Minimal Models for Baut

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① Baut(M)

Recall: We are interested in the rational cohomology of $B\mathrm{Diff}^\partial(W)$
where $W = (\#_g S^n \times S^n) \setminus D^{2n}$ is a "high dimensional surface"

↪ Use the map $B\mathrm{Diff}^\partial(W) \to B\,\mathrm{aut}^\partial(W)$

Defn For a space X $\mathrm{aut}(X)$ is the topological monoid of those maps $f: X \to X$ that are homotopy equivalences. ⇝ $\pi_0\, \mathrm{aut}(X)$ is a group

Q: Why aut? If W is a surface of genus g then $B\mathrm{Diff}(W) \simeq B\mathrm{aut}(W)$
Now: W is not $K(\pi,1)$, but simply con. $\Gamma_g \cong \mathrm{Out}(\pi_1 W)$

Q: What is Baut? Adele's talk: $B\mathrm{Diff}(W)$ classifies smooth W-bundles
⇝ $B\mathrm{aut}(W)$ classifies "homotopical W-bundles"

Defn X a space, an X-fibration is a Serre fibration $E \xrightarrow{p} B$ such that each $p^{-1}(b) \simeq X$.
An elementary equivalence is $\begin{array}{ccc} E & \xrightarrow{f} & E' \\ & {\scriptstyle p}\searrow \; B \; \swarrow{\scriptstyle p'} & \end{array}$ with f homotopy equivalence.
$\mathrm{Fib}(B;X) = \{X\text{-fibrations } E \to B\}/\sim$

Thm There is an X-fibration $\begin{array}{c} E_X \\ \downarrow \\ B_X \end{array}$ equivalent to $\begin{array}{c} B\mathrm{aut}_*(X) \\ \downarrow \\ B\mathrm{aut}(X) \end{array}$ such that for all B

the map
$$[B, B_X] \longrightarrow \mathrm{Fib}(B;X)$$
$$f \longmapsto [f^*E_X \to B]$$
is a bijection.

WE WANT A LIE VERSION OF THIS!

② Minimal Models

Recall from Chris' talk: There is a functor $\lambda: \mathrm{Top}_*^{\geq 1} \longrightarrow \mathrm{dgla}_{\mathbb{Q}}$
that remembers exactly the "rational homotopy type"

$\lambda(X)$ might be very large in general, but we can find small models.

Eg. $\lambda(S^n)$ there is $x \in H_{n-1}\lambda(S^n) \rightsquigarrow \mathbb{L}x \xrightarrow{\sim} \lambda(S^n)$ this is an equivalence!

A more complicated example: $\lambda(\mathbb{C}P^2)$

$\mathbb{C}P^2$ is built by attaching a 4-cell to S^2:

$$\begin{tikzcd}
\mathbb{L}y \arrow[rr, "y \mapsto \frac{1}{2}[x,x]"] \arrow[ddd, "y \mapsto y"'] & & \mathbb{L}x \arrow[ddd] \\
\lambda(S^3) \arrow[rr, "\eta"] \arrow[d, tail] & & \lambda(S^2) \arrow[d] \\
\lambda(D^4) \arrow[rr] \arrow[urr, phantom, "\ulcorner h"] & & \lambda(\mathbb{C}P^2) \\
(\mathbb{L}(y,z), dz = y) \arrow[rr] & & (\mathbb{L}(x,z), dz = \frac{1}{2}[x,x])
\end{tikzcd}$$

$|x| = 1$
$|y| = 2$
$|z| = 3$

From this we could now compute $\pi_*^{\mathbb{Q}}(\mathbb{C}P^2)$ (if we understood free Lie algs...)

Recall: The homology of X can be read of from the Chevailler-Eilenberg complex of $\lambda(X)$

Thm For a dgla of the form $(\mathbb{L}V, d)$ the CE complex is equivalent to:
$$C^{CE}(\mathbb{L}V, d) \simeq (\mathbb{Q} \oplus sV, d_0)$$ where d_0 is the linear part of d

Eg: $H_*(\mathbb{L}(x,z), dz = \frac{1}{2}[x,x]) = \begin{cases} \mathbb{Q} & \text{if } * = 0,2,4 \\ 0 & \text{o/w} \end{cases}$

Def A free model for X is a dgla of the form $(\mathbb{L}V, d)$ together with a weak equivalence $(\mathbb{L}V, d) \xrightarrow{\sim} \lambda(X)$. (= the cofibrant objects of dgla)

A minimal model is a free model where $d_0 : V \to V$ is trivial.

Thm Every X admits a unique minimal model up to isomorphism.

There is a bijection: $\left\{ \begin{matrix} \text{simply connected} \\ \text{spaces } X \end{matrix} \right\} \Big/ \begin{matrix} \text{rational} \\ \text{htp. equ.} \end{matrix} \xleftrightarrow{1:1} \left\{ \begin{matrix} \text{minimal} \\ \text{dglas} \end{matrix} \right\} \Big/ \text{isomorphism}$

We can find free / minimal models just like we did for $\mathbb{C}P^2$: by attaching cells.

Thm If $(\mathbb{L}V, d)$ is a free model for X, and $f: S^d \to X$ is an attaching map with $[f] \in \pi_d X$ represented by $\omega \in (\mathbb{L}V)_{d-1}$, then a free model for $X \cup_{S^d} D^{d+1}$ is given by $(\mathbb{L}(V \oplus \underset{|x|=d}{\mathbb{Q}\langle x \rangle}), d + (dx = \omega))$

(3) Derivations and Baut.

Q: When do two morphisms $(\mathbb{L}_X, d) \overset{\varphi}{\underset{\psi}{\rightrightarrows}} (\mathbb{L}_Y, d')$ represent the same homotopy class $X \xrightarrow[[f]]{} Y$?

