Condensed Mathematics

Goal: Do algebra with topological groups / rings / modules.

Problem: This is more subtle than one might think.

Instances of this subtlety:

(A) **Topological abelian gps**

Topological abelian gps are not an abelian category.

Ex: $\mathbb{R}^{disc} \xrightarrow{\ \iota\ } \mathbb{R}$ is $\begin{cases} \text{an epi} & f\iota = g\iota \Rightarrow f=g \\ \text{a mono} & \iota f = \iota g \Rightarrow f=g \\ \underline{\text{not}} \text{ an iso.} & \end{cases}$

Problem for studying continuous representations of topological groups (e.g. $GL_n(\mathbb{Q}_p)$) on topological vector spaces.

(B) Pontryagin biduality map

Given a topological abelian gp A, Set $A^* = \mathrm{Hom}(A, \mathbb{T})$, $\mathbb{T} \hookleftarrow \mathbb{R}/\mathbb{Z}$.
with the compact open topology.

Have nat'l biduality map

$$\begin{aligned} A &\longrightarrow A^{**} \\ a &\longmapsto (\varphi \mapsto \varphi(a)) \end{aligned}$$

Problem: Not always continuous (even if it's an iso of abstract gps)

Ex: Define a top gp $\tilde{\mathbb{Z}}$ by equipping $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ with the coarsest topology making all characters $\mathbb{Z} \to S^1$ continuous.

(C) Continuous group cohomology

Continuous group cohomology does not induce long exact sequences.

Example: Take $G = S^1$ and the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 0,$$ where S^1 acts trivially.

Then $\underbrace{H^1_c(S^1,\mathbb{Z})}_{=0} \to \underbrace{H^1_c(S^1,\mathbb{R})}_{=0} \to \underbrace{H^1_c(S^1,S^1)}_{\neq 0} \to \underbrace{H^2_c(S^1,\mathbb{Z})}_{=0}$ is not exact!

(D) The Whitehead problem

Natural question: Given an abelian gp A with $\mathrm{Ext}^1(A,\mathbb{Z}) = 0$, is A necessarily free?

Shelah (1974): This is undecidable in ZFC!

Ⓔ 6 functor formalism

Recall that a 6 functor formalism consists of:

Data:

$$\begin{array}{ccc}
\underset{\text{(some cat of)}}{\{\text{Schemes}\}} & \longrightarrow & \{\overset{\text{closed sym monoidal}}{\text{Stable}}\ \infty\text{-categories}\} \\
X & \longmapsto & DX \\
X \xrightarrow{f} Y & \longmapsto & f^* : DY \rightleftarrows DX : f_* \\
\begin{array}{c} X \xrightarrow{f} Y \\ \text{separated} \\ + \\ \text{finite type} \end{array} & \longmapsto & f_! : DX \rightleftarrows DY : f^!
\end{array}$$

Such that: f proper $\Rightarrow f_! \cong f_*$, f étale / open immersion $\Rightarrow f^! \cong f^*$ $(\Rightarrow f_! \dashv f^*)$

$+$ Verdier duality, projection formula, Poincaré duality, Proper base change

Ex: ℓ-adic sheaves : X a scheme in char $p \rightsquigarrow D_{et}(X, \mathbb{Z}_\ell)$, $\ell \neq p$.

<u>Problem</u>: Quasicoherent sheaves ($X \mapsto QC(X)$) does not admit a 6 functor formalism.

<u>Why?</u> If it did, then $i_! \dashv i^! \simeq i^*$ for i the open immersion

$$i: \mathrm{Spec}(A[\tfrac{1}{f}]) \to \mathrm{Spec}(A)$$

But i^* :
$$\begin{array}{ccc} QC(\mathrm{Spec}(A)) & \longrightarrow & QC(\mathrm{Spec}(A[\tfrac{1}{f}])) \\ \shortparallel\wr & & \shortparallel\wr \\ \mathrm{Mod}_A & \longrightarrow & \mathrm{Mod}_{A[\frac{1}{f}]} \\ M & \longmapsto & M[\tfrac{1}{f}] \end{array}$$
does not admit a left adjoint, as i^* does not preserve limits:

example: $A[[t]][\tfrac{1}{f}] = \varprojlim (A[t]/t^n)[\tfrac{1}{f}] \neq \varprojlim (A[t]/t^n[\tfrac{1}{f}])$

