Describe $H^*(BDiff^+(M_{g,1}); \mathbb{Q})$.

$M_{g,1} = \#^g S^d \times S^d \setminus int(D^{2d})$.

1. $BSO(n)$ classifies oriented vector bundles of rank n
2. $BDiff^+(M)$ classifies smooth oriented M-bundles
3. $H^*(BDiff^+(M_{g,1}); \mathbb{Q})$.

Recall: if G is a topological group, acting freely on a contractible EG, then $BG := EG/G$

Let ξ_n be the functor taking
$$X \mapsto \left\{\begin{matrix}\text{rank } n \text{ vector bundles} \\ \text{on it up to equiv.}\end{matrix}\right\}$$
ξ_n naturally isomorphic to $[-, B]$.

$f \in [X, B]$

$$\begin{array}{ccc} f^*E & \dashrightarrow & E \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & B \end{array}$$

$G_n(\mathbb{R}^\infty)$

Rmk: As $V_n(\mathbb{R}^\infty)$ is weakly contractible, $BO(n) = G_n(\mathbb{R}^\infty)$.

$\boxed{\text{What is the universal bundle?}}$

Let $E_n^q = \{(\bar{x}, v) \in G_n(\mathbb{R}^q) \times \mathbb{R}^q \mid v \in \bar{x}\}$

Let γ_n^q = projection to $G_n(\mathbb{R}^q)$

then E_n, γ_n are colimits as $q \to \infty$.

$$\begin{array}{c} E_n \\ \downarrow \gamma_n \\ BO(n) = G_n(\mathbb{R}^\infty). \end{array}$$

This is the universal rank n bundle.

Oriented: use $\tilde{G}_n(\mathbb{R}^\infty)$

comes from $V_n(\mathbb{R}^\infty)$

quotient by $SO(n)$.

$BSO(n) = \tilde{G}_n(\mathbb{R}^\infty)$.

Thm. $H^*(BSO(2n); \mathbb{Q}) \cong$

$$\mathbb{Q}[p_1, \dots, p_{n-1}, e]/(e^2 - p_n)$$

where p_i are Pontrjagin classes
e is the Euler class.

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② Classifying M-bundles.

Let M be a d-dimensional, closed, smooth manifold.

Def. A smooth oriented M-bundle, $\pi: E \to X$, is a smooth fibre bundle whose fibre is M such that the tangent along the fibre bundle, $T_\pi E$, is oriented.

Def. The vector bundle $T_\pi E$ is
$$\{x \in TE \mid \pi_*(x) = 0\}.$$
This is a $2d$-dimensional bundle over E.

$B\mathrm{Diff}^+(M)$ classifies oriented M-bundles.

Def. Let $\mathrm{Emb}(M, \mathbb{R}^{d+n})$ be the space of smooth embeddings into $\mathbb{R}^{d+n}$.

Let $\mathrm{Emb}(M, \mathbb{R}^{d+n})$ and $\mathrm{Diff}^+(M)$ have the Whitney C^∞ topology.

Let $B_n^+(M) = \mathrm{Emb}(M, \mathbb{R}^{d+n}) / \mathrm{Diff}^+(M)$

Have an M-bundle

$$\begin{array}{c} E_n^+(M) \\ \downarrow \\ B_n^+(M) \end{array}$$

where $E_n^+(M) = \mathrm{Emb}(M, \mathbb{R}^{d\times n}) \times M \big/ (f,x) \sim (\varphi \circ f, \varphi(x))$ for $\varphi \in \mathrm{Diff}^+(M)$.

$$E_n^+(M) \hookrightarrow B_n^+(M) \times \mathbb{R}^{d+n}$$

$$\{e, y\} \longmapsto (\pi(e), e(y))$$

Note $\mathrm{Emb}(M, \mathbb{R}^{d+n}) \subset \mathrm{Emb}(M, \mathbb{R}^{d+n+1}) \subset \cdots$

$\mathrm{Emb}(M, \mathbb{R}^{d+\infty})$ is colimit $n \to \infty$

$E_\infty^+(M)$ ----

$B_\infty^+(M)$ ----

$$\begin{array}{c} E_\infty^+(M) \\ \downarrow \\ B_\infty^+(M) \end{array}$$

is the universal M-bundle.

$$B_\infty^+ \simeq B\mathrm{Diff}^+(M).$$

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(3) Cohomology of $B\mathrm{Diff}^+(M_{g,1})$.

$M_{g,1} = \#^g S^d \times S^d \setminus \mathrm{int}(D^{2d})$

Thm. The stable cohomology is

$$H^*(B\mathrm{Diff}^+_\partial(M_{\infty,1});\mathbb{Q}) \cong \mathbb{Q}[\kappa_c \mid c \in B].$$

$$\begin{array}{ccc} E & & T_\pi E \\ \pi\downarrow & \rightsquigarrow & \downarrow \\ X & & E \end{array}$$

$T_\pi E$ is identified with some $f: E \to BSO(2n)$.

Def. Let c be in $H^k(BSO(2d))$ (i.e. a characteristic class). Then $\kappa_c \in H^{k-2d}(B\mathrm{Diff}^+(M))$ is characterised by

$$\kappa_c(\pi) = \pi_!(c(T_\pi E)).$$

Gysin homomorphism

Let $\pi : E \to X$ with fibre M^d.

Then $\pi_! : H^k(E;\mathbb{Z}) \to H^{k-d}(X;\mathbb{Z})$ with the following properties:

1. for any $\alpha \in H^n(X;\mathbb{Z})$ and $\beta \in H^m(E;\mathbb{Z})$,
$$\pi_!(\pi^*\alpha \cup \beta) = \alpha \cup \pi_! \beta$$

2.
$$\begin{tikzcd} D \arrow[r, "\bar f"] \arrow[d] & E \arrow[d] \\ A \arrow[r, "f"'] & B \end{tikzcd} \qquad \begin{tikzcd} H^*(D;\mathbb{Z}) \arrow[d, "\pi_!"] & H^*(E;\mathbb{Z}) \arrow[l, "\bar f^*"'] \arrow[d, "\pi_!"] \arrow[dl, phantom, "\circlearrowright"] \\ H^*(A;\mathbb{Z}) & H^*(B;\mathbb{Z}) \arrow[l, "f^*"] \end{tikzcd}$$

Recall $\kappa_c \in H^{k-2d}(M_g)$

Take through the pullback from

$\#^g S^d \times S^d$

$B\mathrm{Diff}^+_\partial(M_{g,1}) \to B\mathrm{Diff}^+(M_g)$.

<u>Thm</u> (Madsen–Weiss, Galatius–Randal-Williams)

For $g > 2* + 4$ and $2d \neq 4$, the stable cohomology is

$$H^*(B\mathrm{Diff}_\partial(M_{g,1}); \mathbb{Q}) \cong \mathbb{Q}[\kappa_c \mid c \in B]$$

where $B = \left\{ \begin{array}{l} \text{monomials } c \text{ of total degree } |c| > 2d \\ \text{in Euler class } e \text{ and} \\ \text{Pontrjagin classes } p_{d-1}, \dots, p_{\lceil \frac{d+1}{4} \rceil} \end{array} \right\}$