^{k(n-k)}$.
Next, we show that the inclusions $F_A \to F$ are open immersions. Fix a natural map $\mathrm{Spec}(R) \to F$, i.e. an element $q : R^n \to Q \in F(R)$ and consider the pullback

$$\begin{array}{ccc} U_A & \longrightarrow & F_A \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R) & \longrightarrow & F \end{array}$$

We want to identify U with a subset of $\mathrm{Spec}(R)$ the locally ringed space, and we want to show that it is open. To identify the subset, we need to see which points $\iota_{\mathbf{p}} : \mathrm{Spec}(\kappa(\mathbf{p})) \to \mathrm{Spec}(R)$ factor through the functor U_A. Since maps from representables are well understood, we need to study the pullback of sets

$$\begin{array}{ccc} U_A(\kappa(\mathbf{p})) & \longrightarrow & F_A(\kappa(\mathbf{p})) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R)(\kappa(\mathbf{p})) & \longrightarrow & F(\kappa(\mathbf{p})) \end{array}$$

And what we want to understand is the set of $\mathbf{p}$ the element of $\mathrm{Spec}(R)(\kappa(\mathbf{p}))$ corresponding to the residue field at $\mathbf{p}$ is in the image of $U_A(\kappa(\mathbf{p})) \to \mathrm{Spec}(R)(\kappa(\mathbf{p}))$. Notice U_A depends on A and on the

map $\mathrm{Spec}(R) \to F$ (i.e. on $q : R^n \to Q$).
So $U_A = \{\mathbf{p} \mid (q \circ s_A) \otimes_R \kappa(\mathbf{p})$ is surjective$\}$, since $(q \circ s_A) \otimes_R \kappa(\mathbf{p})$ is a map of vector spaces of same dimension $(n-k)$, it is surjective if and only if the determinant is non-vanishing. The determinant is polynomial, and so its non-vanishing locus is an open, which is what we set out to prove.

Now we want to show the F_A cover F, which is to say for any $\mathrm{Spec}(R) \to F$, the U_A cover $\mathrm{Spec}(R)$. By the previous understanding of U_A this amounts to showing that for any $\mathbf{p} \in \mathrm{Spec}(R)$ seen as a locally ringed space, there is some s_A such that

$$\kappa(\mathbf{p})^{n-k} \xrightarrow{s_A} \kappa(\mathbf{p})^n \xrightarrow{q} Q \otimes_R \kappa(\mathbf{p})$$

is surjective. The canonical basis of $\kappa(\mathbf{p})^n$ gets sent to a generating family by $q \otimes_R \kappa(\mathbf{p})$. This family can be refined to a basis, which means there is a subbasis of the cannonical basis of $\kappa(\mathbf{p})$ which is sent to a basis by $q \otimes_R \kappa(\mathbf{p})$. Pick the s_A corresponding to this subbasis, and the claim is proven and in particular so is the whole proof. □

Definition 2.2.3. A map in $\mathrm{Fun}(\mathrm{Aff}^{\mathrm{op}}, \mathrm{Set})$ is called an open immersion if the pullback of this map along any map from a representable is open in the category Aff. These in turn are homeomorphisms onto an open subset such that the sheaf on the domain is a restriction of the sheaf on the former.

Remark 2.2.4. This in particular means we can define $Gr(k,n)_S$ by base change.

The tautological bundle on $Gr(k,n)$ is obtained by taking the universal element of $Gr(k,n)(Gr(k,n))$ (in the Yoneda sense), which is a surjection $\mathcal{O}^{\oplus n}_{Gr(k,n)} \to Q$. Taking the kernel of this yields a quasi coherent sheaf, which we call the tautological bundle of rank d over $Gr(k,n)$ and denote it by $E(k,n)$.