Ⓕ Structure sheaves of adic spaces

equ. classes of valuations on stalks

Huber pairs (A, A^+) give triples $\underbrace{\mathrm{Spa}}_{\text{"adic spectrum"}}(A,A^+) = (\underbrace{X}_{\text{top space}}, \underbrace{\mathcal{O}_X}_{\text{presheaf of top. rings}}, |-|_x)$

Problem: $\mathcal{O}_X$ is not always a sheaf!

Example (Buzzard–Verberkmoes)

$A = \mathbb{Q}_p[T, \frac{1}{T}, Z]/Z^2$

$A_0 = \mathbb{Z}_p$-module gen by $p^{|m|}T^m$, $p^{-|m|}T^m Z$ for $m \in \mathbb{Z}$.

$X = \mathrm{Spa}(A, A_0)$ covered by $U = \{v \in X \mid v(T) \leq 1\}$
$V = \{v \in X \mid v(T) \geq 1\}$.

Then $Z|_U = Z|_V = 0$ but $Z \neq 0$.

(G) Spectra with actions by top. gps

What does it mean for a top. gp to act continuously on a spectrum?

<u>Example</u>: Want a continuous action of the Morava stabiliser gp on Morava E-theory.

($\rightsquigarrow$ Devinatz-Hopkins spectral sequence

$$E_2^{s,t} = H^s_{cts}(\mathbb{G}_n^{\pm}, \pi_t E) \Rightarrow \pi_{t-s} S_{K(n)})$$

Will replace Top by Cond(Set)

top spaces — condensed sets.

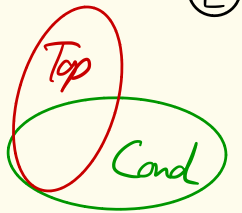
Slogan: Categorically like Set, phenomena like Top
(interacts well with algebra) (contains κ-comp. gen. spaces)

Remark: We will follow Clausen-Scholze's approach.
Barwick-Haine have a set-theoretically slightly different setup ("Pyknotic sets")

This will lead to an abelian category Cond(Ab) (condensed abelian groups)
with closed symmetric monoidal structure $\otimes$ $\left(\mathrm{Hom}(A, \underbrace{\underline{\mathrm{Hom}}}_{\text{Hom object}}(B,C)) = \mathrm{Hom}(A \otimes B, C) \right)$
and enough projectives containing those top. ab. gps
with underlying space κ-compactly generated for some κ.

This will help with the problems raised above:

E



A TAb not abelian: Cond(Ab) (abelian cat) is bigger and contains objects like $\operatorname{coker}(\mathbb{R}^{disc} \to \mathbb{R})$

B Pontryagin biduality not continuous: Cond(Ab) smaller, does not contain very pathological spaces like $\tilde{\mathbb{Z}}$

(other approach: Segal-Mitchison)

C Continuous gp coh. (also addressed by Segal) does not have l.e.s's: Replace $H^*_{cts}(G, M)$ by $\underline{\mathrm{Ext}}^*_{\mathbb{Z}[G]}(\mathbb{Z}, M)$ ← in condensed sets!

D Whitehead problem undecidable: Replace $\mathrm{Ext}^1(A, \mathbb{Z}) = 0$ by stronger condition $\underline{\mathrm{Ext}}^1(A, \mathbb{Z}) = 0$, (condensed!) prove this implies A free.

E No coherent 6 functor formalism: Embed $QC(X)$ into the larger solid (completeness condition) derived category.

F $\mathrm{Spa}(A, A^+)$ not always sheafy: Develop algebraic geometry in condensed setting ⇝ analytic spaces

G Cts actions on spectra: Use ∞-cat. of condensed spectra.

How do we define condensed sets?

