Charney's theorem

$$W_{g,1} := \left(\#_g S^d \times S^d\right) \setminus \mathring{D}^{2d}$$

Recall We are studying the rational homology of $B\,\mathrm{aut}_\partial(W_{g,1})$.

$W_{g,1}$ embeds in $W_{g+1,1}$ with complement $W_{1,2} = (S^d \times S^d) \setminus \coprod^2 \mathring{D}^{2d}$

$\rightsquigarrow$ $\mathrm{aut}_\partial(W_{g,1}) \longrightarrow \mathrm{aut}_\partial(W_{g+1,1})$

extends given automorphism on $W_{g,1}$ by the identity on $W_{1,2}$

Goal Show that the induced map

$$H_k(B\,\mathrm{aut}_\partial(W_{g,1});\mathbb{Q}) \longrightarrow H_k(B\,\mathrm{aut}_\partial(W_{g+1,1});\mathbb{Q})$$

is an isomorphism when $g \gg k$.

[Explicitly, iso if $g > 2k+4$ and epi if $g = 2k+4$.]

Recall from Jan's talk:

Let $V_g := s^{-1}\tilde{H}_*(W_{g,1};\mathbb{Q})$

$\mathbb{L}(V_g)$ free graded Lie algebra on V_g

$\omega_g := \frac{1}{2}\sum_i [\alpha_i^\#, \alpha_i] \in \mathbb{L}(V_g)$ where α_i a basis of V_g and $\alpha_i^\#$ the dual basis

$\mathfrak{g}_g := \mathrm{Der}^+_{\omega_g}\mathbb{L}(V_g)$ positive derivations θ of $\mathbb{L}(V_g)$ s.t. $\theta(\omega_g) = 0$

There is δ on $\mathbb{L}(V_g)$ s.t. $(\mathbb{L}(V_g), \delta)$ minimal model for $W_{g,1}$. [Here $\delta = 0$.]

Also, $(\mathfrak{g}_g, [\delta, -]) = (\mathfrak{g}_g, 0)$ is a dg Lie model for $B\,\mathrm{aut}_{\partial,\circ}(W_{g,1})$.

Hence $H_*(B\,\mathrm{aut}_{\partial,0}(W_{g,1});\mathbb{Q}) \cong H_*^{CE}(\mathfrak{g}_g)$ [Chris' talk]

$B\,\mathrm{aut}_{\partial,0}(W_{g,1})$ is the univ. cover of $B\,\mathrm{aut}_{\partial}(W_{g,1})$, so there is a fibration

$$\begin{tikzcd} B\,\mathrm{aut}_{\partial,0}(W_{g,1}) \arrow[r] & B\,\mathrm{aut}_{\partial}(W_{g,1}) \arrow[r] & B\pi_1 B\,\mathrm{aut}_{\partial}(W_{g,1}) \arrow[d, equal] \\ & & B\pi_0\,\mathrm{aut}_{\partial}(W_{g,1}) \end{tikzcd}$$

$\rightsquigarrow$ Serre s.s.

$$E^2_{pq}(g) = H_p\big(B\pi_0\,\mathrm{aut}_\partial(W_{g,1});\, H_q(B\,\mathrm{aut}_{\partial,0}(W_{g,1});\mathbb{Q})\big)$$

$$\Rightarrow H_{p+q}(B\,\mathrm{aut}_\partial(W_{g,1});\mathbb{Q})$$

By the above, $E^2_{pq}(g) \cong H_p\big(\pi_0\,\mathrm{aut}_\partial(W_{g,1});\, H_q^{CE}(\mathfrak{g}_g)\big)$

