Rational homotopy theory of automorphisms of manifolds

(following Berglund - Madsen)

Overview talk

1) Diffeomorphisms and automorphisms of manifolds

Let M be a compact (smooth) manifold with boundary ∂M.

Mapping Class Groups

- $\mathrm{Diff}_\partial(M)$: orientation-preserving diffeomorphisms $f : M \longrightarrow M$ s.t. $f|_{\partial M} = id$.

  (A topological gp using the Whitney topology)

- $B\mathrm{Diff}_\partial(M)$: classifying space of $\mathrm{Diff}_\partial(M)$, which classifies (smooth oriented) M-bundles (with ∂)

- $H^*(B\mathrm{Diff}_\partial(M), R)$: Any $\kappa \in H.(B\mathrm{Diff}(M), R)$ gives invariant for M-bundles on X, valued in $H.(X, R)$

$H^*(B\mathrm{Diff}_\partial(M), R)$ is very hard to compute.

For $M = M_{g,1} = \left(\#_g S^d \times S^d \right) \setminus \mathring{D}^{2d}$ and $R = \mathbb{Q}$,

we know $H^k(B\mathrm{Diff}_\partial(M_{g,1}), \mathbb{Q})$ if k is small compared to g

$(2k+4 < g)$.

Setting $M_\infty = \operatorname{colim} M_{\infty,1}$, we have

<u>Theorem</u>: (Madsen-Weiss, Galatius-Randall-Williams)

Morita-Miller-Mumford-classes

$$H^*(B\mathrm{Diff}_\partial(M_{\infty,1}), \mathbb{Q}) = \mathbb{Q}[\underbrace{\kappa_c}_{} \mid c \in B]$$

monomials $e^n p_{i_1} \cdots p_{i_s}$

Euler class

Pontryagin classes

with $\frac{d}{4} < i_t < d$ if $n+s \geq 2$

$\frac{d}{2} < i_1 < d$ if $n=0$ and $s=1$

<u>Open problem</u>: Computing $\pi_*(B\mathrm{Diff}_\partial(M_{g,1}))$

A coarser moduli space:

Homotopy automorphisms

- $\mathrm{aut}_\partial(M) = \{ f : M \to M \text{ cts} \mid f|_{\partial M} = \mathrm{id},\ f \text{ a htpy equiv} \} \subseteq \mathrm{Map}_{\partial M}(M,M)$, equipped with the compact-open topology, is a topological monoid.

- $B\mathrm{aut}_\partial(M)$ classifies M-fibrations (with boundary ∂)

- Any $\kappa \in H_\bullet(B\mathrm{aut}_\partial(M), R)$ gives invariant for M-fibrations on X valued in $H_\bullet(X,R)$

There is a sequence $B\mathrm{Diff}_\partial(M) \longrightarrow B\widetilde{\mathrm{Diff}}{}^b_\partial(M) \longrightarrow B\mathrm{aut}_\partial(M)$

where $B\widetilde{\mathrm{Diff}}{}^b_\partial(M)$ is the topological group of <u>block diffeos</u>

with $\pi_0(B\widetilde{\mathrm{Diff}}{}^b_\partial(M)) = \{\text{diffeos } M \to M \text{ fixing } \partial M\} / \text{pseudoisotopy}$

Main goal:

Compute the rational cohomology and homotopy groups of

$$B\,\mathrm{aut}_\partial(M_{\infty,1}) \qquad B\mathrm{Diff}^b_\partial(M_{\infty,1})$$

in high dimensions and prove homological stability.

We will need the following tools:

① Quillen's dg Lie models for rational spaces

② Koszul duality

③ The universal cover spectral sequence

Tool 1: Rational Lie models for spaces

Fundamental observation:

$S^a \times S^b$ can be obtained from $S^a \vee S^b$ by attaching an $(a+b)$-cell $\rightsquigarrow$ attaching map $S^{a+b-1} \to S^a \vee S^b$.

Given a pointed space X and two maps

$$S^a \xrightarrow{f} X \qquad S^b \xrightarrow{g} Y$$

define the Whitehead product $[f,g]$ as the composite

$$S^{a+b-1} \to S^a \vee S^b \xrightarrow{f \vee g} X$$

Ex1: For $a=b=1$, this is the commutator on $\pi_1 X$

Ex2: $[id_{S^2}, id_{S^2}] = 2\eta$, where $\eta : S^3 \to S^2$ is the Hopf ma

If X is a simply connected space, the Whitehead product equips $\pi_{*+1}(X)$ with the structure of a graded Lie algebra.