Natural attempt (modulo set-theoretical technicalities)

Consider the category $\mathrm{Cond}(\mathrm{Set}) = \mathrm{Shv}(\mathrm{CHaus})$ of sheaves of sets on the category CHaus of compact Hausdorff spaces for the Grothendieck topology with $\mathrm{Cov}(Y) = \left\{ \begin{matrix} \text{finite jointly surjective families} \\ \{X_i \to Y\}_{i=1}^n \end{matrix} \right\}$,

i.e. presheaves s.t. for every $\{X_i \to Y\}_{i=1}^n \in \mathrm{Cov}(Y)$, the following is an equaliser:

$$F(X) \to \prod_i F(X_i) \rightrightarrows \prod_{i,j} F(X_i \times_Y X_j)$$

Observe: If X a top. space, $S \longmapsto \mathrm{Map}_{cts}(S, X)$ is a sheaf.

Can specify a condensed set by much less data.

<u>Defⁿ</u>: A compact Hausdorff space T is <u>projective</u> if for any epi $E \twoheadrightarrow X$ and any $f: T \to X$
"surjection"

there is a lift
$$\begin{tikzcd} & E \arrow[d, two heads] \\ T \arrow[ur, dashed] \arrow[r] & X \end{tikzcd}$$
.

<u>Gleason's theorem</u>: $T \in \mathrm{CHaus}$ is projective if and only if T is extremally disconnected, i.e. if the closure of opens is open

much stronger than totally disconnected!
= connected compt's are singletons

To see that CHaus has enough projectives, we use:

- Stone–Čech compactification: The forgetful functor $\mathrm{CHaus} \to \mathrm{Set}$ admits a left adjoint $\beta(S) = \{\text{Ultrafilters on } S\}$ + Stone topology.

Observe βS is always projective; we think of βS's as free.

Obtain inclusion of categories

$$\underset{\beta S\text{'s}}{\text{Free}} \subseteq \underset{\substack{\text{extremally}\\ \text{disconnected}}}{\text{Projective}} \subseteq \underset{\substack{\text{profinite sets}\\ =\text{totally disconnected sets}}}{\text{Stone}} \subseteq \text{CHaus}$$

Any sheaf on CHaus is determined by its restriction to these cats:

Warning: products of projectives need not be projective.

Given $X \in \text{CHaus}$, can consider hypercover

$$\beta(\beta X \times_X \beta X) \rightrightarrows \beta X \to X$$

to compute $FX \cong eq(F\beta X \rightrightarrows F(\beta(\beta X \times_X \beta X)))$

<u>Exercise</u>: Show that restriction defines an equivalence

$$\{\text{Sheaves on CHaus}\} \xrightarrow{\simeq} \left\{\begin{array}{l}\text{Functors}\\ F : (\text{extremally disc sets})^{op} \to \text{Set}\\ F(X \amalg Y) \simeq F(X) \times F(Y)\\ F(\emptyset) = *\end{array}\right\}$$

<u>Hint</u>: Any $S \twoheadrightarrow T$ of projectives splits

Def^n: A condensed set is a finite-product-preserving functor $X : \{\text{Extremally disconnected sets}\}^{op} \longrightarrow \text{Set}$ } not locally small yet!

$\circledast$ such that for some uncountable strong limit cardinal κ ($\lambda < \kappa \Rightarrow 2^\lambda < \kappa$), X is left Kan extended from $\{\kappa\text{-small extremally disconnected sets}\}$

Idea: Add sifted colimits to $\beta(S)$'s

Cond(Set) categorically behaves like Set:

(1) finite limits commute w. filtered colimits.

(2) sifted colimits commute w. products.

From Cond(Set) to Top: $X \rightsquigarrow X(*)$ with quotient topology $\coprod_{S \to T} X(S) \twoheadrightarrow X(*)$

S: κ-small extremally disconnected sets, where X is left Kan extended from κ-small ED sets

From Top to Cond(Set): $Y \rightsquigarrow \underline{Y} = \mathrm{Map}_{cts}(-, Y)$ always a sheaf, but not always left Kan extended as in $\circledast$ (Candidate)

But the assignment $Y \longmapsto \underline{Y}$ is well-defined + fully faithful on spaces which are κ-compactly generated for some κ.

Counterexamples: Sierpinski space, indiscrete space on ≥ 2 points

<u>Rk</u>: Barwick-Haine's setup contains these.