Rational Homotopy theory

(Quillen '69)

Outline:
- I. Motivation
- II. Technical prependix
- III. Quillen's Thm

I.

Homotopy groups are hard to compute.

But (Serre) if X simply conn.

Then $\pi_* X$ are
finitely generated
finite
zero ← Hurewicz
if the $H_* X$ are

They've also abelian.

So, for X a simply conn. CW complex w. finitely many cells in each dimension,

$$\pi_k X \cong \mathbb{Z}^r \oplus T$$

Rational homotopy theory studies only the free part

$$\pi_k X \otimes \mathbb{Q} \cong \mathbb{Q}^r$$

Easier to compute!

Eg.:

$$\pi_k S^n \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & n = k \text{ or } n \text{ even } \& \ k = 2n-1 \\ 0 & \text{else} \end{cases}$$

← Hopf & higher

Moreover,

Thm. (Serre)

basepoint preserving map of simply connected spaces

Given $f: X \to Y$

$$\pi_*(f)\otimes\mathbb{Q} : \pi_*(X)\otimes\mathbb{Q} \xrightarrow{\simeq} \pi_*(Y)\otimes\mathbb{Q}$$

iff

$$H_*(f,\mathbb{Q}) : H_*(X;\mathbb{Q}) \xrightarrow{\simeq} H_*(Y;\mathbb{Q})$$

"rational homotopy equivalence"

So rational htpy. types are distinguished by rational homology.

II. Technical prependix

Def.: $Top_{*,1}$ category of simply connected, pointed spaces & basepoint preserving cont. maps

Want this to be a model category with

weak equivalences: rational htpy. equivalences

(cofibrations: usual Hurewicz cofibrations

fibrations: Serre fibration with rational htpy. fiber)

Doesn't have finite limits, e.g. pullbacks (it contains BS^1, but not S^1)

But it still defines a htpy. theory. (Serre)

$\Rightarrow$ Can use simplicial model instead.

(Quillen shows that the resulting $\underbrace{\text{htpy. theories}}$ end up being the same.)

$\hookrightarrow$ $\mathrm{Ho}\,\mathcal{C}$ = invert w.eq.
+ what are (co)fiber sequences

We will see that we can model $\mathrm{Top}_{*,1}^{\mathbb{Q}}$ purely algebraically using rational htpy. or homology groups + some extra structure.

DG-stuff:

All over $\mathbb{Q}$ here

__Def.:__ A dg vector space is a pair (V, ∂)

$$V = \bigoplus_{q \in \mathbb{Z}} V_q \qquad \partial : V_\bullet \longrightarrow V_{\bullet - 1} \qquad \text{s.t. } \partial^2 = 0$$

Here:

- $V_q = 0$ for $q < 0$
- maps are degree preserving

They form a symmetric monoidal category with $(V \otimes W)_k = \bigoplus_{i+j=k} V_i \otimes W_j$

"k-reduced" if $V_i = 0$ for $i < k$

1-reduced =: reduced

Def.: A dgla is a dg vector sp. (V, ∂) + a Lie bracket

$[\,,\,] : V \otimes V \longrightarrow V$, in particular $V_i \otimes V_j \to V_{i+j}$

satisfying (graded versions of antisymmetry, Jacobi & Leibniz)

- $[x,y] + (-1)^{|x|\cdot|y|}[y,x] = 0$

$[x, -]$ is a graded derivation

- $(-1)^{|x||z|}[x,[y,z]] + (-1)^{|y||x|}[y,[z,x]] + (-1)^{|z||y|}[z,[x,y]] = 0$
- $d[x,y] = [dx,y] + (-1)^{|x|}[x,dy]$

If $d = 0$, we call this a graded Lie algebra.

Example: There is an adjunction
$$\begin{tikzcd} DGLA \arrow[r, bend right, "forget"'] \arrow[r, phantom, "\perp"] & Ch \arrow[l, bend right, "\mathbb{L}"'] \end{tikzcd}$$

The <u>free</u> dgla $\mathbb{L}(V)$ on a dg vector space V is the sub-Lie algebra of $T(V)$ generated by V.