Quillen: There is a functor

$$\begin{tikzcd}
 & \mathfrak{g}_X \arrow[r, dashed] & \{\text{d.g. LA's}/\mathbb{Q}\} \arrow[dd, "H_*(-)"] \\
X \arrow[ur, mapsto] & \boxed{\pi_{*+1}(X)\otimes\mathbb{Q} = H_\cdot(\mathfrak{g}_X)} & \\
S_{*,\geq 2} \arrow[r, "\pi_*[-1]"'] \arrow[uurr, dashed] \arrow[rr, bend right, Rightarrow, "(\mathbb{Q}\otimes\pi_*(-))[-1]"'] & \{\text{Graded Lie algebras}\} \arrow[r, "\mathbb{Q}\otimes -"'] & \{\text{Graded LA's}/\mathbb{Q}\}
\end{tikzcd}$$

We call $\mathfrak{g}_X$ the Lie model for X.

The category dgla is equipped with a model structure where $(\mathfrak{g},d) \xrightarrow{f} (\mathfrak{g}',d')$ is a cofibration if, as graded Lie algebras, $\mathfrak{g}' \cong \mathfrak{g} \oplus \mathrm{Free}_{Lie}(V)$ with f = inclusion.

The assignment $X \longmapsto \mathfrak{g}_X$ preserves hocolims.

We give several examples for Lie models.

Examples

Free graded Lie algebra on a generator in degree n-1.

- Spheres : $\mathfrak{g}_{S^n} = \mathbb{L}(x^{n-1})$ for $n \geq 2$

This encapsulates Serre's computation of the rational homotopy groups of spheres:

$$\pi_i(S^a) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q}[a] & \text{if } a \text{ odd} \\ \mathbb{Q}[a] \oplus \mathbb{Q}[2a-1] & \text{if } a \text{ is even} \end{cases}$$

- Wedges of spheres: $\mathfrak{g}_{\bigvee_k S^n} = \mathbb{L}(x_1^{n-1}, \dots, x_k^{n-1})$ (Rational Hilton-Milnor)

- Attaching a cell: given a htpy pushout

$$\begin{tikzcd} S^a \arrow[r, "f"] \arrow[d] & \bigvee_k S^n \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "h \ulcorner"] & M \end{tikzcd}$$

with $f \otimes \mathbb{Q} = w \in \mathbb{L}(x_1, \dots, x_k)$, we have

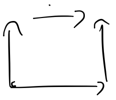
$$\mathfrak{g}_M = (\mathbb{L}(x_1^{n-1}, \dots, x_n^{n-1}, \overset{a-1}{y}), dx_i = 0, dy = w).$$

The assignment $(X \longmapsto \mathfrak{g}_X)$ preserves products

- Products of spheres : $\mathfrak{g}_{S^k \times S^k} \simeq \mathfrak{g}_{S^k} \times \mathfrak{g}_{S^k}$ is modelled by $(\mathbb{L}x \oplus \mathbb{L}y, 0)$

  A cofibrant model of this dg lie algebra is given by

$$\left(\mathbb{L}(x^{k-1}, y^{k-1}, z^{2k-1}),\ dx = 0,\ dy = 0,\ dz = [x, y]\right)$$



- Product of space and sphere

  Let $\mathfrak{g} = (\mathbb{L}U, d_y)$ be a cofibrant model for a s.c. space Y

  Then $S^k \vee Y$ has lie model $\mathfrak{g}_{S^k} \amalg \mathfrak{g}_Y = (\mathbb{L}(x \oplus U), d_y)$

  coproduct in lie algebras

  To obtain a lie model for $S^k \times Y$, we must glue in higher cells ensuring that if $y \in \mathbb{L}U$ has $dy = 0$, then $[x, y]$ is a boundary.

  Set $\mathfrak{g}_{X \times S^k} = \left(\mathbb{L}(\overbrace{x \oplus U}^{X \vee S^k} \oplus \overbrace{U[k-1]}^{\text{higher cells}}), \partial\right)$

  To define ∂, consider the inclusion $i : \mathbb{L}U \longrightarrow \mathbb{L}(x \oplus U \oplus U[k-1])$

  the i-derivation $S : \mathbb{L}U \longrightarrow \mathbb{L}(x \oplus U \oplus U[k-1])$

  $\left(S([a,b]) = [Sa, ib] + (-1)^{(n-1)|a|}[ia, Sb]\right)$ with $S(u) = u[k-1]$

  and set $\partial(x) = 0$, $\partial(y) = d_y(y)$ for $y \in \mathbb{L}U$

  $\partial(y[k-1]) = [x, y] + (-1)^n S(d_y(y))$

- Simply connected covers of Baut-spaces.

