The Graph Complex via Category Theory

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Recollections I

Notations

Fix a positive integer $d \geq 2$. We will use the following notations for g a positive integer:

- $M_g = \sharp^g S^d \times S^d$, $M_{g,1}$ is M_g with an open $2d$-disk removed.
- $H_g = H_d(M_g; \mathbb{Z})$ (thought of as having degree 0)
- $V_g = s^{d-1} H_g \otimes \mathbb{Q}$
- $\Gamma_g = Aut(H_g, \mu, q)$, where μ is the intersection pairing.
- $\mathbb{L}(V_g)$ the free Lie algebra on V_g is a minimal Quillen model for $M_{g,1}$.
- $\omega_g = \frac{1}{2}\Sigma_i [\alpha_i^\sharp, \alpha_i] \in \mathbb{L}(V_g)$ represents (up to sign) the homotopy class of the inclusion of the boundary $S^{2d-1} \hookrightarrow M_{g,1}$.

remark

The intersection pairing μ makes V_g into a **graded anti-symmetric** inner product space.

Recollections II

Notations

With d and g as above:

- $X_g = Baut_\partial(M_{g,1})$
- $\mathfrak{g}_g = Der^+_{\omega_g}\mathbb{L}(V_g)$, the positive truncation of the dg Lie algebra of derivations of $\mathbb{L}(V_g)$ that kill ω_g.

Theorem [BM]

The Lie algebra $\mathfrak{g}_g$ is a Quillen model for $Baut_{\partial,\circ}(M_{g,1})$, the connected component of the identity of X_g.

Theorem [BM]

The stabilization map $M_{g,1} \hookrightarrow M_{g+1,1}$ induces an isomorphism

$$H_k(X_g;\mathbb{Q}) \to H_k(X_{g+1};\mathbb{Q})$$

for $g > 2k+4$, and an epimorphism for $g = 2k+4$.

Recollections III

Notation

Letting $g \to \infty$, we let $X_\infty = hocolim_g X_g$, $\Gamma_\infty = colim_g \Gamma_g$, and $\mathfrak{g}_\infty = colim_g \mathfrak{g}_g$, under the stabilization maps.

Observation

The canonical map $X_g \to X_\infty$ induces an isomorphism

$$H^k(X_\infty; \mathbb{Q}) \to H^k(X_g; \mathbb{Q})$$

for $g > 2k + 4$.

Theorem [BM]

Let $d \geq 3$. There is an isomorphism of graded rings

$$H^*(X_\infty; \mathbb{Q}) \cong H^*(\Gamma_\infty; \mathbb{Q}) \otimes H^*_{CE}(\mathfrak{g}_\infty)^{\Gamma_\infty}.$$

remark

As graded ring, $H^*(\Gamma_\infty; \mathbb{Q}) = \mathbb{Q}[x_1, x_2, ...]$ by results of Borel.

First Reduction

remark

Instead of computing $H^*_{CE}(\mathfrak{g}_\infty)^{\Gamma_\infty}$, we may dually compute the coinvariants $H_*^{CE}(\mathfrak{g}_\infty)_{\Gamma_\infty}$.

remark

Given that, for $g \geq 2$, $C_*^{CE}(\mathfrak{g}_g)$ is chain homotopic to $H_*^{CE}(\mathfrak{g}_g)$ as complexes of $\mathbb{Q}[\Gamma_g]$-modules, it is enough to compute $C_*^{CE}(\mathfrak{g}_\infty)_{\Gamma_\infty}$.

Theorem 9.1 [BM]

There is an isomorphism of chain complexes

$$C_*^{CE}(\mathfrak{g}_\infty)_{\Gamma_\infty} \cong \Lambda \mathscr{G}^d(0).$$

remark

We have $H_*^{CE}(\mathfrak{g}_\infty)_{\Gamma_\infty} \cong \Lambda(H_* \mathscr{G}^d(0))$.

Sp-modules

Definition

Fix a positive integer d. Let $\mathbf{Sp}^d$ denote the category of graded finite dimensional anti-symmetric inner product spaces concentrated in degree d.

remarks

The spaces V_g are objects of $\mathbf{Sp}^d$. All the morphisms of $\mathbf{Sp}^d$ are monomorphisms.

Definition

An $\mathbf{Sp}^d$-module in a category $\mathscr{V}$ is a functor $\mathbf{Sp}^d \to \mathscr{V}$.