2.3 MGL

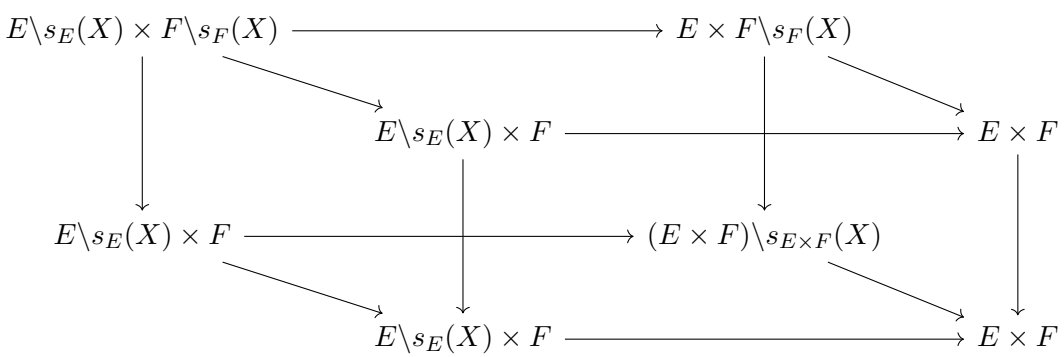
Direct sum with the structure sheaf can be used to take elements of $Gr(k,n)(R)$ and send them to elements of $Gr(k,n+1)(R)$, thus yielding a map of corresponding schemes. These maps induce maps of tautological line bundles, and this way a map of between the Thom spaces $Th(E(k,n)) \to Th(E(k,n+1))$.
The colimit of the above maps yields what we call $MGL(k)$.
We now want to assemble these into a spectrum. Notice that we also have maps of Grassmanians $Gr(k,n) \to Gr(k+1,n+1)$ (given by precomposing a surjection $R^n \to Q$ by the surjection that throws out the $n+1$ summand $R^{n+1} \to R$). Pulling back the tautological bundle on $Gr(k+1,n+1)$ yields the bundle $E(k,n) \times \mathbb{A}^1$.
Using that $Th(\mathbb{A}^1 \oplus F) \simeq \mathbb{P}^1 \wedge Th(F)$ and that $Th(\mathbb{A}^1) \simeq \mathbb{A}^1/\mathbb{A}^1 - \{0\} \simeq \mathbb{P}^1$, we get maps $\mathbb{P}^1 \wedge Th(E(k,n)) \to Th(E(k+1,n+1))$, and thus by taking the colimit in the n-direction we get maps

$$\mathbb{P}^1 \wedge MGL(k) \to MGL(k+1).$$

Proposition 2.3.1.

$$Th(E \oplus F) \simeq Th(E) \wedge Th(F).$$

Proof. We have the following map of pushout squares



By commutativity of colimits, taking the cofiber of each map from back to front yields a pushout square

$$\begin{array}{ccc} (E\backslash s_E(X) \times Th(F))/(E\backslash s_E(X) \times *) & \longrightarrow & (E \times Th(F))/(E \times *) \\ \downarrow & & \downarrow \\ * & \longrightarrow & Th(E \times F) \end{array}$$

This then implies the result.
In order to compute the cofiber of the two top maps from back to front, one uses that motivic spaces are cartesian closed (see section 2 of [4]), at which point one can paste the two following pushouts. □

This yields a spectrum, but to study it we will need to introduce t-structures.

3 t-structures

3.1 Definition

Definition 3.1.1. (1.2.1.4. in [2]) A t-structure on a stable category $\mathcal{C}$ is the data of two full subcategories $\mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq 0}$ such that in the homotopy category $h\mathcal{C}$ we have that $h\mathcal{C}_{\geq 0}$ and $h\mathcal{C}_{\leq 0}$ induce a t-structure on the triangulated homotopy category of $\mathcal{C}$.

Definition 3.1.2. (1.1.2.5. in [2]) A triangulated category C_Δ is an additive category with a translation functor [1] and a distinguished collection of triangles (sequece of composable morphisms $a \to b \to c \to a[1]$) such that

(i) The collection of distinguished triangles is closed under isomorphism

(ii) Any $f : X \to Y$ fits in a distinguished triangle

(iii) A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ is distinguished if and only if $Y \xrightarrow{g} Z \xrightarrow{h} TX \xrightarrow{-Tf} TY$ is distinguished.

(iv) The extension problem below the definition can be solved when the rows are distinguished triangles

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & TX \\ {\scriptstyle\alpha}\downarrow & & \downarrow & & \vdots & & \downarrow{\scriptstyle T(\alpha)} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & TX' \end{array}$$

(v) The octahedral axiom which is cumbersome to write out, but is essentially a statement about the "cofiber of a composite of maps".

