Conclusion

15 June 2020 10:19

Plan for today:
1) Recall what we've seen so far — 15 min
2) Deal with the fundamental groups — 30 min ← ~ 5 min
3) Survey the results — 25 min

PART 1:

Our goal: Understand $BDiff_\partial(M_{g,1}) \to B\widetilde{Diff}_\partial(M_{g,1}) \longrightarrow Baut_\partial(M_{g,1})$

> as $g \to \infty$

> from the perspective of rational htp. theory

Chris & Tom: $M_{g,1} = (\#_g S^d \times S^d) \setminus D^{2d}$ is modelled by $\mathbb{L} V_g$, $V_g = s^{-1}\tilde{H}(M_{g,1};\mathbb{Q})$

d odd, V_g $2g$-dim'l. symplectic vector space

$S^{2d-1} = \partial M_{g,1} \hookrightarrow M_{g,1}$ is modelled by $\omega = \frac{1}{2} \sum [\alpha_i, \alpha_i^\#]$

↳ $Baut_{\partial,\circ}(M_{g,1})$ is modelled by $\boxed{Der^+_\omega \mathbb{L}V_g} =: \mathfrak{g}_g$

Luci: $B\widetilde{Diff}_\partial(M_{g,1})$ behaves like automorphisms respecting the stable tangent bundle τ^s_M

This splitting is not canonical

$B\widetilde{Diff}_\partial(M_{g,1}) \simeq_{\mathbb{Q}} Baut_{\partial,\circ}(\tau^s_M) \simeq_{\mathbb{Q}} map_*(M, BO)_{\tau^s_M} \times Baut_{\partial,\circ}(M_{g,1})$

and this is modelled by: $\Pi := \mathbb{Q}\{q_i \mid 4i > d\} = \pi_{*+1}(BO)$

↳ $\underbrace{(V_g^\vee \otimes \Pi)}_{\mathfrak{t}_g} \oplus \mathfrak{g}_g$ (Q: What is the correct shift?)

$\mathfrak{t}_g$ (think of it as an abelian Lie algebra)

So far we have completely ignored the fundamental groups $\pi_1 Baut_\partial = \pi_0 aut_\partial$, $\pi_1 B\widetilde{Diff}_\partial = \pi_0 \widetilde{Diff}_\partial$

Tom: $\pi_1 Baut_\partial(M_{g,1}) \twoheadrightarrow \Gamma_g < Aut(V_g)$ with finite kernel

Aut as a symplectic vector space

Something similar happens for $B\widetilde{Diff}_\partial(M_{g,1})$.

Warning: The kernel is not finite for $d \equiv 3 \mod 4$

Notation: $X_g = Baut_\partial(M_{g,1}) \rightsquigarrow \tilde{X}_g = Baut_{\partial,\circ}(M_{g,1})$ and $\pi_1 X_g \twoheadrightarrow \Gamma_g$

PART 2: The universal cover spectral sequence

Hty.

Fiber sequence $\tilde{X}_g \longrightarrow X_g \longrightarrow B\pi_1 X_g$ $\rightsquigarrow$ Serre spectral sequence

$$H^p(\pi_1 X_g ; H^q(\tilde{X}_g ; \mathbb{Q})) \Rightarrow H^{p+q}(X_g)$$

We know: $\pi_1 X_g \twoheadrightarrow \Gamma_g$ with finite kernel and $H^q(\tilde{X}_g ; \mathbb{Q}) \cong H^q_{CE}(\mathfrak{g}_g)$

This holds as Γ_g-modules.

So we want to compute: $H^p(\Gamma_g ; H^q_{CE}(\mathfrak{g}_g)) \Rightarrow H^{p+q}(B\mathrm{aut}_\partial(M_{g,1}) ; \mathbb{Q})$

(at least $g \to \infty$)

Let's dream for a moment:

$$\begin{array}{c} F \to E \\ \downarrow \\ B \end{array}$$

If B is simply con. $\rightsquigarrow$ $E_2^{p,q} \cong H^p(F;\mathbb{Q}) \otimes H^q(B;\mathbb{Q})$

and under very favorable circumstances the spectral sequence collapses

$\hookrightarrow$ $H^*(E) = H^*(F) \otimes H^*(B)$

But for us $\pi_1 B = \Gamma_g \neq 0$.

There always is an inclusion:

(B connected)

$$\underbrace{\underbrace{H^0(B ; H^p(F;\mathbb{Q}))}_{H^p(F;\mathbb{Q})^{\pi_1 B}} \otimes H^q(B;\mathbb{Q})}_{H^q(B ; H^p(F,\mathbb{Q})^{\pi_1 B})} \longrightarrow \underset{E_2^{p,q}}{\overset{\shortparallel}{H^q(B ; H^p(F;\mathbb{Q}))}}$$

$\hookrightarrow$ We can hope for this to be an iso, i.e. $E_2^{p,q} = E_2^{p,0} \otimes E_2^{0,q}$

This is an iso.