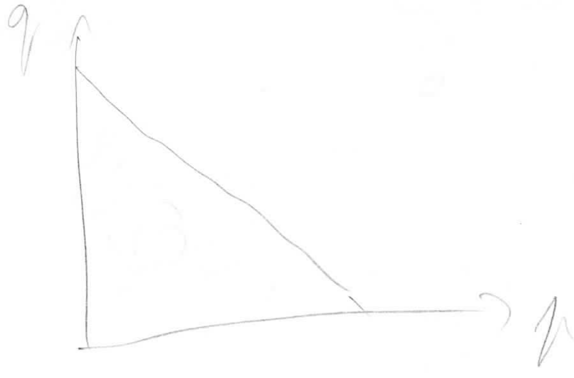
<u>Strategy</u> Show that the "stabilisation map"

$$\mathrm{aut}_\partial(W_{g,1}) \longrightarrow \mathrm{aut}_\partial(W_{g+1,1})$$

induces isos $E^2_{pq}(g) \xrightarrow{\sim} E^2_{pq}(g+1)$ if $g > 2p+2q+4$

and epis $E^2_{pq}(g) \twoheadrightarrow E^2_{pq}(g+1)$ if $g = 2p+2q+4$.

Then the same holds on E^∞, and hence on the abutment.



Homological stability

Suppose we have a sequence

$$G_1 \hookrightarrow G_2 \hookrightarrow G_3 \hookrightarrow \cdots$$

of groups and group homomorphisms.

Question: Does there exist $n(k)$ for every $k \geq 0$ such that

$$H_k(G_n) \longrightarrow H_k(G_{n+1})$$

is an iso for all $n > n(k)$ (and epi for $n = n(k)$)?

If we also have a "coefficient system"

$$\begin{array}{ccccccc} F_1 & \longrightarrow & F_2 & \longrightarrow & F_3 & \longrightarrow & \cdots, \\ \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \\ G_1 & & G_2 & & G_3 & & \end{array}$$

we can ask the same question for twisted homology $H_k(G_n; F_n)$.

Surprising fact: For many "naturally occurring" sequences of groups and "nice" coefficient systems, the answer is yes.

e.g.
- symmetric groups (Nakaoka '60)
- braid groups (Arnol'd '69)
- general linear groups over many rings, many subgroups (Sp, O, U, ...) (van der Kallen, Dwyer, Vogtmann, Charney, ...)
- mapping class groups of surfaces (Harer)
- $\mathrm{Aut}(F_n)$ (Hatcher - Vogtmann)

General principle for proving homological stability

Let G be a group and let X be a simplicial complex with a simplicial G-action (such that if $g \in G$ fixes a simplex σ of X_* as a set, then it fixes σ pointwise).

Consider bigraded complex

$$E_\bullet G \otimes_G \tilde{C}_*(X) \qquad \tilde{C}_{-1}(X) = \mathbb{Z}$$

$E_\bullet G$: $\mathbb{Z}[G]$-projective resolution of $\mathbb{Z}$

$\tilde{C}_*(X)$: augmented chain complex of X

[Intuition: the total complex computes the homology of the mapping cone of $C_*(X/\!/G) \longrightarrow C_*(BG)$.]

The two associated spectral sequences have

$$\left.\begin{aligned} {}^{I}E^1_{pq} &= H_q(G; \tilde{C}_p(X)) \\ {}^{I}E^1_{pq} &= E_p G \otimes_G \tilde{H}_q(X) \\ {}^{II}E^2_{p,q} &= H_p(G; \tilde{H}_q(X)) \end{aligned}\right\} \Rightarrow H_{p+q}(BG, X/\!/G).$$

$$X/\!/G \longrightarrow BG$$

But $\tilde{C}_p(X) \cong \begin{cases} \mathbb{Z} & p=-1 \\ \mathbb{Z}[X_p] \cong \bigoplus_{[\sigma]\in X_p/G} \mathbb{Z}[G/G_\sigma] & p \geq 0 \end{cases}$

Hence

$${}^{I}E^1_{pq} = H_q(G; \tilde{C}_p(X)) \cong \begin{cases} H_q(G) & p=-1 \\ \bigoplus_{[\sigma]\in X_p/G} H_q(G; \mathbb{Z}[G/G_\sigma]) \overset{\text{Shapiro's lemma}}{\cong} \bigoplus_{[\sigma]\in X_p/G} H_q(G_\sigma) & p \geq 0 \end{cases}$$