Proposition

The assignment $V \mapsto Der_\omega \mathbb{L}(V)$ is an $\mathbf{Sp}^d$-module in the category of graded Lie algebras.

proof

$$0 \to Der_\omega \mathbb{L}(V) \to Der\mathbb{L}(V) \to \mathbb{L}^{\geq 2}(V) \to 0$$

The Lie operad I

Recollection

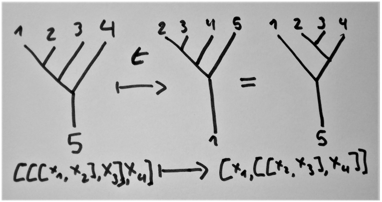
There is a symmetric operad $\mathscr{L}ie$, called the Lie operad, with spaces $\mathscr{L}ie(n)$ spanned by the Lie monomials in $x_1, ..., x_n$ containing every generator exactly once.

Lemma

The Lie operad is a cyclic operad.

proof

Let t denote the permutation $(12...n)$ in Σ_n. We extend the left Σ_{n-1}-action on $\mathscr{L}ie(n-1)$ via:



The Lie operad II

Observation

Let us denote $\mathscr{L}ie((n))$ the space $\mathscr{L}ie(n-1)$ with the Σ_n-action defined previously. The assignment

$$V \mapsto \mathscr{L}ie((V)) := s^{-2d} \bigoplus_{n \geq 2} \mathscr{L}ie((n)) \otimes_{\Sigma_n} V^{\otimes n}$$

defines an $\mathbf{Sp}^d$-module. This is a Schur functor.

Proposition [BM]

There is an isomorphism of $\mathbf{Sp}^d$-modules,

$$\mathscr{L}ie((V)) \cong Der_\omega \mathbb{L}(V).$$

remark

One can describe explicitly the Lie bracket on $\mathscr{L}ie((V))$ induced by the above isomorphism.

Σ-modules I

We fix $\mathscr{V}$ a cocomplete symmetric monoidal category.

Definition

Let Σ denote the groupoid of finite sets and bijections. A left Σ-module in $\mathscr{V}$ is a functor $\Sigma \to \mathscr{V}$.

Example

Any object V of $\mathscr{V}$ defines a right Σ-module via $S \mapsto \bigotimes_S V$.
The functor $S \mapsto \mathscr{L}ie(S)$ is a Σ-module.

Notation

We denote by $(\Sigma \downarrow \Sigma)$ the category with objects the maps $f : S \to T$ of finite sets, and with morphisms pairs of bijections

$$\begin{array}{ccc} S_1 & \xrightarrow{\cong} & S_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ T_1 & \xrightarrow[\cong]{} & T_2. \end{array}$$

Σ-modules II

Notation

Fix a finite set S. We denote by $(S \downarrow \Sigma)$ the category with objects the maps $f : S \to T$ of finite sets, and with morphisms bijections

$$\begin{array}{ccc} & S & \\ {}^{f_1}\swarrow & & \searrow^{f_2} \\ T_1 & \xrightarrow[\cong]{} & T_2. \end{array}$$

Definition

Let $\mathscr{C}$ and $\mathscr{D}$ be Σ-modules. Their composition $\mathscr{C} \circ \mathscr{D}$ is the Σ-module specified by

$$\mathscr{C} \circ \mathscr{D}(S) := colim_{(f:S \to T)} \mathscr{C}(T) \otimes \bigotimes_{s \in S} \mathscr{D}(f^{-1}(s)),$$

where the colimit is taken over the category $(S \downarrow \Sigma)$.

Σ-modules III

remark

Monoids in the category of Σ-modules with monoidal structure given by $\circ$ correspond precisely to symmetric operads.

Definition

Assuming $\mathscr{V}$ is cocomplete and symmetric monoidal. The **Schur functor** associated to a Σ-module $\mathscr{C}$ is the functor $\mathscr{C}[-] : \mathscr{V} \to \mathscr{V}$ specified by

$$\mathscr{C}[V] := colim_{S \in \Sigma} \mathscr{C}(S) \otimes V^{\otimes S} \cong \bigoplus_{n \geq 0} \mathscr{C}(n) \otimes_{\Sigma_n} V^{\otimes n}.$$

remark

There is a natural isomorphism of functors

$$\mathscr{C}[\mathscr{D}[V]] \cong (\mathscr{C} \circ \mathscr{D})[V].$$

Matchings I

Definition

A matching on a finite set S is a set M of disjoint 2-element subsets whose union is S. We denote by $\mathcal{M}_S$ the set of all matchings on S.

remark

Matchings define a Σ-module $\mathcal{M} : \Sigma \to \mathbf{Set}$. If $|S|$ is odd, $\mathcal{M}_S = \emptyset$.

