Motivic Homotopy Theory Week 4

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Our main goal today is that we want to show an equivalence

$$\mathrm{MGL}_{\leq 0} \to H\mathbb{Z}_{\leq 0}$$

for $H\mathbb{Z}$ the motivic Eilenberg Maclane Spectrum. This will help towards the overarching goal which is to produce a canonical equivalence of motivic spectra over S,

$$\mathrm{MGL}/(a_1, a_2, \dots)[1/c] \to H\mathbb{Z}[1/c].$$

From last week, we know that MGL is connective, and this comes from constructing an equivalence to a spectrum $(\mathbf{1}/\eta)$ which we need to define.

The strategy to show this equivalence is as follows: First, we will show, using the connectivity theorem of last week (E is d-connective iff $\pi_{p,q}(E) = 0$ for all $p - q \leq d$) to show this motivic Eilenberg maclane spectrum $H\mathbb{Z}$ is indeed connective. With this, we will proceed to produce an equivalence of the coconnective spectra

$$(1/\eta)_{\leq 0} \simeq H\mathbb{Z}_{\leq 0},$$

which will allow us to surmise that $\mathrm{MGL}_{\leq 0} \simeq H\mathbb{Z}_{\leq 0}$.

First, we will need to construct motivic cohomology, starting with the category of correspondences, to also adequately define $H\mathbb{Z}$. Next, we need to construct two K-theories, Milnor K theory and Milnor-Witt K theory which we need. We then will construct some more technical lemmas that will be needed for our proof.

Armed with all this, we then complete our main proofs.

Before we start, lets quickly define $\mathbf{1}/\eta$. Last week, we looked at the Hopf map $h : \mathbb{A}^2 \to \mathbb{P}^2$ (equivalent to $\Sigma^{3,2} \to \Sigma^{2,1}$) and take the cofiber

$$h : \mathbb{A}^2 - \{0\} \to \mathbb{P}^1 \to C(h) = \mathrm{Th}(E(1,2)).$$

After stabilising, this gives a map $\eta : \Sigma^{1,1}\mathbf{1} \to \mathbf{1}$, and then a spectrum $\Sigma^{-2,-1}\Sigma^\infty C(h) = \mathbf{1}/\eta$ which is connective.

(Connectivity: $Th(1,2)$ is 1-connective. Then $\Sigma^{-2,-1}$ changes connectivity by $-2-(-1) = -1$ so that $1/\eta$ is 0 connective as desired).

For simplicitly, we work today in $\mathcal{SH}(k)$ for k a perfect field. The base scheme hypothesis can be loosened to S an essentially smooth K scheme for K an arbitrary field but we provide no details here.

1 Motivic Cohomology

Definition 1.1. Let Cor_k denote the category of smooth separated schemes of finite type/k.

We first construct the elementary correspondences in the case X is connected. Here an elementary correspondence, an irreducible closed subset $W \subseteq X \times Y$ so that for the associated integral subscheme W and projection map $\pi_1 : X \times Y \to X$ we have that $\pi_1 |_W$ is surjective and finite. Then $Cor(X, Y)$ is the free abelian group generated by the elementary correspondences. In the non connected case, we set $Cor(X, Y) = \bigoplus_i Cor(X_i, Y)$ over all connected components of X.

Composition proceeds as follows. For $V \subseteq X \times Y, W \subseteq Y \times Z$ irreducible closed subsets finite and surjective over X/Y respectively, consider the intersection product

$$T = (V \times Z) \cap (X \times W)$$

which we then project under the map $X \times Y \times Z \to X \times Z$. One should check that this definition makes sense, still finite and surjective, MVW Lemma 1.7 gives this result,

Now with this, we can define

Definition 1.2. A presheaf with transfers is a (contravariant) additive functor $F : Cor_k^{op} \to \mathbf{Ab}$.
These assemble into the category $\mathbf{PST}(k)$ on Sm/k (T denotes the transfer part).

The reason for the notation transfer here is as all morphisms $F(W)$ are defined using transfer maps. Firstly, we pull back along the projection $\pi_2 : X \times Y \to Y$. Then we use that by definition $W \to X$ is a finite map, and so we can perform a standard transfer homomorphism, and the composition is $F(W)$.

Particularly of importance is the presheaf $\mathbb{Z}_{tr}(X)$, for X a smooth scheme. It is constructed as $\mathbb{Z}_{tr}(X)(U) = Cor(U, X)$ (schemes considered without base scheme/over Spec $\mathbb{Z}$.). (Will always have that $\mathrm{Hom}_{PST(k)}(\mathbb{Z}_{tr}, F) \simeq F(X)$ by Yoneda, i.e. it is a projective object).