Definition 3.1.3. (1.2.1.1. in [2]) Let C_Δ be a triangulated category. A t-structure on C_Δ is the data of two subcategories $C_{\Delta,\geq 0}$ and $C_{\Delta,\leq 0}$, such that, defining $C_{\Delta,\geq n} := C_{\Delta,\geq 0}[n]$ and $C_{\Delta,\leq n} := C_{\Delta,\leq 0}[n]$, we have

(i) For $X \in C_{\Delta,\geq 0}$ and $Y \in C_{\Delta,\leq -1}$ we have $Hom_{C_\Delta}(X, Y) = 0$.

(ii) We have $C_{\Delta,\geq 1} \subset C_{\Delta,\geq 0}$ and $C_{\Delta,\leq -1} \subset C_{\Delta,\leq 0}$.

(iii) For all $X \in C_\Delta$ there is a fiber sequence $X' \to X \to X''$ with $X' \in C_{\Delta,\geq 0}$ and $X'' \in C_{\Delta,\leq -1}$

Remark 3.1.4. Specifying one of $\mathcal{C}_{\leq 0}$ or $\mathcal{C}_{\geq 0}$ determines the other (if you know you have a t-structure) by the fact that $\mathcal{C}_{\geq 0}$ is exactly $\mathcal{C}_{\leq -1}^{\perp}$ with respect to the Hom pairing.
Indeed take Y such that for all $X \in \mathcal{C}_{\geq 0}$ we have $\mathrm{Hom}(X, Y) = 0$. This is in particular true for $X = Y'$ the space given by the third axiom. In particular we have a fiber sequence $Y' \to Y \to Y''$ where the first map is 0, this means the second is an isomorphism, proving the claim.

Example 3.1.5. The category of chain complexes with t-structure given by those chain complexes with homology concentrated in positive (resp. negative) degrees.

3.2 Stray Results to get a vibe

Proposition 3.2.1. *(1.2.1.5. in [2]) We have a diagram of adjoints*

$$\mathcal{C}_{\leq n} \underset{\tau_{\leq n}}{\overset{\perp}{\rightleftarrows}} \mathcal{C} \underset{\tau_{\geq n}}{\overset{\perp}{\rightleftarrows}} \mathcal{C}_{\geq n}$$

Remark 3.2.2. In particular we can interpret $\tau_{\leq n}$ as a localization and $\tau_{\geq n}$ as a colocalization.

Proposition 3.2.3. *(1.2.1.10 in [2]) Let $\mathcal{C}$ be stable with t-structure, the commutative square*

$$\begin{array}{ccc} \mathcal{C}_{\geq n} & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ \mathcal{C}_{\geq n} \cap \mathcal{C}_{\leq m} & \longrightarrow & \mathcal{C}_{\leq m} \end{array}$$

induces a "push-pull" natural map $\theta : \tau_{\leq m} \circ \tau_{\geq n} \to \tau_{\geq n} \circ \tau_{\leq m}$ which is an equivalence.

Notation 3.2.4. By mild abuse of notation we write this functor as $\tau_{n \leq m}$ or $\tau_{=n}$ if $n = m$.

In some ways this is the part of the t-structure of interest (as far as my limited impression of the subject can see), as it is the gateway to the slogan "t-structures let you do homological algebra in stable categories".

Definition 3.2.5. (1.2.1.11. in [2]) Let $\mathcal{C}$ be a stable category with t-structure. Then $\mathcal{C}_{=0} := \mathcal{C}^{\heartsuit}$ is equivalent to the nerve of an abelian category (for which we abusively use the same notation). We define the functors $H^t_n := \tau_{=0} \circ \Sigma^{-n}$.

Remark 3.2.6. I would like to mention in passing one result to justify this intuition that t-structures enable homological algebra that in quasi-full generality we have spectral sequences from filtered objects converging to the t-homology of the filtered object.

3.3 Important Result

We'll use the following result to get a t-structure on our stable motivic category.

