Block Diffeomorphisms

Monday, 1 June 2020 11:00

I. What are block diffeomorphisms?

Def: Let M be a manifold, we define the topological group of block diffeomorphisms of M, $\widetilde{\mathrm{Diff}}(M)$, to be the geometric realization of the semi-simplicial group $\widetilde{\mathrm{Diff}}_\bullet(M)$ with k-simplices being given by the diffeomorphisms

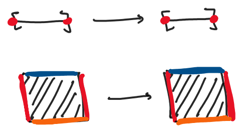
$$\varphi : \Delta^k \times M \longrightarrow \Delta^k \times M$$

such that
(1) If F is a face of Δ^k, $\varphi(F \times M) = F \times M$;
(2) $\Delta^k \times \partial M$ is fixed pointwise by φ.

⊕ We want a collar of $F \times M$ and $\Delta^k \times \partial M$ to be fixed.

Examples:

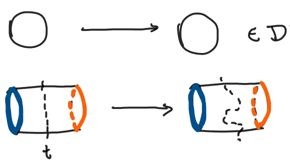
$\underline{M = I}$: $\widetilde{\mathrm{Diff}}(I)_0 =$



$\in \mathrm{Diff}_\partial(I)$

$\widetilde{\mathrm{Diff}}(I)_1 =$

$\underline{M = S^1}$: $\widetilde{\mathrm{Diff}}(S^1)_0 =$



$\in \mathrm{Diff}(S^1)$

$\widetilde{\mathrm{Diff}}(S^1)_1 =$

Remark: $\pi_0(\widetilde{\mathrm{Diff}}(M))$ is the set of pseudoisotopy classes of diffeos of M rel ∂.

? Why look at block diffeomorphisms?

① These are built to be more easily studied by surgery theoretic techniques.

② $\widetilde{\mathrm{Diff}}(M)$ can be seen as a midway between $\mathrm{Diff}(M)$ and $\mathrm{aut}(M)$.

$$\mathbf{BDiff(M)} \xrightarrow{(\mathrm{I})} \mathbf{B\widetilde{Diff}(M)} \xrightarrow{(\mathrm{II})} \mathbf{Baut(M)}$$

classifies smooth fibre bundles with fibre M.	classifies block bundles with fibre M.	classifies Serre fibrations with fibre M.

THE MOST BASIC BLOCK BUNDLE: $p: \Delta^k \times M \longrightarrow \Delta^k$ taking $F \times M \longrightarrow F$ for any face F of Δ^k.
More generally, $E \longrightarrow |K|$ is a block bundle if it is built out of maps like this ↑ glued together by block diffeomorphisms.

(I) We know that $\mathrm{Diff}(M)$ is weakly equivalent to the geometric realization of its singular simplicial space,

$$S_k\mathrm{Diff}(M) := \mathrm{maps}(\Delta^k, \mathrm{Diff}(M))$$

$f: \Delta^k \to \mathrm{Diff}(M)$ gives a map $\Delta^k \times M \longrightarrow \Delta^k \times M$, $(t, m) \longmapsto (t, f(t)(m))$

$\mathrm{Diff}(M)$ includes into $\widetilde{\mathrm{Diff}}(M)$ as those diffeos

$$\Delta^k \times M \longrightarrow \Delta^k \times M$$

that commute with the projection to Δ^k.

(II) We can do something analogous for $\mathrm{aut}(M)$:

$$\widetilde{\mathrm{Diff}}(M)_k \hookrightarrow \widetilde{\mathrm{aut}}(M)_k \xleftarrow{\simeq} S_k\mathrm{aut}(M)$$

$\Delta^k \times M \longrightarrow \Delta^k \times M$ diffeo; $\Delta^k \times M \longrightarrow \Delta^k \times M$ homotopy equivalence; $\Delta^k \longrightarrow \mathrm{aut}(M)$

Claim: $\mathrm{aut}(M) \hookrightarrow \widetilde{\mathrm{aut}}(M)$ is a homotopy equivalence.