More generally we can define the same for any ring R, which we denote R_{tr}, which we can fit into an adjunction.

In fact, we can say some more. Recall from an earlier talk, $Spc_{S,*}$. Here, $Spc_S = \mathcal{P}_{\mathbb{A}^1}(Sm_S) \cap \mathcal{P}_{Nis}(Sm_S)$, and $Spc_{S,*}$ comes with an additional basepoint. (We require here simplicial presheaves)

We further construct a category $Spc_{tr}(S, R)$ which we build from our correspondence category. We build R valued correspondences over S by looking at $Cor_S(X, Y; R) := Cor_S(X, Y) \otimes_{\mathbb{Z}} R$. Then $Cor(S, R)$ is the category of smooth S schemes with hom sets $Cor_S(X, Y; R)$.

From this we get $Spc_{tr}(S, R)$ is the category of additive (simplicial) presheaves on $Cor(S, R)$ which are $\mathbb{A}^1$-invariant and Nisnevich.

From this we have a functor $R_{tr} : Spc_{S,*} \to Spc_{tr}(S, R)$ which is colimit preserving and is the the unique simplicial functor satisfying that such that $R_{tr}X_+$ is represented by X on $Cor(S, R)$ We then have an adjunction $(R_{tr} \dashv u_{tr})$, where u_{tr} is its restriction along Γ.

Definition 1.3. In particular, we can construct the motivic Eilenberg Maclane space $HR := u_{tr}R_{tr}\mathbb{1}$ in general, and in particular $H\mathbb{Z} := u_{tr}\mathbb{Z}_{tr}\mathbb{1}$

Definition 1.4. The motivic complexes are defined as $\mathbb{Z}(q) = C_*\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q})[-q]$. Here

$$\mathbb{Z}(q)^i = \begin{cases} C_{q-i}\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q}) \text{ vanishes} & i > q \\ \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q}) & i = q. \end{cases}$$

Where here $C_n\mathcal{F}(X) = \mathcal{F}(X \times \Delta^n)$ With this, we define the motivic cohomology groups $H^{p,q}(X, \mathbb{Z}) = \mathbb{H}^p_{Zar}(X, \mathbb{Z}(q))$.

This is computed by taking an injective resolution of the entire chain complex, take global sections, and we then take cohomology (can imagine looking along the totalisation for this computation) (working in the Zariski topology).

This will then be representable by an E_∞ ring spectrum, the motivic Eilenberg Maclane space $H\mathbb{Z}$, as we have defined earlier $H\mathbb{Z} := u_{tr}c_{tr}\mathbb{Z}$. We can then write that

$$H^{p,q}(X, \mathbb{Z}) \cong [\Sigma^\infty X_+, \Sigma^{p,q} H\mathbb{Z}]$$

Now we have these definitions, we want to have some ways to work with this, first we need to introduce two K-theories to be able to provide technical lemmas later.

2 K Theories

Definition 2.1. Milnor K-theory of a field F is defined as follows. Consider the tensor algebra $T(F^*)$ over $\mathbb{Z}$, and then quotient by the ideal generated by elements of the form $x \otimes (1-x), x \in F^*$. We denote it by $K_*^M(F)$ (grading, every element has degree 1, tensoring increases it in the obvious way).

Theorem 2.2.

$$H^{n,n}(\operatorname{Spec} F, \mathbb{Z}) \cong K_n^M(F)$$

We need to consider a slightly more general form of this as follows

Definition 2.3. Milnor Witt K-Theory of a field, denoted $K_*^{MW}(F)$ is the graded associative ring generated by the classes $[u]$ for $[u] \in F^*$ of degree 1, and a symbol η of degree -1, subject to the relations

1. For $a \in F^* - \{1\}, [a] \cdot [1-a] = 0$
2. For all $a, b \in F^*$, $[ab] = [a] + [b] + \eta \cdot [a] \cdot [b]$
3. For all $a \in F^*$, $[a] \cdot \eta = \eta \cdot [a]$
4. Denoting $h := \eta \cdot [-1] + 2$, $\eta \cdot h = 0$. (2 is twice the multiplicative identity of $K_*^{MW}(F)$

We can also see by looking at these definitions that the quotient $K_*^{MW}/\eta \cong K_*^M(F)$. ($\eta$ being the notation here is intentional and we see later how this corresponds to our earlier map η.)