Notations

For a finite set S, we denote sgn_S the sign representation of Σ_S. Given $\sigma \in \Sigma_n$, V a graded vector space, and $x = x_1 \otimes ... \otimes x_n \in V^{\otimes n}$. We denote by $sgn(\sigma, x)$ the sign for which

$$(x_1 \otimes ... \otimes x_n)\sigma = sgn(\sigma, x) x_{\sigma_1} \otimes ... \otimes x_{\sigma_n}.$$

Matchings II

Theorem [BM]

Let V be a graded anti-symmetric inner product space concentrated in degree $d-1$. There is a pairing

$$\langle -,-\rangle : \mathcal{M}_{2k} \otimes V^{\otimes 2k} \to sgn_{2k}$$

$$\langle \{\{\sigma_1,\sigma_2\},...,\{\sigma_{2k-1},\sigma_{2k}\}\}, x_1 \otimes ... \otimes x_n\rangle = sgn(\sigma)sgn(\sigma,x)\langle x_{\sigma_1}, x_{\sigma_2}\rangle ... \langle x_{\sigma_{2k-1}}, x_{\sigma_{2k}}\rangle$$

This pairing gives rise to a morphism of Σ_{2k}-modules of degree $-2k(d-1)$:

$$\psi : (V^{\otimes 2k})_{\mathbf{Sp}(V)} \to \mathcal{M}_{2k} \otimes sgn_{2k}$$

$$\psi([x]) := \sum_{M \in \mathcal{M}_{2k}} \langle M, x\rangle M,$$

that is an isomorphism if $dim(V) \geq 2k$.

Matchings III

Definition

From now on, we fix $\mathscr{V}$ as the category of graded $\mathbb{Q}$-vector spaces. Fix a positive integer d. We denote by **Sp** the category of graded anti-symmetric inner product spaces concentrated in degree $d-1$. Given an **Sp**-module $M : \mathbf{Sp} \to \mathscr{V}$, we denote

$$M_{\mathbf{Sp}} := colim_{V \in \mathbf{Sp}} M(V).$$

Example

There is an $\mathbf{Sp} - \Sigma$-bimodules given by

$$\mathbf{Sp} \times \Sigma^{op} \to \mathscr{V}, \quad (V, S) \mapsto V^{\otimes S}.$$

Corollary

There is an isomorphism of right Σ-modules

$$(V^{\otimes S})_{\mathbf{Sp}} \cong s^{|S|(d-1)} \mathcal{M}_S \otimes sgn_S.$$

The Graph Complex I

Definition

A graph $G = (f : F \to V, E)$ consists of a set F of half-edges, a set V of vertices, a function $f : F \to V$, and a matching E on F. The elements of E are thought of as the edges of the graph.

Notation

We denote by $\mathscr{G}raph$ the groupoid of graphs and their isomorphisms.

Construction

Let $\mathscr{C} : \Sigma \to \mathscr{V}$. There is an induced functor $\mathscr{C} : \mathscr{G}raph \to \mathscr{V}$, with value at $G = (f : F \to V, E)$ given by

$$\mathscr{C}(G) = \bigotimes_{v \in V} \mathscr{C}(f^{-1}(v)).$$

Recollection: The Grothendieck Construction

Definition

Let I be a category, and $F : I \to \textbf{Cat}$ be a functor to the category of small categories. The Grothendieck construction $I \int F$ is the category whose objects are pairs (i, x), with $i \in I$ and $x \in F(i)$.

Lemma

For every functor $D : I \int F \to \mathscr{V}$ into a cocomplete category, there is a canonical isomorphism

$$colim_{(i,x) \in I \int F} D(i, x) \cong colim_{i \in I} colim_{x \in F(i)} D(i, x).$$

Example

There is a functor $(- \downarrow \Sigma) : \Sigma \to \textbf{Cat}$ that sends S to $(S \downarrow \Sigma)$. The Grothendieck construction $\Sigma \int (- \downarrow \Sigma)$ is isomorphic to $(\Sigma \downarrow \Sigma)$ (as a category over Σ).

The Graph Complex II

Example

Let $\mathcal{M} : (\Sigma \downarrow \Sigma) \to \mathbf{Cat}$ denote the functor that sends $f : S \to T$ to $\mathcal{M}_S$ viewed as a discrete category. The Grothendieck construction on $\mathcal{M}$ is isomorphic to $\mathscr{G}raph$.

Notation

Given a Σ-module $\mathscr{C}$, the Schur functor $V \mapsto \mathscr{C}((V))$ is an **Sp**-module with

$$\mathscr{C}((V)) := s^{2-2d} \bigoplus_{n \geq 0} \mathscr{C}(n) \otimes_{\Sigma_n} V^{\otimes n}.$$

remark

For $\mathscr{C} = \mathscr{L}ie((-))$ this agrees with the previously used notation, as $\mathscr{L}ie((0)) = \mathscr{L}ie((1)) = 0$.