So we have a map

$$\widetilde{\mathrm{Diff}}(M) \longrightarrow \mathrm{aut}(M)$$

GOAL → Use the tools we have developed the past weeks to determine the rational homotopy type of $\widetilde{\mathrm{Diff}}(M)$.

II. AUTOMORPHISMS OF FIBRE BUNDLES

Let $\xi: E \longrightarrow X$ be a vector bundle over a CW-complex.
We define $\mathrm{aut}(\xi)$ to be the topological monoid of diagrams

$$\begin{tikzcd} E \arrow[r, "\hat{f}"] \arrow[d, "\xi"'] & E \arrow[d, "\xi"] \\ X \arrow[r, "f"] & X \end{tikzcd}$$

where f is a homotopy equivalence and $\hat{f}$ is a fibrewise isomorphisms. We topologise $\mathrm{aut}(\xi)$ as a subspace of $\mathrm{aut}(X) \times \mathrm{aut}(E)$.

Some more notation:

- Given $D \subseteq C \subseteq X$, denote by $\mathrm{aut}^D_C(\xi)$ the subspace of those $(f, \hat{f})$ st $f|_C$ is the identity and and $\hat{f}$ is the identity on $\xi|_D$.
- $\mathrm{aut}_C(\xi) := \mathrm{aut}^C_C(\xi)$.
- $\mathrm{aut}^D_{C,\circ}(\xi)$ is the subspace of those $(f, \hat{f})$ with f homotopic to the identity.

Lemma: $B\mathrm{aut}^{\mathcal{D}}_c(\xi) \longrightarrow B\mathrm{aut}_c(X)$ is a Serre Fibration, with homotopy fibre $\mathrm{map}_{\mathcal{D}}(X, BO(n))_\xi$.

dim of ξ

Idea of proof: Let $\gamma_n : U_n \longrightarrow BO(n)$ be the universal bundle and $\kappa : X \longrightarrow BO(n)$ a classifying map for ξ. Then the pull back square

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\xi"'] & U_n \arrow[d, "\gamma_n"] \\ X \arrow[r, "\kappa"] & BO(n) \end{tikzcd}$$

induces a pullback square

contractible

$$\begin{tikzcd} \mathrm{aut}^{\mathcal{D}}_c(\xi) \arrow[r, "\kappa_*"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Bun}(\xi, \gamma_n) \arrow[d] \\ \mathrm{aut}_c(X) \arrow[r, "\kappa_*"] & \mathrm{map}_{\mathcal{D}}(X, BO(n)) \end{tikzcd}$$

So we get a homotopy fibre sequence

$$\mathrm{aut}^{\mathcal{D}}_c(\xi) \longrightarrow \mathrm{aut}_c(X) \longrightarrow \mathrm{map}_{\mathcal{D}}(X, BO(n)) \qquad \triangle$$

$\boxed{\text{IDEA}}$ → Relate the block diffeomorphisms to the automorphisms they induce on the tangent bundle.

$$\mathrm{DIFF}(M) \ni \varphi \longrightarrow \begin{tikzcd} \tau_M \arrow[r, "D\varphi"] \arrow[d] & \tau_M \arrow[d] \\ M \arrow[r, "\varphi"] & M \end{tikzcd}$$

$$\mathrm{DIFF}(\Delta^k \times M) \ni \psi \longrightarrow \cdots$$

We define the stabilization map:

$$\mathrm{aut}^{\mathcal{D}}_c(\xi) \longrightarrow \mathrm{aut}^{\mathcal{D}}_c(\xi \times \mathbb{R})$$
$$(f, \hat{f}) \longmapsto (f, \hat{f} \times id_{\mathbb{R}})$$

We define $\mathrm{aut}^{\mathcal{D}}_c(\xi^s)$ to be the homotopy colimit

$$\mathrm{aut}^{\mathcal{D}}_c(\xi) \longrightarrow \mathrm{aut}^{\mathcal{D}}_c(\xi \times \mathbb{R}) \longrightarrow \cdots$$