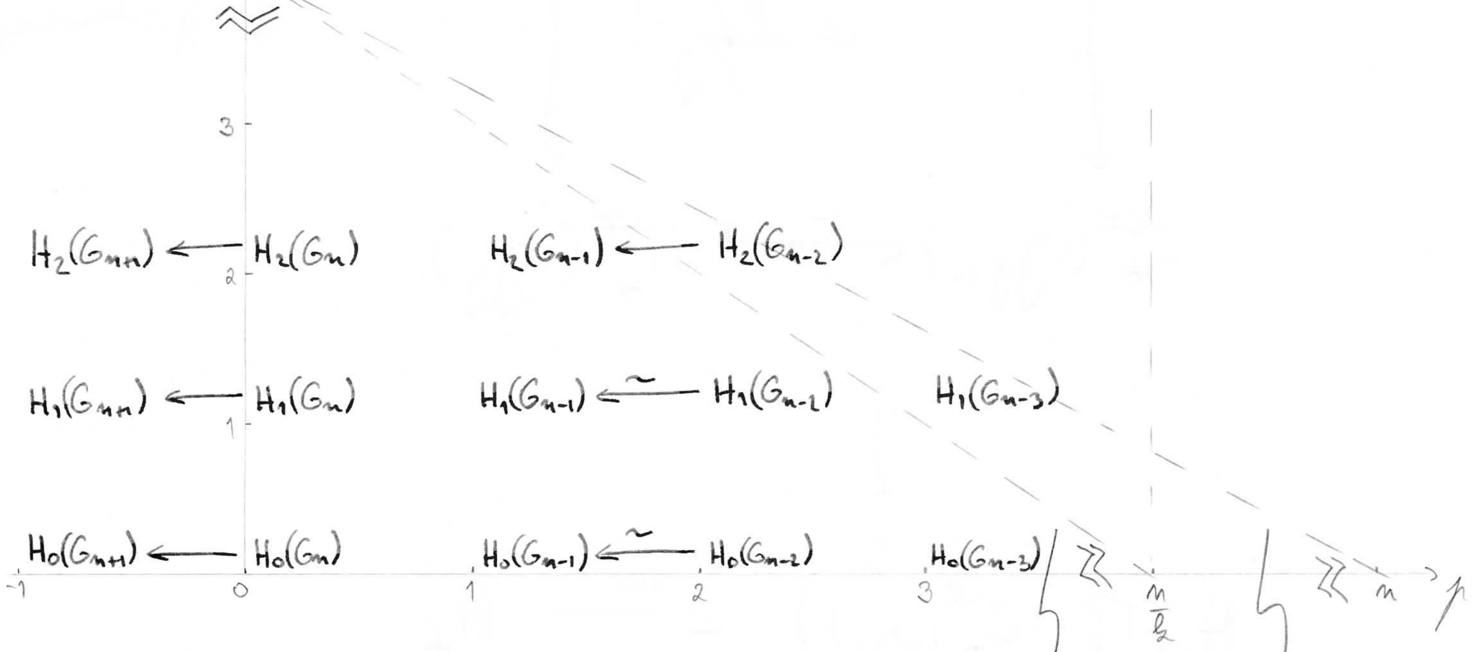
Strategy of proof

For each G_n in sequence, find simplicial complex X^n s.t.

- $X^n \curvearrowleft G_n$ (simplicial, nice)
- $X^n_p \curvearrowleft G_n$ transitively for $p<n$
- $\exists\ \sigma \in X^n_p$ s.t. $\mathrm{Stab}_{G_n}(\sigma) = G_{n-p-1}$ for $p<n$

  (so $X^n_p \cong G_n / G_{n-p-1}$ as a G-set)
- X^n is $\left(\frac{n-2}{k}\right)$-connected, for some $k \geq 2$ [independent of n]

<u>Claim</u> $H_i(G_n) \longrightarrow H_i(G_{n+1})$ iso for $i \leq \frac{n-1}{k}$, epi for $i \leq \frac{n}{k}$.



