Condensed Abelian Groups

Recap:

Fix κ an uncountable strong limit cardinal (ie. for $\lambda<\kappa$, $2^\lambda<\kappa$).

Define ProFin_κ to be the category of κ-small profinite spaces (ie. totally disconnected compact Hausdorff) with the the Grothendieck topology with covers given by finite collections $(f_i : Y_i \to X)$ which are jointly surjective.

$$\mathrm{Cond}_\kappa(\mathrm{Set}) := \mathrm{Shv}(\mathrm{ProFin}_\kappa)$$

In more detail... a κ-condensed set is a functor $X : \mathrm{ProFin}_\kappa^{op} \to \mathrm{Set}$ such that (i) $X(\emptyset) = *$ and $X(S_1 \amalg S_2) = X(S_1) \times X(S_2)$;

(ii) For any surjection $S' \to S$ of κ-profinite sets with fibre product $S' \times_S S'$ and projections p_1, p_2 to S',

$$X(S) \longrightarrow \{ x \in X(S') \mid p_1^*(x) = p_2^*(x) \in X(S' \times_S S') \}$$

is a bijection.

How can we think about condensed sets?

Think of $X(S)$ as encoding the continuous maps $S \to X$:

For $S = *$, this implies that $X(*)$ would give us the underlying set;

For $\boxed{S = \mathbb{N} \cup \{\infty\}}$, $X(S)$ would give us the collection of converging sequences on $X(*)$; more general S detect different types of convergence.

$$\boxed{\mathbb{N} \cup \{\infty\} = \lim_n \{0, \dots, n\} \cup \{\infty\}}$$

Topological space $\longrightarrow$ Condensed Sets

For X a topological space, define $\underline{X} \in \mathrm{Cond}_\kappa(\mathrm{Set})$ by

$$\underline{X}(S) := \mathrm{Continuous}(S, X)$$

? Why is condition (ii) hold?

Fix a surjection $s: S' \longrightarrow S$ in ProFin_κ. Then

$$\{ x \in \underline{X}(S') \mid p_1^*(x) = p_2^*(x) \in \underline{X}(S' \times_S S') \}$$

consists of $F \in \mathrm{Cont}(S', X)$ s.t. $F(x) = F(y)$ if $s(x) = s(y)$.

Define $g: S \longrightarrow Y$ by $g(x) = F(y)$ for some $y \in s^{-1}(x)$.

The condition of F guarantees this is well-defined, but why would it be continuous? Any surjection between compact Hausdorff spaces is a quotient map so

$$S \longrightarrow X \text{ is continuous iff } S' \xrightarrow{s} S \longrightarrow X \text{ is continuous.}$$

More generally...

<u>Def:</u> A κ-condensed group/ring/... is a sheaf of group/rings/... over the site ProFin_κ.

κ-Condensed Abelian Groups, $\mathrm{Cond}_\kappa(\mathrm{Ab})$:

Category of functors $X: \mathrm{ProFin}_\kappa^{op} \longrightarrow \mathrm{Ab}$ such that

(i) $X(\emptyset) = *$ and $X(S_1 \sqcup S_2) = X(S_1) \times X(S_2)$;

(ii) For any surjection $S' \longrightarrow S$ of κ-profinite sets with fibre product $S' \times_S S'$ and projections p_1, p_2 to S',

$$X(S) \longrightarrow \{ x \in X(S') \mid p_1^*(x) = p_2^*(x) \in X(S' \times_S S') \}$$

is a bijection.

! Categories of sheaves of abelian groups always form an abelian category.

This is already much better than the category of topological abelian groups (TopAb):

In TopAb: $\mathbb{R}^{disc} \to \mathbb{R}$ has trivial kernel and cokernel, but it's not an isomorphism.

In Cond(Ab): $\underline{\mathbb{R}^{disc}} \longrightarrow \underline{\mathbb{R}}$ has non-trivial cokernel Q:

Although $Q(*) = 0$, for general S

$$Q(S) = \frac{\{\text{continuous } S \to \mathbb{R}\}}{\{\text{locally constant } S \to \mathbb{R}\}} \quad \text{generally } \neq 0.$$

! But Cond(Ab) behaves better than general sheaf categories of abelian groups.

Grothendieck's Axioms for Abelian Categories

→ (AB3) All colimits exist

→ (AB4) Arbitrary direct sums are exact

→ (AB5) Filtered colimits are exact

(AB6) For any index set J and filtered categories $(I_j)_{j\in J}$ with functors $i \mapsto M_i$ from I_j to $\mathrm{Cond}_\kappa(\mathrm{Ab})$,

$$\varinjlim_{i_j \in I_j} \prod_{j\in J} M_{i_j} \xrightarrow{\cong} \prod_{j\in J} \varinjlim_{i_j\in I_j} M_{i_j}.$$

→ (AB3*) All limits exist

(AB4*) Arbitrary products are exact

(AB5*) Filtered limits are exact

(AB6*) For any index set J and filtered categories $(I_j)_{j\in J}$ with functors $i \mapsto M_i$ from I_j to $\mathrm{Cond}_\kappa(\mathrm{Ab})$,

$$\sum_{j\in J} \varprojlim_{i_j\in I_j} M_{i_j} \xrightarrow{\cong} \varprojlim_{i_j\in I_j} \sum_{j\in J} M_{i_j}$$

Axioms satisfied by Ab

Axioms satisfied by Sheaves of Ab.

