

CONDENSED ABELIAN GROUPS

Recap:

Fix κ an uncountable strong limit cardinal (i.e. for $\lambda < \kappa$, $2^\lambda < \kappa$).

Define PreFin_κ to be the category of κ -small profinite spaces (i.e. totally disconnected compact Hausdorff) with the Grothendieck topology with covers given by finite collections $(F_i: Y_i \rightarrow X)$ which are jointly surjective.

$$\text{Cond}_\kappa(\text{Set}) := \text{Shv}(\text{PreFin}_\kappa)$$

In more detail... a κ -condensed set is a functor $X: \text{PreFin}_\kappa^{\text{op}} \rightarrow \text{Set}$

such that

(i) $X(\emptyset) = *$ and $X(S_1 \sqcup S_2) = X(S_1) \times X(S_2)$;

(ii) For any surjection $S' \rightarrow S$ of κ -profinite sets with

fibre product $S' \times_S S'$ and projections p_1, p_2 to S' ,

$$X(S) \longrightarrow \{ x \in X(S') \mid p_1^*(x) = p_2^*(x) \in X(S' \times_S S') \}$$

is a bijection.

How can we think about condensed sets?

Think of $X(S)$ as encoding the continuous maps $S \rightarrow X$:

For $S = *$, this implies that $X(*)$ would give us the underlying set;

For $S = \mathbb{N} \cup \{\infty\}$, $X(S)$ would give us the collection of converging

sequences on $X(*)$; more general S detect different types of

convergence.

$$\mathbb{N} \cup \{\infty\} = \lim_n \{0, \dots, n\} \cup \{\infty\}$$

Topological space \rightarrow Condensed Sets

For X a topological space, define $\underline{X} \in \text{Cond}_\kappa(\text{Set})$ by

$$\underline{X}(S) := \text{Continuous}(S, X)$$

② Why is condition (ii) held?

Fix a surjection $s: S' \rightarrow S$ in ProFin_κ . Then

$$\{x \in \underline{X}(S) \mid p_1^*(x) = p_2^*(x) \in \underline{X}(S' \times_S S')\}$$

consists of $F \in \text{Cont}(S', X)$ s.t. $F(x) = F(y)$ if $s(x) = s(y)$.

Define $g: S \rightarrow X$ by $g(x) = F(y)$ for some $y \in s^{-1}(x)$.

The condition of F guarantees this is well-defined, but why would it be continuous? Any surjection between compact Hausdorff spaces is a quotient map so

$$S \rightarrow X \text{ is continuous IFF } S' \xrightarrow{s} S \rightarrow X \text{ is continuous.}$$

More generally...

Def: A κ -condensed group/ring/... is a sheaf of group/rings/... over the site ProFin_κ .

κ -Condensed Abelian Groups, $\text{Cond}_\kappa(\text{Ab})$:

Category of functors $X: \text{ProFin}_\kappa^{\text{op}} \rightarrow \text{Ab}$ such that

(i) $X(\emptyset) = *$ and $X(S_1 \amalg S_2) = X(S_1) \times X(S_2)$;

(ii) For any surjection $S' \rightarrow S$ of κ -profinite sets with

fibre product $S' \times_S S'$ and projections p_1, p_2 to S' ,

$$X(S) \rightarrow \{x \in X(S') \mid p_1^*(x) = p_2^*(x) \in X(S' \times_S S')\}$$

is a bijection.

! Categories of sheaves of abelian groups always form an abelian category.

☁ This is already much better than the category of topological abelian groups (TopAb):

In TopAb: $\mathbb{R}^{\text{disc}} \rightarrow \mathbb{R}$ has trivial kernel and cokernel, but it's not an isomorphism.

In Cond(Ab): $\mathbb{R}^{\text{disc}} \rightarrow \mathbb{R}$ has non-trivial cokernel Q :

Although $Q(*) = 0$, for general S

$$Q(S) = \frac{\{\text{continuous } S \rightarrow \mathbb{R}\}}{\{\text{locally constant } S \rightarrow \mathbb{R}\}} \text{ generally } \neq 0.$$

⚠ But Cond(Ab) behaves better than general sheaf categories of abelian groups.

Grothendieck's Axioms For Abelian Categories

→ (AB3) All colimits exist

→ (AB4) Arbitrary direct sums are exact

→ (AB5) Filtered colimits are exact

(AB6) For any index set J and filtered categories $(I_j)_{j \in J}$ with functors

$i \mapsto M_i$ from I_j to $\text{Cond}_{\mathbb{Z}}(\text{Ab})$,

$$\varinjlim_{j \in J} \prod_{i \in I_j} M_{i,j} \xrightarrow{\cong} \prod_{j \in J} \varinjlim_{i \in I_j} M_{i,j}.$$

• Axioms satisfied by Ab

→ (AB3*) All limits exist

(AB4*) Arbitrary products are exact

(AB5*) Filtered limits are exact

(AB6*) For any index set J and filtered categories $(I_j)_{j \in J}$ with functors

$i \mapsto M_i$ from I_j to $\text{Cond}_{\mathbb{Z}}(\text{Ab})$,

$$\sum_{j \in J} \varinjlim_{i \in I_j} M_{i,j} \xrightarrow{\cong} \varinjlim_{i \in I_j} \sum_{j \in J} M_{i,j}$$

• Axioms satisfied by Sheaves of Ab.