${}^{I}E^1_{pq}$ for $G_{n+1} \curvearrowright X^{n+1}$

slope -1 slope $-\frac{1}{k}$

<u>Claim</u>

$$d^1 : \underset{{}^{I}E^1_{pq}}{\underset{\parallel}{H_q(G_{n-p})}} \longrightarrow \underset{{}^{I}E^1_{(p-1),q}}{\underset{\parallel}{H_q(G_{n-p+1})}} = \begin{cases} 0 & p \text{ odd} \\ \mathrm{incl}_* & p \text{ even} \end{cases}$$

e.g. • $G_n = \Sigma_n$ $\qquad X^n_p = \mathrm{Inj}([p+1],[n])$

"complex of injective words"

$|X^n| \simeq$ a wedge of $(n-1)$-spheres

• $G_n =$ MCG of genus n oriented surface $\qquad X^n$ related to the complex of curves on the surface

• $G_n = Sp_{2n}\mathbb{Z},\ O_{n,n}\mathbb{Z}, \dots$ $\qquad X^n_p = \overset{\mathrm{Emb}}{\mathrm{Iso}}(V_{p+1}, V_n)$

where $V_i = \mathbb{Z}^{2i}$ + suitable non-degenerate form

[twisted coefficients]

<u>Back to $B\mathrm{aut}_\partial(W_{g,1})$</u>!

We would like to apply this theory to the groups

$$G_g := \pi_1 B\mathrm{aut}_\partial(W_{g,1}) \cong \pi_0 \mathrm{aut}_\partial(W_{g,1})$$

<u>Problem</u> These groups are too complicated.

<u>Solution</u> Consider the action of G_g on $H_g := H_d(W_{g,1};\mathbb{Z}) \cong \mathbb{Z}^{2g}$

$$\begin{tikzcd} & \mathrm{Aut}(H_g, \langle\cdot,\cdot\rangle) \arrow[d, hook] \arrow[r, dash] & \begin{cases} O_{g,g}(\mathbb{Z}) & d \text{ even} \\ Sp_{2g}(\mathbb{Z}) & d \text{ odd} \end{cases} \\ \rightsquigarrow \varphi_g : G_g \arrow[ur] \arrow[r] & \mathrm{Aut}(H_g) & \end{tikzcd}$$

<u>Th^m</u> (Kreck'78 ? Berglund–Madsen) Let $d \geq 3$. Then $\ker \varphi_g$ is finite.

Hence if $\Gamma_g := \mathrm{im}\,\varphi_g \leq \mathrm{Aut}(H_g)$ and M is a $\mathbb{Q}[\Gamma_g]$-module, then $H_*(G_g;M) \cong H_*(\Gamma_g;M)$. $\qquad K \to G_g \to \Gamma_g$

Hence the spectral sequence from the beginning takes the form

$$E^2_{pq} = H_p(\Gamma_g ; H_q^{CE}(\mathfrak{g}_g)) \Rightarrow H_{p+q}(B\,\mathrm{aut}_\partial(W_{g,1}); \mathbb{Q})$$

[This is well-defined since $\mathfrak{g}_g = \mathrm{Der}^+_{\omega_g} \mathbb{L}(s^{d-1} H_g)$.]

Charney's theorem

Let $P : \mathrm{Ab} \to \mathrm{Ab}$ be a polynomial functor of degree $\leq l$ of the form

$$P(H) = \bigoplus_{k=0}^{l} P(k) \otimes_{\Sigma_k} H^{\otimes k}$$

for some Σ_k-modules $P(k)$, $0 \leq k \leq l$.

Applying P to the coefficient system

$$\begin{array}{ccccccc} \overset{H_1}{\overset{\shortparallel}{H_d(W_{1,1};\mathbb{Q})}} & \hookrightarrow & \overset{H_2}{\overset{\shortparallel}{H_d(W_{2,1};\mathbb{Q})}} & \hookrightarrow & \overset{H_3}{\overset{\shortparallel}{H_d(W_{3,1};\mathbb{Q})}} & \hookrightarrow & \cdots \\ \circlearrowright & & \circlearrowright & & \circlearrowright & & \\ \Gamma_1 & & \Gamma_2 & & \Gamma_3 & & \end{array}$$

yields another coefficient system

$$\begin{array}{ccccccc} P(H_1) & \longrightarrow & P(H_2) & \longrightarrow & P(H_3) & \longrightarrow & \cdots \\ \circlearrowright & & \circlearrowright & & \circlearrowright & & \\ \Gamma_1 & & \Gamma_2 & & \Gamma_3 & & \end{array}$$

Th^m (Charney)

$$H_k(\Gamma_g ; P(H_g)) \longrightarrow H_k(\Gamma_{g+1} ; P(H_{g+1}))$$

is an iso for $g > 2k + l + 4$ and epi for $g = 2k + l + 4$.