3 Technical Results

Definition 3.1. Take $X \in \mathcal{H}_\bullet(k)$. Then $\pi_p^{\mathbb{A}^1}(X)$ is the sheaf of groups (in the Nisnevich topology) associated to the presheaf

$$U \mapsto [\Sigma^{p,0} U_+, X]_\bullet$$

Definition 3.2. A sheaf of groups (in the Nisnevich topology) $\mathcal{G}$ is strictly $\mathbb{A}^1$-invariant if for all X smooth schemes over k, the map $H^*_{Nis}(X, \mathcal{G}) \to H^*_{Nis}(X \times \mathbb{A}^1, \mathcal{G})$ is an isomorphism. (Here, H^*_{Nis} is precisely analogous to etale cohomology, where we replace etale with Nisnevich in our requirements for open covers)

Remark 3.3. For $E \in \mathcal{SH}(k)$ a ring spectrum, $\pi_{p,q}(E)$ is strictly $\mathbb{A}^1$-invariant for any p, q.

Theorem 3.4. Consider $f : \mathcal{F} \to \mathcal{G}$ a morphism of strictly $\mathbb{A}^1$-invariant Nisnevich sheaves on Sm/k for k a perfect field. Then f is an isomorphism if and only if for all finitely generated field extensions $k \subseteq L$, $\mathcal{F}(L) \to \mathcal{G}(L)$ is an isomorphism

Theorem 3.5 (Vanishing Theorem)**.** For all smooth schemes X $H^{a,b}(X, \mathbb{Z}) = 0$ (vanishes) for $a > b + \dim X$

This comes from the following facts: $\mathbb{Z}(b)$ is zero in degrees $> b$. $H^i_{Zar}(X, F)$ will vanish for any sheaf F in $i > dim X$. Then Hypercohomology spectral sequence gives the results.

Theorem 3.6 (Colimit Theorem)**.** For a finitely generated field extension $k \subseteq F$, and a smooth scheme $X \in Sm/k$, there is an isomorphism

$$H^{*,*}(X_F, A) = \text{colim}_{k \subseteq E \subseteq F, E finitetype} H^{*,*}(X_E, A)$$

Remark 3.7. This is demonstrated by first computing the similar thing for correspondences.

4 Main Results

Theorem 4.1. The motivic Eilenberg Maclane spectrum $H\mathbb{Z}$ is connective in $\mathcal{SH}(k)$

Proof. We only need to check this on finitely generated field extensions as $\pi_{p,q}(H\mathbb{Z})$ is strictly $\mathbb{A}^1$-invariant. This comes directly from motivic cohomology being $\mathbb{A}^1$ invariant. Also, by our connectivity theorem, we only need to check that $\pi_{p,q}(H\mathbb{Z})(\operatorname{Spec} L) = 0$ whenever $p - q < 0$.

We know that

$$\pi_{p,q}(H\mathbb{Z})(\operatorname{Spec} L) = [\Sigma^{p,q}\Sigma^\infty_{\mathbb{P}^1}(\operatorname{Spec} L)_+, H\mathbb{Z}]$$

and while we work in $\mathcal{SH}(k)$ we can freely write this

$$= [\Sigma^\infty_{\mathbb{P}^1}(\operatorname{Spec} L)_+, \Sigma^{-p,-q} H\mathbb{Z}]$$

using our inverse.

With this, we can just apply our vanishing theorem, knowing that the spectrum of a field has dimension zero. To extend this from finite type fields to finitely generated field extensions, we can just use our colimit theorem meaning $H^{p,q}(\operatorname{Spec} L, \mathbb{Z})$ is written as a colimit of $H^{p,q}(\operatorname{Spec} E, \mathbb{Z})$ once we set $X = \operatorname{Spec} k$ in the theorem. □

This will now allow us to show

Theorem 4.2. The unit $\mathbb{1} \to H\mathbb{Z}$ induces an equivalence $(1/\eta)_{\leq 0} \simeq H\mathbb{Z}_{\leq 0}$

Proof. We now know both spectra are connective, so the only information we need to check is in the $\pi_{p,p}$ (Connectivity theorem plus coconnective truncation means everything else is trivial), specifically we want to verify that

$$\pi_{n-1,n-1}(\mathbb{1}) \xrightarrow{\eta} \pi_{n,n}(\mathbb{1}) \to \pi_{n,n}(H\mathbb{Z}) \to 0$$

is exact for all n. We also know it will suffice to check exactness on stalks at finitely generated field extensions L of k.

One important technical detail is that for a finitely generated field extension L, $\operatorname{Spec} L$ is not necessarily smooth over $\operatorname{Spec} k$ even in the perfect case. However, as we can write L as a colimit of field extensions of finite type (which will be smooth as k is perfect), we have that $\operatorname{Spec} L = \varprojlim X_\alpha$ where each X_α is smooth over k.