The Graph Complex III

Theorem [BM]

One can describe the complex $\Lambda s\mathscr{C}((V))_{\mathbf{Sp}}$ explicitly.

proof

The **Sp**-module $V \mapsto \Lambda sV$ is identified with the Schur functor associated to the Σ-module Λs given by $\Lambda s(T) := s^{|T|} sgn_T$. We compute:

$$\begin{aligned} \Lambda s\mathscr{C}((V)) &\cong (\Lambda s \circ \mathscr{C})[V] \\ &= colim_{S\in\Sigma}(\Lambda s \circ \mathscr{C})(S) \otimes V^{\otimes S} \\ &= colim_{S\in\Sigma} colim_{f:S\to T} \Lambda s(T) \otimes \mathscr{C}((f)) \otimes V^{\otimes S} \\ &\cong colim_{f\in(\Sigma\downarrow\Sigma)} \Lambda s(T) \otimes \mathscr{C}((f)) \otimes V^{\otimes S}. \end{aligned}$$

The Graph Complex IV

proof continued

As colimits commute with colimits and tensor products of graded vector spaces, we find:

$$\begin{aligned}\Lambda s\mathscr{C}((V))_{\mathbf{Sp}} &\cong colim_{f\in(\Sigma\downarrow\Sigma)}\Lambda s(T)\otimes\mathscr{C}((f))\otimes(V^{\otimes S})_{\mathbf{Sp}}\\ &\cong colim_{f\in(\Sigma\downarrow\Sigma)}\Lambda s(T)\otimes\mathscr{C}((f))\otimes s^{|S|(d-1)}\mathcal{M}_S\otimes sgn_S\end{aligned}$$

Viewing $\mathcal{M}_S$ as a discrete category, we may rewrite the above expression as:

$$colim_{f\in(\Sigma\downarrow\Sigma)}colim_{M\in\mathcal{M}_S}s^{|S|(d-1)}\Lambda s(T)\otimes\mathscr{C}((f))\otimes sgn_S$$

Using the above description of $\mathscr{G}raph$ as a Grothendieck construction, and some changes of notations yield:

$$\mathscr{G}^d\mathscr{C} := colim_{G\in\mathscr{G}raph}s^{(3-2d)|V|+|F|(d-1)}sgn_V\otimes sgn_F\otimes\mathscr{C}((G)).$$

Concluding remarks

remark

Any object of $\mathscr{G}^d\mathscr{C}$ is represented by a graph $(f : F \to V, E)$ together with an orientation of the vertices, an orientation of the half-edges, and for every vertex v, an element $\xi_v \in \mathscr{C}(f^{-1}(v))$.

remark

If $\mathscr{C}$ is a cyclic operad, the **Sp**-module $V \mapsto \mathscr{C}((V))$ admits a Lie algebra structure. Thus, we are entitled to consider the chain complex

$$C_*^{CE}(\mathscr{C}((V)))_{\mathbf{Sp}} = (\Lambda s\mathscr{C}((V))_{\mathbf{Sp}}, \partial).$$

Warning: This depends on d.

remark

For $d = 1$, this is the complex of $\mathscr{C}$-labeled graphs. If, in addition, $\mathscr{C} = \mathscr{L}ie((-))$, we recover the graph complex.

A proof of Theorem 9.1

We have:

$$C_*^{CE}(\mathfrak{g}_\infty)_{\Gamma_\infty} \cong C_*^{CE}(\mathscr{L}ie((V)))_{\mathbf{Sp}} = \mathscr{G}^d\mathscr{L}ie.$$

This isomorphism follows from

$$Der^+_{\omega_g}\mathbb{L}(V_g) = Der_{\omega_g}\mathbb{L}(V_g) \cong \mathscr{L}ie((V_g)),$$

together with the fact that $\Gamma_g \subseteq Sp(V_g)$ is dense, which implies that

$$C_*^{CE}(\mathfrak{g}_g)_{\Gamma_g} \cong C_*^{CE}(\mathscr{L}ie((V_g)))_{\mathbf{Sp}(V_g)}.$$

If we denote by $\mathscr{G}^d(0)$ the subcomplex of $\mathscr{G}^d\mathscr{L}ie$ on the connected graphs, we have

$$\mathscr{G}^d\mathscr{L}ie \cong \Lambda\mathscr{G}^d(0),$$

as disjoint union of graphs endow $\mathscr{G}^d\mathscr{L}ie$ with a differential graded commutative product.