Eg. if $V = \mathbb{Q}x^{\leftarrow \deg n}$

$$\Rightarrow \mathbb{L}V = \begin{cases} \mathbb{Q}x, & n \text{ even} \\ \mathbb{Q}x \oplus \mathbb{Q}[x,x], & n \text{ odd} \end{cases}$$

Can take homology in two ways:

- "naively" - disregarding Lie algebra structure

  ⇝ get a graded Lie algebra

- Chevalley Eilenberg homology ⇝ graded coalgebra

Def.: L a dg Lie algebra

$$C_*^{CE}(L) = (\Lambda sL, \delta)$$

free graded commutative algebra on: L shifted up one degree

$$(sL)_i = (L[1])_i = L_{i-1}$$

where

$$\delta(sx_1 \wedge sx_2 \cdots \wedge sx_k) =$$

$$-\sum_i (-1)^{\epsilon_i} sx_1 \wedge \cdots sdx_i \wedge \cdots \wedge sx_n$$

$$+\sum_{i<j} (-1)^{|sx_i|+\eta_{ij}} s[x_i, x_j] \wedge \cdots \hat{sx_i} \cdots \hat{sx_j} \cdots \wedge sx_n$$

ϵ_i & η_{ij} just implement graded commutativity

are called the Chevalley-Eilenberg chains of L.

Chevalley-Eilenberg homology is

$$H_*^{CE}(L) := H_*(C_*^{CE}(L))$$

The <u>shuffle</u> coproduct on ΛsL

s.t. ΛsL is a graded Hopf algebra with $P(\Lambda sL) = sL$

makes $H_*^{CE}(L)$ into a graded coalgebra.

Example: $L = \mathbb{L}(V)$ for $V = \mathbb{Q}\underset{n}{x}$

Recall: Hopf algebra has grouplike elem's and primitive elements

$\mathcal{G}(H)$: form group $\Delta(x) = x \otimes x$

$\mathcal{P}(H)$: Like in Lie algebras $\Delta(x) = x \otimes 1 + 1 \otimes x$

n even: $\Lambda_\delta L = \Lambda(\underset{\sim}{sx}) = \underset{0}{\mathbb{Q}} \oplus \underset{n+1}{\mathbb{Q}(sx)}$, $\delta = 0$

n odd: $\Lambda_\delta L = \Lambda(\overset{\text{even}}{sx}, \overset{\text{odd}}{s[x,x]}) = Sym(sx) \otimes \Lambda(s[x,x]) \overset{\text{as u.sp}}{\cong} Sym(sx) \otimes (\mathbb{Q} \oplus \mathbb{Q}(s[x,x])) = Sym(sx) \oplus Sym(sx).\, s[x,x]$

$\delta : sx \wedge sx \longmapsto s[x,x]$

$\Rightarrow H_*^{CE}(L) = \mathbb{Q} \oplus V[1]$

III. Quillen's Theorem

Why care?

The Whitehead product on the htpy. groups of a space

$$\pi_k X \times \pi_l X \longrightarrow \pi_{k+l-1} X$$

comes from

$$\begin{tikzcd} S^{k+l-1} \arrow[r] \arrow[d, "i_0"'] & S^k \vee S^l \arrow[d] \arrow[r, "f \vee g"] & X \\ D^{k+l} \arrow[r] & S^k \times S^l \arrow[ul, phantom, "\ulcorner"] & \end{tikzcd}$$

makes

$$\pi_*^{\mathbb{Q}} X := \pi_{*+1} X \otimes \mathbb{Q}$$

a graded Lie algebra.

(for $X \in Top_{*,1}$)

(And the diagonal makes H_* a graded coalgebra.)