Theorem: $\text{Cond}_{\kappa}(\text{Ab})$ is an abelian category satisfying (AB3), (AB4), (AB5), (AB6),

(AB3*) and (AB4*). Moreover $\text{Cond}(\text{Ab})$ is generated by compact projective objects.

\exists a collection (G_i) of objects s.t. for any object X there exists an epimorphism $\bigoplus_i G_i \twoheadrightarrow X$.

$\text{Hom}(X, -)$ commutes with filtered colimits

$\text{Hom}(X, -)$ commutes with reflexive coequalizers

We need to check that $\text{Shv}(\text{ProFin}_{\kappa})$ satisfies all the properties above...

(?) Why ProFin_{κ} ?

Lemma: $\text{Shv}(\text{ProFin}_{\kappa}) \cong \text{Shv}(\text{CHaus})$

Lemma: $\text{Shv}(\text{ProFin}_{\kappa}) \cong \text{Shv}(\text{ED}_{\kappa})$

κ -small Extremely Disconnected CHaus = Projective objects in CHaus

Using this ... $X \in \text{Cond}(\text{Ab})$ is a functor $X: \text{ED}_{\kappa}^{\text{op}} \rightarrow \text{Ab}$ s.t.

(i) $X(\emptyset) = *$ and $X(S_1 \sqcup S_2) = X(S_1) \times X(S_2)$;

(ii) For any surjection $S' \rightarrow S$ of κ -prefinite sets with

fibre product $S' \times_S S'$ and projections p_1, p_2 to S' ,

$$X(S) \longrightarrow \{ z \in X(S') \mid p_1^*(z) = p_2^*(z) \in X(S' \times_S S') \}$$

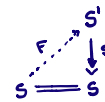
is a bijection.

CLAIM: condition (ii) is always immediately satisfied in this case.

This follows from the fact that ED_{κ} -spaces are the projective spaces in

CHaus_{κ} . Then any such a surjection $S' \rightarrow S$ splits. So the

maps $X(S)$ and $X(F)$ will automatically give such a bijection.



To sum up: $X \in \text{Cond}(\text{Ab})$ is a functor $X: \text{ED}_{\kappa}^{\text{op}} \rightarrow \text{Ab}$ s.t. $X(\emptyset) = *$ and

X takes finite disjoint union to products.

\mathbf{Caus}

\supset

\mathbf{PreFin}

\supset

\mathbf{ED}

More intuitive spaces

Hard sheaf condition to work with

Nicer sheaf condition

Slightly less intuitive spaces

Awesome sheaf condition

Horrible spaces to describe

Proof of Theorem: We can identify $\mathbf{Cond}_k(\mathbf{Ab})$ with functors $X: \mathbf{ED}_k^{\text{op}} \rightarrow \mathbf{Ab}$ taking

$X(\emptyset) = *$ and finite disjoint unions to products.

Recall that in \mathbf{Ab} all limits and colimits commute with finite products.

Then the category $\mathbf{Cond}(\mathbf{Ab})$ is stable under forming pointwise limits and colimits.

So for a functor $I \rightarrow \mathbf{Cond}(\mathbf{Ab})$ taking $i \mapsto M_i$, the (co)limit of the M_i is given on S by the (co)limit of $M_i(S)$.

Then Grothendieck's axioms follow directly from being satisfied in \mathbf{Ab} .

About the generators: by the adjoint functor thm, the forgetful functor $\mathbf{Cond}_k(\mathbf{Ab}) \rightarrow \mathbf{Cond}_k(\mathbf{Set})$ has a left adjoint $T \mapsto \mathbb{Z}[T]$.

We'll show the generators are the $\mathbb{Z}[\underline{S}]$ where S is an \mathbf{ED}_k set.

- Why are these compact and projective?

$\text{Hom}_{\mathbf{Cond}(\mathbf{Ab})}(\mathbb{Z}[\underline{S}], T) \cong \text{Hom}_{\mathbf{Cond}(\mathbf{Set})}(\underline{S}, T) \cong T(S)$. Since $T \mapsto T(S)$ commutes with all limits and colimits, $\text{Hom}(\mathbb{Z}[\underline{S}], -)$ does too $\Rightarrow \mathbb{Z}[\underline{S}]$ is compact and projective.

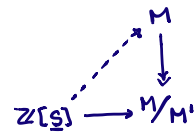
\hookrightarrow Yoneda

- $\mathbb{Z}[\underline{S}]$ generate:

For any condensed abelian group M , let M' be the maximal subobject which admits a surjection from a coproduct of $\mathbb{Z}[\underline{S}]$'s (such M' exists by Zorn's Lemma)

If $M/M' \neq 0$, there exists S in \mathbf{ED}_k s.t. $M/M'(S) \neq 0$. Since $M/M'(S) \cong \text{Hom}(\mathbb{Z}[\underline{S}], M/M')$

this implies there is a non-zero map $\mathbb{Z}[\underline{S}] \rightarrow M/M'$. Since $\mathbb{Z}[\underline{S}]$ is projective, this map admits a lift to M contradicting the maximality of M' .



■