We need a notion of chain homotopy!

Def • Given $\varphi, \psi : (L, d) \to (L', d')$ a (φ,ψ)-derivation of degree k, is a linear degree k map $\theta : L_\bullet \to L'_{\bullet+k}$ satisfying:

$$\theta([x,y]) = [\theta x, \psi y] + (-1)^{k \cdot |x|} [\varphi(x), \theta y]$$

Ex $\varphi = \psi = id$
$\hookrightarrow$ d is a derivation of degree -1

• We say φ and ψ are homotopic if there is a degree 1 derivation θ such that $\theta \circ d + d' \circ \theta = \varphi - \psi$.

$$D(\theta) = \varphi - \psi$$

IDEA: Homotopies are paths in $map(X,Y)$. We can now study loops in $map(X,Y)$. In particular $\Omega_{id}\, map_*(X,X) \simeq \Omega\, aut_*(X)$.

Defn: Given $f : L \to L'$ we let $Der_f(L, L')$ be the chain complex

$$Der_f(L,L')_k = \{ \text{degree } k \ (f,f)\text{-derivations } \theta : L_\bullet \to L'_{\bullet+k} \}$$

with differential $D(\theta) = d_{L'} \circ \theta - (-1)^{|\theta|} \theta \circ d_L$

Thm (Lupton-Smith) $f : X \to Y$ a map of simply con. CW complexes, X finite, $\varphi : \mathbb{L}_X \to \mathbb{L}_Y$ a Lie model for f, then $\pi_k(map_*(X,Y), f) \cong H_k(Der_\varphi(\mathbb{L}_X, \mathbb{L}_Y))$ $(k \geq 2)$

Construction of the map:

$\alpha \in \pi_k(map_*(X,Y), f) \rightsquigarrow$ want to find a degree k derivation $\theta : \mathbb{L}_X \to \mathbb{L}_Y$

Represent α by $h : S^k_+ \wedge X \to Y$.

There's a Lie model $(\mathbb{L}(V \oplus s^k V), \delta')$ for $S^k_+ \wedge X$ such that $(\mathbb{L}V, \delta'_{|\mathbb{L}V})$ is a model for X

Find $\varphi_h : \mathbb{L}(V \oplus s^k V) \to \mathbb{L}_Y$ representing h and set $\theta(v) = \varphi_h(s^k v)$.

$\llcorner$

Back to Baut! $\rightsquigarrow$ set $X = Y$, $f = id$.

Def • $\mathrm{Der}(L) = \mathrm{Der}_{id}(L,L)$ is a dgla under $[\theta,\eta] = \theta\circ\eta - (-1)^{|\theta||\eta|}\eta\circ\theta$

$\boxed{D(\theta) = [\delta,\theta]}$

• L^+ is the positive truncation: $(L^+)_k = \begin{cases} L_k & k\geq 2 \\ \ker(L_1\to L_0) & k=1 \\ 0 & k\leq 0\end{cases}$

There is a map $L \xrightarrow{ad} \mathrm{Der}\, L$ sending x to $[x,\_]$.
This is well-defined by Jacobi. ↪ "inner derivations"

Def $\mathrm{Der}\, L /\!\!/_{ad} L$ is $sL \oplus \mathrm{Der}\, L$ with $\delta(\theta) = D(\theta)$, $\delta(sx) = ad_x - s\,dx$

Thm (Tanré) X a finite simply connected CW complex with Lie model $(\mathbb{L}_X, d)$.
Then $\mathrm{Der}(\mathbb{L}_X)^+ \longrightarrow (\mathrm{Der}\,\mathbb{L}_X /\!\!/_{ad}\mathbb{L}_X)^+$ is a Lie model for
$B\mathrm{aut}^\circ_*(X) \longrightarrow B\mathrm{aut}^\circ(X)$ where $\mathrm{aut}^\circ(X) \subset \mathrm{aut}(X)$ is the identity component
$(B\mathrm{aut}_*(X))\langle 1\rangle \longrightarrow (B\mathrm{aut}(X))\langle 1\rangle$

Back to manifolds.

M^n simply con. oriented compact n-mfd with $\partial M \cong S^{n-1}$ $\boxed{M = (\#_g S^d\times S^d)\setminus D^{2d}}$

$V = s^{-1}\tilde{H}_*(M;\mathbb{Q})$ has a non-degenerate pairing.
Pick a basis α_i with dual basis $\alpha_i^\#$ and define a special element

$$\omega = \frac{1}{2}\sum_i [\alpha_i, \alpha_i^\#] \in (\mathbb{L}V)_{n-2}$$

Thm (Stasheff) There is a δ on $\mathbb{L}V$ such that $(\mathbb{L}V, \delta)$ is a minimal model for M and $\omega \in (\mathbb{L}V)_{n-2}$ is a cycle representing $\underset{\cong\partial M}{S^{n-1}} \hookrightarrow M$.

Def $\mathrm{Der}^+_\omega(\mathbb{L}V) = (\text{the dgla of those derivations } \theta : \mathbb{L}V \to \mathbb{L}V \text{ with } \theta(\omega) = 0)^+$

Thm M, V as above then $(\mathrm{Der}_\omega \mathbb{L}V, [\delta,\_])^+$ is a Lie model for $B\mathrm{aut}^{\partial,\circ}(M)$.

Cor If for some $d\geq 2$ M is $(d-1)$-connected and $\dim(M) \leq 3d-2$, then

$$\pi_*^{\mathbb{Q}}(B\mathrm{aut}^{\partial,\circ}(M)) \cong \mathrm{Der}^+_\omega(\mathbb{L}V)$$

$\boxed{M = (\#_g S^d\times S^d)\setminus D^{2d}}$