Hope $H_q^{CE}(\mathfrak{g}_g) = H_q^{CE}(\mathrm{Der}^+_{\omega_g} \mathbb{L}(s^{d-1} H_g))$ is a polynomial functor in H_g.

<u>Disappointment</u> It isn't.

However, the stability of

$$H_p(\Gamma_g ; H_q^{CE}(\mathfrak{g}_g)) \longrightarrow H_p(\Gamma_{g+1} ; H_q^{CE}(\mathfrak{g}_{g+1}))$$

for $g \geq 2p + 2q + 4$ is a technical consequence of

<u>Lemma</u> $\mathcal{C} : H_g \longmapsto C_{\cancel{*}q}^{CE}(\mathfrak{g}_g) = C_q^{CE}(\mathcal{D}er^+_{\omega_g} \mathbb{L}(s^{d-1} H_g))$ is polynomial of degree $\leq \left[\frac{3q}{d}\right] \leq 2q$ if $d \geq 2$

<u>Proof</u>

$H_g \longmapsto s^{d-1} H_g$ is polynomial with structure chains

$$I(k) = \begin{cases} 0 & k \neq 1 \\ \mathbb{Q}[d-1] & k = 1 \end{cases}$$

$V \longmapsto \mathcal{D}er^+_\omega \mathbb{L}(V)$ is polynomial (as a functor on graded antisymmetric inner product spaces V)

$L \longmapsto C_q^{CE}(L)$ is polynomial (as a functor on graded Lie algebras)

$$C_*^{CE}(L) = \bigoplus_{k \geq 0} \Lambda(k) \otimes_{\Sigma_k} L^{\otimes k}$$

sign representation of Σ_k in degree k.

$$L \text{ Lie alg} \overset{CE}{\rightsquigarrow} \quad L \leftarrow \Lambda^2 L \leftarrow \Lambda^3 L \leftarrow \cdots$$

$$\mathrm{Der}\,\mathbb{L}(V) \cong \mathrm{Hom}(V, \mathbb{L}(V)) \cong \mathbb{L}(V) \otimes V^*$$

If V inner product space, then

$$\cong \mathbb{L}(V) \otimes V$$

$$\begin{tikzcd}
 & \mathbb{L}(V) \otimes V \arrow[d, "s"] \arrow[rd, bend left, "{[\cdot,\cdot]}"] & \\
\mathrm{Der}_\omega \mathbb{L}(V) \arrow[r] & \mathrm{Der}\,\mathbb{L}(V) \arrow[r, two heads, "ev_\omega"] & \mathbb{L}^{\geq 2}(V)
\end{tikzcd}$$