Fix a cofibration $A \hookrightarrow X$ of s.c. finite CW cx's modelled by a cofibration $g_A \to g_X$ between cofibrant dgla's.

We have the following result

Theorem (Tanré)

The simply connected cover $B := \widetilde{B\,\mathrm{aut}_A(X)}$ of $B\mathrm{aut}_A(X)$ is modelled by the dgla $g_B = \mathrm{Der}_{g_A}(g_X)^+$ with

$$\mathrm{Der}_{g_A}(g_X)_i = \left\{ \theta : g_* \to g_{*+i} \;\middle|\; \theta[x,y] = [\theta x, y] + (-1)^{i|x|}[x, \theta y],\ \theta|_{g_A} = 0 \right\}$$

with differential $D\theta = d \circ \theta - (-1)^{|\theta|} \theta \circ d$
and bracket $[\theta, \eta] = \theta \circ \eta - (-1)^{|\theta||\eta|} \eta \circ \theta$.

and

$$\mathrm{Der}_{g_A}(g_X)^+_i = \begin{cases} \mathrm{Der}_{g_A}(g_X)_i & \text{if } i \geq 2 \\ \ker\left(d : \mathrm{Der}_{g_A}(g_X)_1 \to \mathrm{Der}_{g_A}(g_X)_0\right) & \text{if } i = 1 \\ 0 & \text{if } i \leq 0 \end{cases}.$$

For instance, take $A=*$. For $k>1$, we have

If G is a top. gp.,

$\pi_pG\times\pi_qG\longrightarrow\pi_{p+q}G$

$((S^p\xrightarrow{f}G),(S^q\xrightarrow{g}G))$

$$\begin{tikzcd}
 & G\times G \arrow[d, "{[-,-]}"] \\
\mapsto S^{p+q}=\frac{S^p\times S^q}{S^p\vee S^q} \arrow[ru, dashed] \arrow[r] & G
\end{tikzcd}$$

$$\begin{tikzcd}
\text{Whitehead product} \arrow[r, bend left] & \pi_{k+1}(\widetilde{B\mathrm{aut}_*(X)})\otimes\mathbb{Q} \arrow[r, phantom, "="] \arrow[d, equal] & H_k(\mathrm{Der}_{g_A}(g_X)^+) \arrow[dd, equal] \\
 & \pi_{k+1}(B\mathrm{aut}_\circ(X))\otimes\mathbb{Q} \arrow[d, equal] & \\
\text{Samelson product} \arrow[r, bend left] & \pi_k(\mathrm{aut}_\circ(X))\otimes\mathbb{Q} \arrow[r, Rightarrow, "\beta"'] & H_k(\mathrm{Der}_{g_A}(g_X))
\end{tikzcd}$$

How is β defined?

Let $g_X=(\mathbb{L}V,d_X)$. Given $f: S^k\longrightarrow\mathrm{aut}_\circ(X)$, we obtain a map $S^k\times X\longrightarrow X$ modelled by

$$(\mathbb{L}(x\oplus V\oplus V[k-1]),\partial)\longrightarrow(\mathbb{L}(V),d)$$

Then $\beta(f)$ is represented by the derivation

$$\mathbb{L}(V)\xrightarrow{S}\mathbb{L}(x\oplus V\oplus V[k-1])\longrightarrow\mathbb{L}(V)$$

Exercise:

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, equal] & X \arrow[d, equal] & & g_A \arrow[r] \arrow[d, equal] & g_X \arrow[d, equal] \\
S^3 \arrow[r, hook] & D^4 & \text{is modelled by} & (\mathbb{L}(x),0) \arrow[r, hook] & (\mathbb{L}(\overset{2}{x},\overset{3}{y}),dy=x)
\end{tikzcd}$$

Using that $S^3\hookrightarrow D^4$ is modelled by $(\mathbb{L}(x),0)\hookrightarrow(\mathbb{L}(x,y),dy=x)$, show that $\widetilde{\mathrm{aut}_{S^3}(D^4)}$ is rationally contractible.