Thm This splitting of the E_2-page indeed happens for the X_∞-spectral sequence

moreover the spectral sequence collapses to give:

$$H^* X_\infty \cong H^*(\Gamma_\infty ; \mathbb{Q}) \otimes H^*_{CE}(\mathfrak{g}_\infty)^{\Gamma_\infty}$$

First part: Polynomial functor "magic" + Borel's vanishing theorem

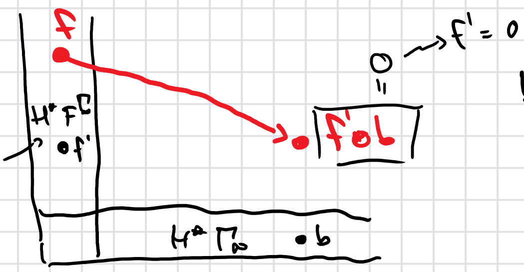
Second part: This will follow from the following observations:

1) $H^*(X_\infty)$ and $H^*(\Gamma_\infty)$ are free graded com. algebras

2) $H^*(\Gamma_\infty) \longrightarrow H^*(X_\infty)$ is injective

Idea

$H^*_{CE}(\mathfrak{g}_\infty)^{\Gamma_\infty}$



$b \in H^* \Gamma_\infty$, $e \in H^* X_\infty$

$\pi^* b \cdot e = 0 \Rightarrow e = 0$

? (d odd)

$H^*(\Gamma_\infty;\mathbb{Q}) \overset{?}{\cong} H^*(Sp_\infty(\mathbb{Z});\mathbb{Q})$ (d odd) It's a classical computation by Borel that $H^*(Sp_\infty(\mathbb{Z});\mathbb{Q}) = \mathbb{Q}[x_2, x_2, \dots]$ $|x_i| = 4i-2$.

$H^*(X_\infty;\mathbb{Q})$ This is what we want to compute. So how do we show that it's free without already computing it ??

$A := \coprod_{g\geq 0} B\mathrm{aut}_\partial(M_{g,1})$ this can be made into a E_{2d}-algebra

$\hookrightarrow \Omega B A$ this is a 2d-fold loop space

The group completion theorem tells us that $X_\infty \times \mathbb{Z} \longrightarrow A$ H_*-iso

$\hookrightarrow X_\infty$ has the same homology as a 2d-fold loop space $\square$

The map $H^*(B\Gamma_\infty) \longrightarrow H^*(X_\infty) \xrightarrow{} H^*B\mathrm{Diff}(M_\infty)$ is injective.

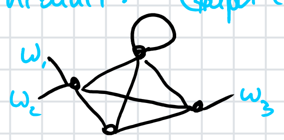
$x_i \longmapsto \qquad \longmapsto \tilde{\kappa}_{L_{i,s}}$

Thm 1.3 $2d \geq 6$ $H^*(B\mathrm{aut}_\partial(M_{\infty,1});\mathbb{Q}) \cong H^*(\Gamma_\infty;\mathbb{Q}) \otimes H^*_{CE}(\mathfrak{g}_\infty)^{\Gamma_\infty}$

1.4 " $H^*(B\widetilde{\mathrm{Diff}}_\partial(M_{\infty,1});\mathbb{Q}) \cong H^*(\Gamma_\infty;\mathbb{Q}) \otimes H^*_{CE}(\mathfrak{g}_\infty \oplus \mathfrak{t}_\infty)^{\Gamma_\infty}$

PART 3: Conclusion

We need to compute $H^*_{CE}(\mathfrak{g}_\infty)^{\Gamma_\infty}$ Thibault's Graph complexes



$k = 4$
rank $= 4$
leaves $= 3$

$\mathcal{G}^d(0)$ = graph complex w/o leaves

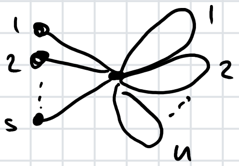
$\mathcal{G}^d[W]$ = graph complex w/ leaves labelled in some vector space W

Thm $H_*^{CE}(\mathfrak{g}_\infty)_{\Gamma_\infty} \cong \Lambda(H_*\,\mathcal{G}^d(0))$ $\mathcal{G}$ ~ connected

$H_*^{CE}(\mathfrak{t}_\infty \oplus \mathfrak{g}_\infty)_{\Gamma_\infty} \cong \Lambda(H_*\,\mathcal{G}^d[\pi])$