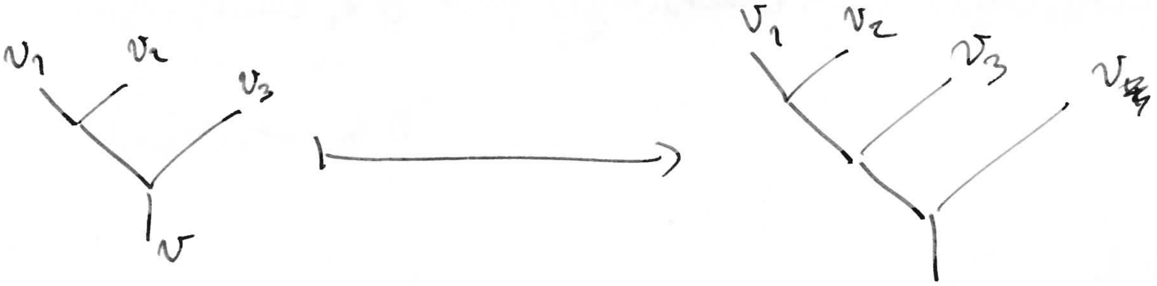
$\mathbb{L}^{\geq 2}(V)$

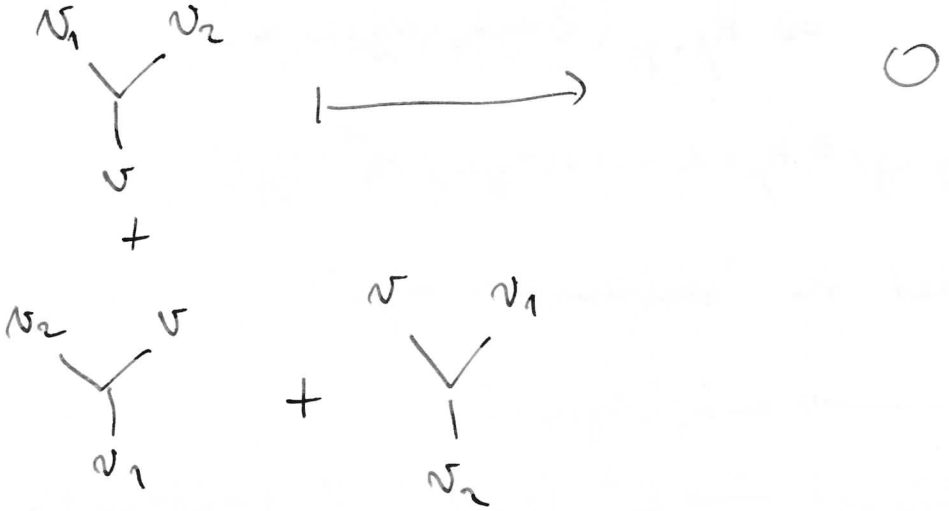
$\mathcal{L}ie(3) \otimes_{\Sigma_3} V^{\otimes 3}$

$$\mathbb{L}^{\geq 2}(V) = \bigoplus_{k \geq 2} \mathcal{L}ie(k) \otimes_{\Sigma_k} V^{\otimes k}$$

$$\mathbb{L}(V) \otimes V = \bigoplus_{k \geq 1} \left(\mathcal{L}ie(k) \otimes_{\Sigma_k} V^{\otimes k} \right) \otimes V$$

$$
\begin{array}{ccc}
\mathbb{L}(V)\otimes V & \xrightarrow{[\cdot,\cdot]} & \mathbb{L}^{\geq 2}(V) \\
\cup & & \cup \\
\left(\mathcal{L}ie(k-1)\otimes_{\Sigma_{k-1}} V^{\otimes k-1}\right)\otimes V & \longrightarrow & \mathcal{L}ie(k)\otimes_{\Sigma_k} V^{\otimes k}
\end{array}
$$





$$\mathcal{D}er_\omega\, \mathbb{L}(V) \cong \bigoplus_{k\geq 1} \mathcal{L}ie((k)) \otimes_{\Sigma_k} V^{\otimes k}$$

$\mathcal{L}ie(k-1)$ with Σ_k-action

$$H_p(\Gamma_{\cancel{q}}; \cancel{H} C_q^{CE}(\mathfrak{g}_g)) \Rightarrow \mathbb{H}_{p+q}(\Gamma_g; C_*^{CE}(\mathfrak{g}_g))$$

stable for $g \geq 2(p+q)+4$

(A. Borel) Γ_g is $\mathbb{Q}$-perfect

$$\begin{tikzcd}
\rightsquigarrow \exists\ C_*^{CE}(\mathfrak{g}_g) \arrow[r, leftrightarrow, "\sim"] \arrow[d] \arrow[dr, phantom, "= \text{htpy}"] & H_*^{CE}(\mathfrak{g}_g) \arrow[d] \\
C_*^{CE}(\mathfrak{g}_{g+1}) \arrow[r, leftrightarrow, "\sim"] & H_*^{CE}(\mathfrak{g}_g)
\end{tikzcd}$$

as chain cxes of Γ_g-modules

$$\begin{tikzcd}
\mathbb{H}_*(\Gamma_g; C_*^{CE}(\mathfrak{g}_g)) \arrow[d, "\wr"] & \mathbb{H}_* \arrow[l, "\sim"'] \arrow[d, "\wr"] \\
? & ? \arrow[l, "\sim"']
\end{tikzcd}$$

$\mathbb{H}_*(\Gamma_g; H_*^{CE}(\mathfrak{g}_g))$ stabilises