Tool 2: Koszul duality

Natural question: Given a s.c. space X with Lie model $\mathfrak{g}_X = (\mathfrak{g}, d)$, what is $H_*(X, \mathbb{Q})$?

Using that the Koszul dual of the Lie operad is the commutative operad, we will prove:

$$H_*(X, \mathbb{Q}) = H_*^{CE}(\mathfrak{g}_X) = \mathrm{Tor}_*^{U(\mathfrak{g}_X)}(\mathbb{Q}, \mathbb{Q})$$

Where $H^{CE}(\mathfrak{g}_X)$ denotes Lie algebra homology, which can be computed using the Chevalley-Eilenberg complex

$$CE(\mathfrak{g}, d) = (\mathrm{Sym}^\bullet(\mathfrak{g}[1]), S)$$

$$S([x_1|\dots|x_n]) = \sum_{1 \le i \le n} (-1)^{p_1 + \dots + p_{i-1}} x_1 \cdots x_{i-1} \, dx_i \, x_{i+1} \cdots x_n$$

$$+ \sum_{1 \le i < j \le n} (-1)^{p_i (p_{i+1} + \dots + p_{j-1})} x_1 \cdots x_{i-1} x_{i+1} \cdots x_{j-1} [x_i, x_j] x_{j+1} \cdots x_n$$

Tool 3 : The Universal Cover Spectral Sequence

If X is a CW complex with universal cover $\widetilde{X}$ there is a spectral sequence of signature

$$E^2_{pq} = H_p(\pi_1 X, H_q(\widetilde{X}, \mathbb{Q})) \Rightarrow H_{p+q}(X).$$

This is a special case of Grothendieck's composite functor spectral sequence.

To compute the homology of $X - \mathrm{Baut}_\partial(M)$, we will implement the following strategy:

① Find a rational Lie model for the universal cover $\widetilde{X}$

② Compute $H_*(\widetilde{X}, \mathbb{Q}) = H^{CE}_\cdot(\mathfrak{g}_{\widetilde{X}})$ as a $\pi_1 X$-module using Koszul duality

③ Compute the group homology $H_\cdot(\pi_1 X, H_\cdot(\widetilde{X}, \mathbb{Q}))$, study differentials in the universal cover spectral sequence.

Back to manifolds

Let M be a $(d-$ $-$connected manifold of dimension $2d > 2$.

We record several simple facts:

- M simply connected $\Rightarrow$ M orientable $\Rightarrow$ Poincaré duality holds.
- As M is $(d-1)$ connected, this shows that $H_\cdot(M, \mathbb{Z})$ is simple:

$$H_i(M, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 2d \\ \mathbb{Z}^{\beta} & \text{if } i = d \\ 0 & \text{else.} \end{cases}$$

- Given an embedded ball $\mathbb{D}^{2d} \hookrightarrow M$, pick a basis $\alpha_1, \dots, \alpha_k$ for the middle-dimensional homology = homotopy

$$\pi_d(M \setminus D^{2d}) = H_d(M \setminus D^{2d}, \mathbb{Z}) = H_d(M, \mathbb{Z})$$

Whitehead's theorem gives an equivalence

$$\bigvee_{i=1}^{k} S^d \xrightarrow{\simeq} M \setminus \mathring{D}^{2d} =: N$$

Writing $w \in \pi_{2d-1}(\bigcup_{i=1}^{k} S^d)$ for the class corresponding to the inclusion of the boundary of D^{2d} into N

- The homotopy type of M is determined by the following (homotopy) pushout:

$$\begin{tikzcd} S^{2d-1} \arrow[r, "w"] \arrow[d] & \bigvee_{i=1}^{k} S^d \arrow[d] \\ D^{2d} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & M \end{tikzcd}$$

The abelian group $V = H_d(M, \mathbb{Z})$ is equipped with the intersection pairing.

Pick a dual basis $\bar{\alpha}_1, \dots, \bar{\alpha}_k$ to $\alpha_1, \dots, \alpha_k$.

<u>Theorem:</u>

(1) The boundary class $\omega \in \pi_{2d-1}(\bigvee_{i=1}^{k} S^d) \otimes \mathbb{Q}$ is given by $\omega = \frac{1}{2} \sum_{i=1}^{k} [\alpha_i, \bar{\alpha}_i]$

(2) The Lie models of M and $N = M \setminus \mathring{D}^{2d}$ are given by

$$\mathfrak{g}_M = (\mathrm{Free}(x_1, \dots, x_k, y), \delta y = \omega)$$
$$\mathfrak{g}_N = (\mathrm{Free}(x_1, \dots, x_n), 0)$$

(3) The Lie model of $\widetilde{\mathrm{Baut}}_\partial(N)$ is formal and given by $(\mathrm{Der}^+_\omega(LV), 0)$, where V is concentrated in degree $d-1$.