Proposition 3.3.1. *(1.2.1.6. in [2] and 1.4.4.11. in [2]) Suppose $\mathcal{C}$ is presentable and stable, and $\mathcal{C}' \subset \mathcal{C}$ is a full presentable subcategory, closed under small colimits and extensions. Then there exists a t-structure on $\mathcal{C}$ such that $\mathcal{C}' = \mathcal{C}_{\geq 0}$*

Proof. First notice that by the adjoint functor theorem, there is a right adjoint functor $\lrcorner : \mathcal{C} \to \mathcal{C}'$. We define $\mathcal{C}_{\geq 0} := \{A \in \mathcal{C} \mid \lrcorner A \simeq A\} \simeq \mathcal{C}'$ and $\mathcal{C}_{\leq -1} := \{A \in \mathcal{C} \mid \lrcorner A \simeq 0\}$.
The first thing to verify is that given $X \in \mathcal{C}_{\geq 0}$ and $Y \in \mathcal{C}_{\leq -1}$, we have $\mathrm{Hom}(X, Y) = 0$. To see this, note

$$\mathrm{Hom}(X, Y) = \mathrm{Hom}(\lrcorner X, Y) = \mathrm{Hom}_{\mathcal{C}'}(\lrcorner X, \lrcorner Y) = \mathrm{Hom}_{\mathcal{C}'}(\lrcorner X, 0) = 0.$$

The second thing to prove is that $A \in \mathcal{C}_{\geq 0}$ implies that $\Sigma A \in \mathcal{C}_{\geq 0}$. Specifically we want to show $\lrcorner \Sigma A \simeq \Sigma A$. This follows by the following chain of equivalences (traveled backwards) :

$$\lrcorner \Sigma A \simeq \Sigma A \Leftrightarrow \Omega \lrcorner \Sigma A \simeq \Omega \Sigma A \simeq A \Leftrightarrow \lrcorner \Omega \Sigma A \simeq A \Leftrightarrow \lrcorner A \simeq A.$$

Similarly we need to show that for $A \in \mathcal{C}_{\leq -1}$ we have $\Omega A \in \mathcal{C}_{\leq -1}$. This follows from

$$\lrcorner \Omega A \simeq \Omega \lrcorner A \simeq \Omega 0 \simeq 0.$$

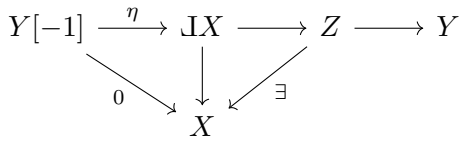
We now work on proving the final axiom of triangulated categories. Let $X \in \mathcal{C}$ be arbitrary, and consider the cofiber sequence

$$\lrcorner X \to X \to CX.$$

We want to show that $\lrcorner CX = 0$, this is equivalent to showing the claim for any $Y \in \mathcal{C}_{\geq 0}$ that $\mathrm{Ext}^0(Y, \lrcorner CX) = 0$, which allows us to work concretely with abelian groups. In particular we consider the following long exact sequence

$$\mathrm{Ext}^0(Y, \lrcorner X) \to \mathrm{Ext}^0(Y, X) \to \mathrm{Ext}^0(Y, C(X)) \to \mathrm{Ext}^1(Y, \lrcorner X) \to \mathrm{Ext}^1(Y, X).$$

We want to show the object in the middle is 0, for which it suffices to show that the last map is injective and the first surjective. The latter claim is direct, using that Y is $\lrcorner$-colocal, the first map is bijective and in particular surjective.
Injectivity of the second map is a smidge more involved. Choose some $\eta : Y[-1] \to \lrcorner X \in \mathrm{Ext}^1(Y, \lrcorner X)$, classifying an extension $\lrcorner X \to Z \to Y$. And suppose this η is mapped to 0 when post composing by $\lrcorner X \to X$. What we want to show is that η itself is 0 in this case, that is to say it classifies the trivial extension. To do this we should produce a section of the map $\lrcorner X \to Z$.
For this notice the following factorization



And notice that Z is an extension of two objects in $\mathcal{C}_{\geq 0}$, and so in particular is in $\mathcal{C}_{\geq 0}$ itself. Apply $\lrcorner$ to the sequence $\lrcorner X \to Z \to X$ and get a factorization of the identity $\lrcorner X \to Z \to \lrcorner X$, in particular a section of $\lrcorner X \to Z$ and so we are done. □

Definition 3.3.2. (Text after 1.2.1.17. in [2]) A t-structure is left complete if $X \simeq \lim_{(\mathbb{Z},\leq)} \tau_{\leq n} X$ and right complete if $X \simeq \mathrm{colim}_{(\mathbb{Z},\leq)} \tau_{\geq n} X$.