The categories dgla$_1$ & dgca$_2$ admit model structures:

- weak equiv's: maps that induce isomorphisms on H_* (naive)

dgla$_1$

- fibrations: surjective in degree > 1
- cofibrations: retracts of free maps, i.e. $\mathfrak{g} \to$ $$\begin{tikzcd} \mathfrak{g} \arrow[r] \arrow[dr, "(id,0)"'] & \mathfrak{g}' \\ & \mathfrak{g} \oplus \mathbb{L}V \arrow[u, "\cong"'] \end{tikzcd}$$ (forgetting ∂)

(dgca$_2$:
- cofibrations: injective)

Thm. (Quillen)

∃ equivalences of categories

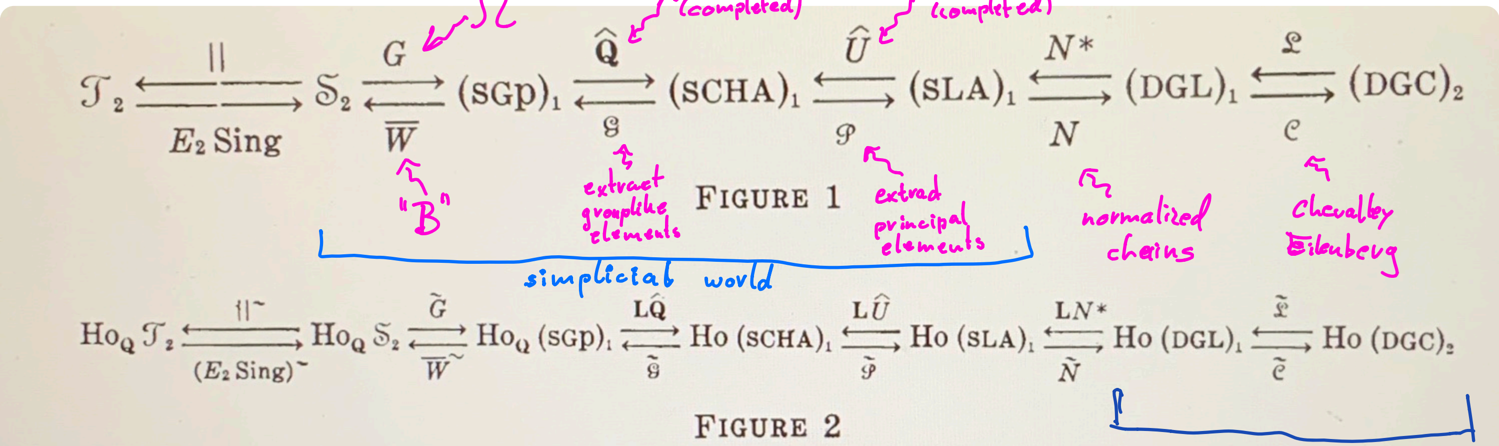
$$h\,\mathrm{Top}^{\mathbb{Q}}_{*,1} \xrightarrow[\simeq]{\lambda} h\,\mathrm{dgla}_1 \xrightarrow[\simeq]{C^{CE}_*} h\,\mathrm{dgca}_2$$

instance of Koszul duality

such that

$$\pi_*^{\mathbb{Q}} X \simeq H_*(\lambda X) \quad \text{and} \quad H_*(X;\mathbb{Q}) \simeq H_*^{CE}(\lambda X)$$
$$:= H_*(C_*^{CE}(\lambda X))$$

Pf. Sketch: A row of Quillen adjunctions between model categories.



$$\begin{tikzcd}
\mathcal{T}_2 \arrow[r, shift right, "E_2\,\mathrm{Sing}"'] & \mathcal{S}_2 \arrow[l, shift right, "|\,|"'] \arrow[r, shift left, "G"] & (\mathrm{sGp})_1 \arrow[l, shift left, "\overline{W}"] \arrow[r, shift left, "\widehat{\mathbf{Q}}"] & (\mathrm{SCHA})_1 \arrow[l, shift left, "\mathcal{G}"] \arrow[r, shift right, "\mathcal{P}"'] & (\mathrm{SLA})_1 \arrow[l, shift right, "\widehat{U}"'] \arrow[r, shift right, "N"'] & (\mathrm{DGL})_1 \arrow[l, shift right, "N^*"'] \arrow[r, shift right, "\mathcal{C}"'] & (\mathrm{DGC})_2 \arrow[l, shift right, "\mathcal{L}"']
\end{tikzcd}$$