Back to $N = M_{g,1} = \#_g (S^d \times S^d) \setminus \mathring{D}^{2d}$

Set $V_g = H_d(M_{g,1}, \mathbb{Q})$ and $\Gamma_g = \mathrm{Aut}(V_g, \langle -,- \rangle)$

$\left(= O_{g,g} \text{ for } d \text{ even } \left(\begin{array}{c|c} 0 & \mathbb{1} \\ \hline \mathbb{1} & 0 \end{array}\right)\right)$

Using Koszul duality and our Lie model for $\widetilde{B\mathrm{aut}_\partial(M_{g,1})}$,

we observe

$$H_*\left(\widetilde{B\mathrm{aut}_\partial(M_{g,1})}, \mathbb{Q}\right) \cong H_*^{CE}\left(\mathrm{Der}^+_\omega(\mathbb{L}(V_g))\right)$$

Fact: This isomorphism is $\pi_1(B\mathrm{aut}_\partial(M_{g,1}))$-equivariant, where the action on the RHS uses the canonical morphism

$$\begin{tikzcd}
\pi_1(B\mathrm{aut}_\partial(M_{g,1})) \arrow[r] \arrow[d, equal] & \Gamma_g \arrow[d, equal] \\
\pi_0(\mathrm{aut}_\partial(M_{g,1})) \arrow[r, "\circledast"] & \mathrm{Aut}(H_d(M_{g,1}, \mathbb{Q}), \langle -,- \rangle)
\end{tikzcd}$$

Theorem (Berglund-Madsen): The kernel of $\circledast$ is finite.

As the rational homology of finite groups vanishes, we can rewrite the E_2-page of the universal cover spectral sequence in terms of Γ_g:

$$E^2_{p,q} = H_p\left(\Gamma_g, H_q^{CE}\left(\mathrm{Der}^+_\omega(LV_g)\right)\right) \Longrightarrow H_{p+q}\left(B\,\mathrm{aut}_\partial(M_{g,1}), \mathbb{Q}\right)$$

To establish homological stability, we need the following general result:

Theorem (Charney, Vogtman)

If $P : Ab \longrightarrow Ab$ is a polynomial functor on abelian groups of degree $\leq l$ $\left(\text{i.e. } P(-) \simeq \bigoplus_{k=0}^{l} \underset{\text{ab. gp.} + \Sigma_k\text{-action}}{P(k)} \otimes_{\Sigma_k} (-)^{\otimes k} \right)$

then $H_q(\Gamma_g, P(H_g)) \longrightarrow H_q(\Gamma_{g+1}, P(H_{g+1}))$

is an isomorphism for all $g > 2q + l + 4$.

Berglund and Madsen apply this theorem to the "modified" E_2-page $E^2_{p,q} = H_p(\Gamma_g, H^{CE}_q(\mathrm{Der}^+_\omega(LV_g)))$.

Using Borel-vanishing, they prove:

<u>Theorem (Berglund-Madsen)</u>. (Let $d \geq 3$).

(1) For $g > 2k+4$, the following map is an isomorphism:

$$H_k(B\mathrm{aut}_\partial(M_{g,1}), \mathbb{Q}) \longrightarrow H_k(B\mathrm{aut}_\partial(M_{g+1,1}), \mathbb{Q})$$

(2) Writing

$M_{\infty,1} = \mathrm{colim}\, M_{g,1}$, $\Gamma_\infty = \mathrm{colim}\, \Gamma_g$, $\mathfrak{g}_\infty = \mathrm{colim}\, \mathfrak{g}_g$

for $\mathfrak{g}_g = \mathrm{Der}^+_\omega(LV_g)$

there is an isomorphism

$$H^*(B\mathrm{aut}_\partial(M_{\infty,1})) \cong H^*(\Gamma_\infty, \mathbb{Q}) \otimes H^\bullet_{CE}(\mathfrak{g}_\infty)^{\Gamma_\infty}$$

The factor $H^*(\Gamma_\infty, \mathbb{Q})$ is known by a result of Borel, and $H^\bullet_{CE}(\mathfrak{g}_\infty)^{\Gamma_\infty}$ can be computed using graph cohomology.