Lemma: We have a homotopy fibre sequence

$$\mathrm{map}_{\mathcal{D}}(X, BO)_\xi \longrightarrow B\mathrm{aut}^{\mathcal{D}}_c(\xi^s) \longrightarrow B\mathrm{aut}_c(X)$$

III. BACK TO BLOCK DIFFEOMORPHISMS

An element of $\widetilde{\mathrm{Diff}}(M)_k$ is given by

$$(\varphi, \psi) : \Delta^k \times M \longrightarrow \Delta^k \times M$$

and its differential gives a map

$$\begin{tikzcd}
\Delta^k \times \mathbb{R}^k \times \tau_M \arrow[r, "D(\varphi,\psi)"] \arrow[d] & \Delta^k \times \mathbb{R}^k \times \tau_M \arrow[d] \\
\Delta^k \times M \arrow[r, "(\varphi,\psi)"] & \Delta^k \times M
\end{tikzcd}$$

which gives an element of $\mathrm{aut}_\partial(\tau_M^S)$.

! There are some subtilties here as before with $\mathrm{aut}(M)$ and $\widetilde{\mathrm{aut}}(M)$

This process gives us a map

$$\widetilde{\mathrm{Diff}}_0(M) \longrightarrow \mathrm{aut}_{\partial,0}(\tau_M^S)$$

called the differential map.

Theorem: For a simply-connected smooth compact manifold M with $\partial M = S^{n-1}$, $n \geq 5$, the differential induces a rational homotopy equivalence

$$B\widetilde{\mathrm{Diff}}_{\partial,0} \longrightarrow B\,\mathrm{aut}^*_{\partial,0}(\tau_M^S)$$

basepoint on the boundary

Idea of the proof: compare homotopy fibre sequences

$$\begin{tikzcd}
{} \arrow[r] \arrow[d, dashed] & B\widetilde{\mathrm{Diff}}_{\partial,0} \arrow[r] \arrow[d] & B\mathrm{aut}_{\partial,0}(M) \arrow[d, equal] \\
\mathrm{map}_*(M, BO)_{\tau_M^S} \arrow[r] & B\mathrm{aut}^*_{\partial,0}(\tau_M^S) \arrow[r] & B\mathrm{aut}_{\partial,0}(M)
\end{tikzcd}$$

Δ

we can show this is a rational homotopy equivalence using surgery techniques

? How does this help us with the rational homotopy type of $\widetilde{\mathrm{Diff}}(M)$?

We have a homotopy fibre sequence

$$\mathrm{map}_*(M, BO)_{\tau_M^s} \longrightarrow \underline{B\mathrm{aut}^*_{\partial,0}(\tau_M^s)} \longrightarrow B\mathrm{aut}_{\partial,0}(M)$$

and we know that

SPACE	dg LIE ALGEBRA
M	$\mathbb{L}_M = (\mathbb{L}(V), \delta)$
$B\mathrm{aut}_{\partial,0}(M)$	$\mathrm{Der}^+_\omega \mathbb{L}_M$
BO	$P = \mathbb{Q}[q_1, q_2, \dots]$ ↳ dual of Pontryagin classes

Theorem: If $\tilde{H}_*(M, \mathbb{Q})$ is concentrated in a single degree, then the fibration

$$\mathrm{map}_*(M, BO)_{\tau_M^s} \longrightarrow B\mathrm{aut}^*_{\partial,0}(\tau_M^s) \longrightarrow B\mathrm{aut}_{\partial,0}(M)$$

is rationally trivial.

$$\Rightarrow B\mathrm{aut}^*_{\partial,0}(\tau_M^s) \simeq_{\mathbb{Q}} \mathrm{map}_*(M, BO)_{\tau_M^s} \times B\mathrm{aut}_{\partial,0}(M)$$

therefore a dg Lie algebra model for $B\mathrm{aut}^*_{\partial,0}(\tau_M^s)$ is given by

$$(\underbrace{\tilde{H}^*(M, \mathbb{Q})}_{M} \otimes \underbrace{P}_{BO}) \times \underbrace{\mathrm{Der}^+_\omega \mathbb{L}(V)}_{B\mathrm{aut}_{\partial,0}(M)}$$