Graph complexes have been related to $\mathrm{Out}(F_n)$

Def $A_{n,s} := \pi_0\,\mathrm{aut}(\bigvee_n S^1 \text{ rel } \{1,\dots,s\})$



Ex: $A_{n,1} = \mathrm{Aut}(F_n)$ $A_{n,0} = \mathrm{Out}(F_n)$

$A_{n,s} = F_n^{s-1} \rtimes \mathrm{Aut}(F_n)$

Very little is known about $H^*(A_{n,s};\mathbb{Q})$

Thm (Kontsevich, CKV) $H_k(\mathcal{G}^d(n,s);\partial) = H^{(2(n-1)+s)d-k}(A_{n,s};\mathbb{Q})$

ASSEMBLING HOMOLOGY CLASSES IN AUTOMORPHISM GROUPS OF FREE GROUPS

JAMES CONANT, ALLEN HATCHER, MARTIN KASSABOV, AND KAREN VOGTMANN

Some low-dimensional classes in $H_*(Aut(F_n);\mathbb{Q})$ are known explicitly:

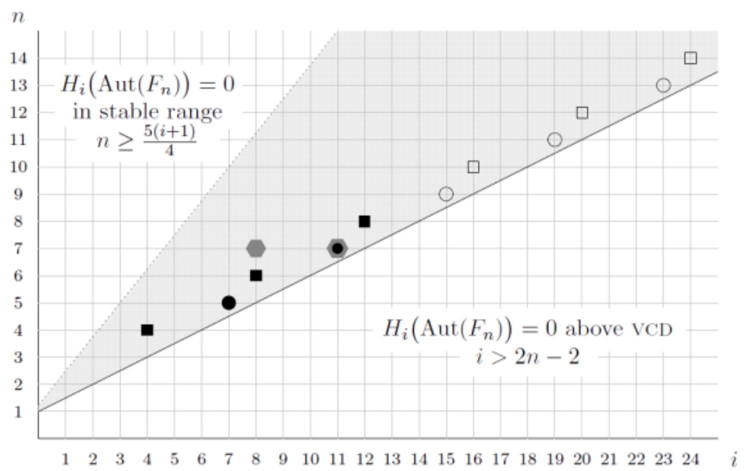


FIGURE 1. Classes in the homology of $Aut(F_n)$ for $n \leq 14$. The Morita classes are shown as squares and the Eisenstein classes are shown as circles, filled in if the classes are known to be nontrivial. The nontrivial classes recently found by Bartholdi are shown as hexagons.

But there are lots of classes we don't really know about.

n	3	4	5	6	7	8	9	10	11	12
$\chi(\mathrm{Out}(F_n))$	1	2	1	2	1	1	-21	-124	-1202	?

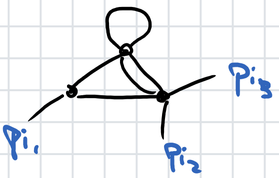
FIGURE 2. Euler characteristic of $Out(F_n)$

Adele:

Thm (G.RW) $H^*(B\mathrm{Diff}(M_{\infty,1});\mathbb{Q}) = \mathbb{Q}[\kappa_{e^n p_{i_1}\cdots p_{i_s}} \mid d/4 < i_j < d \text{ and } n+s \geq 0 \text{ or } d/2 < i_1 < d \text{ and } (n,s)=(0,1)]$

For $B\widetilde{\mathrm{Diff}}$: $\xi \in H^*(A_{n,s};\mathbb{Q})$ $\quad p_{i_1}\cdots p_{i_s} \in \Pi^\vee$

$\rightsquigarrow \tilde{\kappa}^{\xi}_{p_{i_1}\cdots p_{i_s}}$ degree: $2(n-1)d + \sum 4i_j - |\xi|$



Thm $H^*(B\widetilde{\mathrm{Diff}}_\partial(M_{\infty,1});\mathbb{Q}) \cong \mathbb{Q}[\tilde{\kappa}^{\xi}_{p_{i_1}\cdots p_{i_s}} \mid i_\nu > d/4 \text{ and } n+s \geq 2] \otimes \mathbb{Q}[x_1, x_2, \ldots]$ $\quad |x_i| = 4i-2$ (d odd)

$\uparrow$

Thm $H^*(B\mathrm{aut}_\partial(M_{\infty,1});\mathbb{Q})$ is isomorphic to the subalgebra gen. by $\tilde{\kappa}^{\xi}$

$\xi \in H^*(A_{n,0};\mathbb{Q}) = H^*(\mathrm{Out}(F_n);\mathbb{Q})$

Conjecture: The subalgebra of $H^*(B\widetilde{\mathrm{Diff}}_\partial(M_{\infty,1});\mathbb{Q})$ generated by certain $\tilde{\kappa}^{\varepsilon_{n,s}}_{p_1 \dots p_s}$ and x_j maps isomorphically to $H^*(B\mathrm{Diff}_\partial(M_{\infty,1});\mathbb{Q})$

Here $\varepsilon_{n,s}$ is the generator of $H^0(A_{n,s};\mathbb{Q})$.