Now we want to investigate these $\pi_{n,n}$, we know $\pi_{n,n}(H\mathbb{Z})(\operatorname{Spec} L)$ already in terms of Milnor K-theory. On the other side, our Hopf map will yield a global section of the sheaf $\pi_{1,1}(\mathbb{1})$. Further, any element $u \in L^*$ specifies a morphism $\mathbb{G}_m \to L$ and similarly for $\operatorname{Hom}_k(\operatorname{Spec} L, \mathbb{G}_m) = \varinjlim_\alpha \operatorname{Hom}_k(X_\alpha, \mathbb{G}_m)$ giving a germ $[u] \in \pi_{-1,-1}(\mathbb{1})(\operatorname{Spec} L)$ (germ as may come to be defined on some X_α but not the entirety)

One can compute $\pi_n^{\mathbb{A}^1}(S^n \wedge \mathbb{G}_m^{\wedge i}) \cong K_i^{MW}$ (remark 6.42...). In particular, if we look at $\pi_0(S^0 \wedge \mathbb{G}_m^{\wedge i})$, this will be (the sheaf associated to the presheaf

$$U \mapsto [U, \Sigma^{i,i}\mathbb{1}]$$

where this is defined in $\mathcal{H}_\bullet(k)$. If we now consider this in the stable category $\mathcal{SH}(k)$, this will become

$$U \mapsto [\Sigma^\infty U, \Sigma^{i,i}\mathbb{1}]$$

which is just $\pi_{-i,-i}(\mathbb{1})(U)$ as we recall that $\Sigma^{i,i}$ is invertible here.

This means

$$\bigoplus_{n\in\mathbb{Z}} \pi_{n,n}(\mathbb{1})(\operatorname{Spec} L) \cong K_{-*}^{MW}(L)$$

and in particular η and the $[u]$ generate $\bigoplus_{n\in\mathbb{Z}} \pi_{n,n}(\mathbb{1})(\operatorname{Spec} L)$.

From our two K theories ($K_*^{MW}/\eta \cong K_*^M$) we further have an exact sequence

$$\pi_{n-1,n-1}(\mathbb{1})(\operatorname{Spec} L) \xrightarrow{\eta} \pi_{n,n}(\mathbb{1})(\operatorname{Spec} L) \to K_{-n}^M(L) \to 0$$

when we sent u to $\{u\}$ a generator of $K_*^M(L)$.

Then by using our colimit lemma to extend to arbitrary f.g field extensions and knowing that $\pi_{n,n}(H\mathbb{Z})(\operatorname{Spec} F) \cong K_{-n}^M(F)$, we can construct a graded linear isomorphism

$$\lambda : K_{-*}^M(L) \to \bigoplus_{n\in\mathbb{Z}} \pi_{n,n}(H\mathbb{Z})(\operatorname{Spec} L)$$

The only thing that remains to verify our sequences are exact is to check that the following triangle is commutative

$$\begin{array}{ccc} \bigoplus_{n\in\mathbb{Z}} \pi_{n,n}(\mathbb{1})(\operatorname{Spec} L) & \xrightarrow{\quad s \quad} & \bigoplus_{n\in\mathbb{Z}} \pi_{n,n}(H\mathbb{Z})(\operatorname{Spec} L) \\ \Big\downarrow{\scriptstyle t} & \nearrow_{\lambda} & \\ K_{-*}^M(L) & & \end{array}$$

We can do this on generators. First, looking at η, this is by degree reasons. We see $t(\eta)$ will lie in $K_{-1}^M(L) = 0$ by construction. Going the other way, $s(\eta) \in H^{-1,-1}(\operatorname{Spec} L, \mathbb{Z})$. This is necessarily zero, which can be seen by just directly computing the hypercohomology (where $\mathbb{Z}(-1)$ vanishes in the appropriate degrees).

For $[u]$, $t([u]) = \{u\}$, and then we follow through $\lambda\{u\}$. We work again from

$$L^* = \varinjlim_\alpha \operatorname{Hom}_k(X_\alpha, \mathbb{G}_m) \to \varinjlim_\alpha [(X_\alpha)_+, \mathbb{G}_m] \to \varinjlim_\alpha [(X_\alpha)_+, u_{tr}\mathbb{Z}_{tr}\mathbb{G}_m]$$

with this last map induced by the unit $\mathbb{G}_m \to u_{tr}\mathbb{Z}_{tr}\mathbb{G}_m$. But this is also exactly the image of $[u]$ by the unit $\mathbb{1} \to H\mathbb{Z}$, completing that this diagram is commutative, and that the original sequence is exact. □

As a corollary of this we see

Theorem 4.3. $\mathrm{MGL}_{\leq 0} \to H\mathbb{Z}_{\leq 0}$ is an equivalence

This is just applying the two equivalences we already have, and knowing that $\mathrm{M}GL \to H\mathbb{Z}$ is a map of ring spectra.