4 Understanding $\tau_{=0} MGL$

4.1 The homotopy t-structure

It is immediate that $\mathcal{SH}(S)_{\geq 0}$ the full subcategory generated by $\Sigma^{p,q}\Sigma^\infty X_+$ with $p - q \geq 0$ under colimits and extensions satisfies the assumption of the above proposition. We work with the associated t-structure, called the homotopy t-structure.

To get a feel for this t-structure we include.

Theorem 4.1.1. *(2.3 in [1]) Let k be a field and let $E \in \mathcal{SH}(k)$.*
Then E is d-connective if and only if $\pi_{p,q}E = 0$ for all $p - q < d$.
And E is d-coconnective if and only if $\pi_{p,q}E = 0$ for all $p - q > d$.

Proof. By section 5.2 of Morel's introduction to homotopy theory ([3]) the vanishing of homotopy groups in the specified range defines a t-structure, we call it the $\mathcal{T}$-structure.
We prove the first of the two points for $d = 0$. First notice that the right to left implication follows from suspension spectra of scheme and $\Sigma^{p,q}$ shifts thereof when $p - q \geq 0$ being $\mathcal{T}$-non negative as established by Morel, and in particular it is closed under colimits and extensions. This shows the right to left implication of (1).

To show the left to right implication, take $F \in \mathcal{SH}(S)_{\geq 0}$, we want to show $\mathrm{Hom}(F, E) = 0$ for any E in the ≤ -1 part wrt the $\mathcal{T}$-structure. By closure under colimits, and generation of $\mathcal{SH}(S)_{\geq 0}$ by $\Sigma^{p,q}\Sigma^\infty X_+$ under colimits, we may choose $F = \Sigma^{p,q}\Sigma^\infty X_+$. By adjunction juggling

$$[\Sigma^{p,q}\Sigma^\infty X_+, E] = [\Sigma^{p-q,0}\Sigma^\infty X, \Sigma^{-q,-q}E] = [\Sigma^\infty \Sigma^{p-q,0} X, \Sigma^{-q,-q}E]$$

$$= [\Sigma^{p-q,0} X, \Omega^\infty \Sigma^{-q,-q}E] = [\Sigma^{p-q,0} X_+, L_{\mathbb{A}^1}\Omega^\infty \Sigma^{-q,-q}E]$$

Our assumption is that this (the first term in our chain of equalities) is equal to 0 when $X = S$, which implies that $L_{\mathbb{A}^1}\Omega^\infty\Sigma^{-q,-q}E$ is a presheaf of contractible simplicial sets, which implies this (the final

term in our chain of equalities) is always 0, which shows what we wanted to show.

The second statement follows since one half of a t-structure fully determines the other one. □

Lemma 4.1.2. *(2.4 and 2.5. in [1]) The homotopy t-structure is left and right complete.*

4.2 Connectivity estimates

Lemma 4.2.1. *(3.1 in [1]) Let $p : E \to X$ be a rank d vector bundle in Sm/S. Then $\Sigma^\infty Th(E)$ is d-connective.*

Proof. Let $\{U_\alpha\}_{\alpha\in A}$ be a trivializing Zariski cover on X (i.e. $E|_{U_\alpha} = U_\alpha \times \mathbb{A}^d$ and similarly for $E - s_p(X)$.). Notice we engineered the category of motivic spaces so that

$$\begin{array}{ccc} U_{\alpha\beta} & \longrightarrow & U_\beta \\ \downarrow & & \downarrow \\ U_\alpha & \longrightarrow & U_\alpha \cup U_\beta \end{array}$$

is a homotopy pushout. Iterating, we get that X is the homotopy colimit of the simplicial diagram

$$\cdots \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \bigvee_{\alpha,\beta\in A} U_{\alpha\beta} \;\substack{\longrightarrow\\ \longrightarrow}\; \bigvee_{\alpha\in A} U_\alpha$$

Similar results hold for E and $E - s_p(X)$, and so in particular because colimits commute with eachother, we have that $Th(E)$ is the homotopy colimit of

$$\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \bigvee_{\alpha,\beta\in A} Th(E|_{U_{\alpha\beta}}) \;\substack{\longrightarrow\\ \longrightarrow}\; \bigvee_{\alpha\in A} Th(E|_{U_\alpha})$$

Infinite suspension preserves colimits, and so the claim follows from the fact that $Th(U \times \mathbb{A}^d) \simeq \Sigma^{2d,d} U$ which is d-connective. □