Theorem: $\mathrm{Cond}_\kappa(\mathrm{Ab})$ is an abelian category satisfying (AB3), (AB4), (AB5), (AB6), (AB3*) and (AB4*). Moreover $\mathrm{Cond}(\mathrm{Ab})$ is generated by compact projective objects.

generated: $\exists$ a collection (G_i) of objects s.t. for any object X there exists an epimorphism $\bigoplus_{I'} G_i \twoheadrightarrow X$.

compact: $\mathrm{Hom}(X,-)$ commutes with filtered colimits

projective: $\mathrm{Hom}(X,-)$ commutes with reflexive coequalizers

We need to check that $\mathrm{Shv}(\mathrm{ProFin}_\kappa)$ satisfies all the properties above...

? Why ProFin_κ?

Lemma: $\mathrm{Shv}(\mathrm{ProFin}_\kappa) \cong \mathrm{Shv}(\mathrm{CHaus})$

Lemma: $\mathrm{Shv}(\mathrm{ProFin}_\kappa) \cong \mathrm{Shv}(\underbrace{\mathrm{ED}_\kappa})$

κ-small Extremally Disconnected CHaus = Projective objects in CHaus

Using this ... $X \in \mathrm{Cond}(\mathrm{Ab})$ is a functor $X : \mathrm{ED}_\kappa^{op} \longrightarrow \mathrm{Ab}$ s.t.

(i) $X(\emptyset) = *$ and $X(S_1 \amalg S_2) = X(S_1) \times X(S_2)$;

(ii) For any surjection $S' \to S$ of κ-profinite sets with fibre product $S' \times_S S'$ and projections p_1, p_2 to S',

$$X(S) \longrightarrow \{ x \in X(S') \mid p_1^*(x) = p_2^*(x) \in X(S' \times_S S') \}$$

is a bijection.

CLAIM: condition (ii) is always immediately satisfied in this case.

This follows from the fact that ED_κ-spaces are the projective spaces in CHaus_κ. Then any such a surjection $S' \to S$ splits. So the maps $X(s)$ and $X(f)$ will automatically give such a bijection.

$$\begin{tikzcd} & S' \arrow[d, "s"] \\ S \arrow[ur, dashed, "f"] \arrow[r, equal] & S \end{tikzcd}$$

To sum up: $X \in \mathrm{Cond}(\mathrm{Ab})$ is a functor $X : \mathrm{ED}_\kappa^{op} \longrightarrow \mathrm{Ab}$ s.t. $X(\emptyset) = *$ and X takes finite disjoint union to products.

CHaus $\supset$	ProFin $\supset$	ED
More intuitive spaces	Nicer sheaf condition	Awesome sheaf condition
Hard sheaf condition to work with	Slightly less intuitive spaces	Horrible spaces to describe

<u>**Proof of Theorem:**</u> We can identify $\mathrm{Con}_\kappa(\mathrm{Ab})$ with functors $X : \mathrm{ED}_\kappa^{op} \to \mathrm{Ab}$ taking $X(\emptyset) = *$ and finite disjoint unions to products.

Recall that in Ab all limits and colimits commute with finite products.

Then the category $\mathrm{Cond}(\mathrm{Ab})$ is stable under forming pointwise limits and colimits.

So for a functor $I \to \mathrm{Cond}(\mathrm{Ab})$ taking $i \mapsto M_i$, the (co)limit of the M_i is given on S by the (co)limit of $M_i(S)$.

Then Grothendieck's axioms follow directly from being satisfied in Ab.

About the generators: by the adjoint functor thm, the forgetful functor $\mathrm{Cond}_\kappa(\mathrm{Ab}) \to \mathrm{Cond}_\kappa(\mathrm{Set})$ has a left adjoint $T \mapsto \mathbb{Z}[T]$.

We'll show the generators are the $\mathbb{Z}[\underline{S}]$ where S is an ED_κ set.

- Why are these compact and projective?

$\mathrm{Hom}_{\mathrm{Cond}(\mathrm{Ab})}(\mathbb{Z}[\underline{S}], T) \cong \mathrm{Hom}_{\mathrm{Cond}(\mathrm{Set})}(\underline{S}, T) \underset{\text{Yoneda}}{\cong} T(S)$. Since $T \mapsto T(S)$ commutes with all limits and colimits, $\mathrm{Hom}(\mathbb{Z}[\underline{S}], -)$ does too $\Rightarrow$ $\mathbb{Z}[\underline{S}]$ is compact and projective.

- $\mathbb{Z}[\underline{S}]$ generate:

For any condensed abelian group M, let M' be the maximal subobject which admits a surjection from a coproduct of $\mathbb{Z}[\underline{S}]$'s (such M' exists by Zorn's Lemma)

If $M/M' \neq 0$, there exists S in ED_κ s.t. $M/M'(S) \neq 0$. Since $M/M'(S) \cong \mathrm{Hom}(\mathbb{Z}[\underline{S}], M/M')$ this implies there is a non-zero map $\mathbb{Z}[\underline{S}] \to M/M'$. Since $\mathbb{Z}[\underline{S}]$ is projective, this map admits a lift to M contradicting the maximality of M'.

$$\begin{tikzcd} & M \arrow[d, two heads] \\ \mathbb{Z}[\underline{S}] \arrow[ur, dashed] \arrow[r] & M/M' \end{tikzcd}$$

$\blacksquare$