FIGURE 1

$$\begin{tikzcd}
\mathrm{Ho}_{\mathbb{Q}}\mathcal{T}_2 \arrow[r, shift right, "(E_2\,\mathrm{Sing})^{\sim}"'] & \mathrm{Ho}_{\mathbb{Q}}\mathcal{S}_2 \arrow[l, shift right, "|\,|^{\sim}"'] \arrow[r, shift left, "\tilde{G}"] & \mathrm{Ho}_{\mathbb{Q}}(\mathrm{sGp})_1 \arrow[l, shift left, "\overline{W}^{\sim}"] \arrow[r, shift left, "\mathrm{L}\widehat{\mathrm{Q}}"] & \mathrm{Ho}(\mathrm{SCHA})_1 \arrow[l, shift left, "\tilde{\mathcal{G}}"] \arrow[r, shift right, "\tilde{\mathcal{P}}"'] & \mathrm{Ho}(\mathrm{SLA})_1 \arrow[l, shift right, "\mathrm{L}\widehat{U}"'] \arrow[r, shift right, "\tilde{N}"'] & \mathrm{Ho}(\mathrm{DGL})_1 \arrow[l, shift right, "\mathrm{L}N^*"'] \arrow[r, shift right, "\tilde{\mathcal{C}}"'] & \mathrm{Ho}(\mathrm{DGC})_2 \arrow[l, shift right, "\tilde{\mathcal{L}}"']
\end{tikzcd}$$

FIGURE 2

In sCHA$_1$ we have an isomorphism $\exp: \mathcal{P}(H) \longrightarrow \mathcal{G}(H)$

How does this help?

λ is a complicated functor – but we can build models for spaces without knowing its exact form!

Example: $\lambda(S^n)$ is a dgla

with
$$\begin{tikzcd} H^{CE}_*(\lambda(S^n)) \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\shortparallel"] & H_*(S^n;\mathbb{Q}) = \mathbb{Q}[0]\oplus \overset{x}{\overset{\wedge}{\mathbb{Q}}}[n] \\ H_*(\Lambda s\lambda(S^n), \delta^{CE}) & \end{tikzcd}$$

Let $V = \mathbb{Q}\underset{\text{deg } n-1}{\underset{\wr}{\hat{x}}}$ candidate for $\lambda(S^n)$: $\mathbb{L}V$

$$\begin{tikzcd} \mathbb{L}V \arrow[r, "x"] & \lambda(S^n) & \rightsquigarrow & H^{CE}_*(\mathbb{L}V) \arrow[r, "H_* x", "\cong"'] \arrow[d, phantom, "\cong" rotate=90] & H_*(S^n;\mathbb{Q}) \\ & & & \mathbb{Q}\oplus\mathbb{Q}[n] \arrow[dll, Rightarrow] & \end{tikzcd}$$

$$\begin{tikzcd} H_*\mathbb{L}V \arrow[r, "\pi_* x", "\cong"'] \arrow[d, "\cong"'] & H_*(\lambda(S^n)) \cong \pi_*^{\mathbb{Q}}(S^n) \\ \mathbb{L}H_*V \arrow[d, phantom, "\cong" rotate=90] & \\ \mathbb{L}V & \end{tikzcd}$$

Thm.

Hence
$$\pi_*S^n\otimes\mathbb{Q} \overset{\text{v.sp.}}{\cong} \mathbb{L}\,\mathbb{Q}[n-1][1] \cong \begin{cases} \mathbb{Q}[n] & n \text{ odd} \\ \mathbb{Q}[n]\oplus\mathbb{Q}[2n-1] & n \text{ even} \end{cases}$$

+ Whitehead product