Corollary 4.2.2. *(3.2 in [1]) Let $Z \subset X$ be a closed subscheme of codimension d. Then $\Sigma^\infty X/(X\backslash Z)$ is d-connective*

Proof. Follows a result of Morel and Voevodsky, identifying $X/(X\backslash Z)$ with the Thom space of the normal bundle of Z, which is of rank d. □

Work now with vector bundles directly over S, and say we have $W \subset V$ a subbundle and $\{V_i\}$ a collection of bundles.
We have closed immersions compatible with the tautological bundles

$$i_W : Gr(r, W) \to Gr(r, V)$$

and

$$j_{V_1,\ldots,V_t} : Gr(r_1, V_1) \times \cdots \times Gr(r_t, V_t) \to Gr(r_1 + \cdots + r_t, V_1 \times \cdots \times V_t).$$

Specialize now that $W \subset V \simeq \mathbb{A}^n$ a hyperplane, and $L \subset V$ a complementary line bundle (in particular $Gr(1, L) \simeq S$). So our above maps specialize to maps

$$i_W : Gr(r, W) \to Gr(r, V)$$

and

$$j_{L,W} : Gr(r-1, W) \to Gr(r, V).$$

These maps have disjoint image and are "complementary" in the following sense :
The inclusion $i_W : Gr(r, W) \to Gr(r, V)\backslash Im(j_{L,W})$ is the zero section of a rank r vector bundle p (whose fiber over an S-point P is the vector bundle of r planes in $P \oplus L$ not containing L). Similarly the inclusion $j_{L,W} : Gr(r-1, W) \to Gr(r, V)\backslash Im(i_W)$ is the zero section of a rank $n-r$ vector bundle q, whose fiber over an S point P can be identified with the bundle of lines in a complement of P that are not contained in W.

Lemma 4.2.3. *(3.3 in [1]) The immersion i_W induces an equivalence*

$$Th(E(r,W)) \simeq Th(E(r,V)|_{Gr(r,V)\setminus Im(j_{L,W})}).$$

and $j_{L,W}$ induces an equivalence

$$Th(E(1,L)\times E(r-1,W)) \simeq Th(E(r,V)|_{Gr(r,V)\setminus Im(i_W)}).$$

Proof. (only of the first) The two vector bundles are pullbacks of one another, i.e. equivalent in the category of vector bundles and fiberwise isomorphisms, in particular they are still equivalent upon removing the zero section, and so the Thom constructions are equivalent. □

Lemma 4.2.4. *(3.4 in [1])*

(i) The cofiber of $i_W : Gr(r,W) \to Gr(r,V)$ is stably $n-r$ connective

(ii) The cofiber of $j_{L,W} : Gr(r-1,W) \to Gr(r,V)$ is stably r-connective

(iii) The cofiber of $Th(E(r,W)) \to Th(E(r,V))$ is stably n-connective

(iv) The cofiber of $Th(E(1,L)\times E(r-1,W)) \to Th(E(r,V))$ is stably $2r$-connective

Proof. (only of the first) Notice that a vector bundle $E \to X$ is sent to an equivalence in the motivic category (as it is sent to a fiber sequence with fiber $\mathbb{A}^n \simeq *$). So this cofiber is equivalent to th cofiber of $Gr(r,V)\setminus Im(j_{L,W}) \to Gr(r,V)$ whose stable connectivity is the codimension of $Im(j_{L,W})$. Since $j_{L,W}$ can be seen as a 0-section of rank $n-r$, we see that its image has codimension $n-r$, and so we are done. □

4.3 The computation

We have a Hopf Map : $h : \mathbb{A}^2\setminus\{0\} \to \mathbb{P}^1$, and we denote its homotopy cofiber by $C(h)$. As in the above proof, vector bundles being sent to equivalences means $C(h) \simeq Th(E(1,2))$ by the following diagram

$$\begin{array}{ccc} \mathbb{A}^2\setminus\{0\} & \longrightarrow & \mathbb{P}^1 \\ \sim\uparrow & & \sim\uparrow \\ E(1,2)\setminus s(\mathbb{P}^1) & \longrightarrow & E(1,2) \end{array}$$

This identification means we have a map $C(h) \to MGL_1$. Using the bonding maps of MGL we get maps $\Sigma^{2r-2,r-1}C(h) \to MGL_r$. In the stable category, we can take the r-colimit to get a map

$$\Sigma^{-2,-1}\Sigma^\infty C(h) \to \operatorname{colim} \Sigma^{-2r,-r}\Sigma^\infty MGL_r \simeq MGL.$$

Proposition 4.3.1. *(3.7 in [1]) The above map induces an equivalence on the 0-coconnective truncation.*

Proof. We want to show $\Sigma^{-2,-1}\Sigma^\infty C(h) \to \operatorname{colim} \Sigma^{-2r,-r}\Sigma^\infty MGL_r$ is an equivalence after applying $\tau_{\leq}0$. It suffices to show that each map

$$\Sigma^{-2,-1}\Sigma^\infty C(h)_{\leq 0} \to \Sigma^{-2r,-r}\Sigma^\infty MGL_{r,\leq 0}$$

is an equivalence. To this end notice $\Sigma^{2,1} : \mathcal{SH}(S)_{\leq d} \to \mathcal{SH}(S)_{\leq d+1}$ is an equivalence. So that what we want to show is equivalent to showing that the following map is an equivalence

$$\Sigma^{2r-2,r-1}\Sigma^\infty C(h)_{\leq r} \to \Sigma^\infty MGL_{r,\leq r}.$$

This is the r-truncation of the map we constructed using the attaching maps of the spectrum MGL. We can thus identify it as (the infinite suspension of) the composite in the following diagram going down the first column and then across the last row :

$$\begin{array}{ccccccc}
\Sigma^{2r-2,r-1}Th(E(1,2)) & \xrightarrow{\text{“}j_{L,W}\text{”}} & \Sigma^{2r-4,r-2}Th(E(2,3)) & \xrightarrow{\text{“}j_{L,W}\text{”}} & \cdots & \xrightarrow{\text{“}j_{L,W}\text{”}} & Th(E(r,r+1)) \\
\downarrow{\scriptstyle \text{“}i_W\text{”}} & & \downarrow{\scriptstyle \text{“}i_W\text{”}} & & & & \downarrow{\scriptstyle \text{“}i_W\text{”}} \\
\Sigma^{2r-2,r-1}Th(E(1,3)) & \xrightarrow{\text{“}j_{L,W}\text{”}} & \Sigma^{2r-4,r-2}Th(E(2,4)) & \xrightarrow{\text{“}j_{L,W}\text{”}} & \cdots & \xrightarrow{\text{“}j_{L,W}\text{”}} & Th(E(r,r+2)) \\
\downarrow{\scriptstyle \text{“}i_W\text{”}} & & \downarrow{\scriptstyle \text{“}i_W\text{”}} & & & & \downarrow{\scriptstyle \text{“}i_W\text{”}} \\
\vdots & & \vdots & & & & \vdots \\
\downarrow & & \downarrow & & & & \downarrow \\
\Sigma^{2r-2,r-1}MGL_1 & \longrightarrow & \Sigma^{2r-4,r-2}MGL_2 & \longrightarrow & \cdots & \longrightarrow & MGL_r
\end{array}$$

In particular since we have a bunch of commutative squares, we can also show that the composite along the top row and then down the last column is an equivalence.
By our previous connectivity estimates, the arrows along the top row have stably $r+s$ connective cofiber (where s is the integer appearing in the column of the codomain). And the vertical arrows in the last column have $n+1$ connective cofiber. Since $r+s > r$ and $n+1 > r$ these all become equivalences when we r-truncate, and so we are done! □

Finally, we note that after applying $\Sigma^{-2,-1}\Sigma^\infty$, the defining cofiber sequence of $C(h)$ becomes

$$\Sigma^{1,1}\mathbf{1} \xrightarrow{\eta} \mathbf{1} \to \mathbf{1}/\eta.$$

Theorem 4.3.2. *(3.8 in [1]) The unit $\mathbf{1} \to MGL$ induces an equivalence $\mathbf{1}/\eta_{\leq 0} \to MGL_{\leq 0}$*

Proof. Follows from the above upon identifying the composition $\mathbf{1} \to \mathbf{1}/\eta \simeq \Sigma^{-2,-1}\Sigma^\infty C(h) \to MGL$ with the unit of MGL (which we will not do as this would require an explicit understanding of the ring structure of MGL) □

Corollary 4.3.3. *(3.9 in [1]) The spectrum MGL is connective.*

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