Cohomology

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Condensed Abelian Groups

Tensor Product

The category $Cond(Ab)$ of condensed abelian groups has a symmetric monoidal product $- \otimes -$. The object $M \otimes N$ is the sheafification of $S \mapsto M(S) \otimes N(S)$. Observe that for any condensed sets T_1, T_2, we have $\mathbb{Z}[T_1] \otimes \mathbb{Z}[T_1] \cong \mathbb{Z}[T_1 \times T_2]$. Also note that $\mathbb{Z}[T_1]$ is flat.

Enriched Hom's

Given M, N two condensed abelian groups, the assignement

$$S \mapsto Hom(\mathbb{Z}[S] \otimes M, N)$$

for every extremally disconnected space S is a condensed abelian group, which we denote by $\underline{Hom}(M, N)$. Further, we get an adjunction

$$Hom(P, \underline{Hom}(M, N)) \cong Hom(P \otimes M, N).$$

The Derived Category $D(Cond(Ab))$

Properties

1. As $Cond(Ab)$ is generated by compact projective objects, so is $D(Cond(Ab))$.
2. We can use projective resolutions to define $- \otimes^L -$ and $R\underline{Hom}(-,-)$.
3. There is still an adjunction

$$Hom(P, R\underline{Hom}(M, N)) \cong Hom(P \otimes^L M, N).$$

Remark

The category of condensed abelian groups has enough projectives, unlike the categories of sheaves we are used to, which usually have enough injectives, but not enough projectives.

Condensed sets for varying cardinals

Proposition

Given $\kappa' > \kappa$ uncountable strong limit cardinals, the inclusion $\mathrm{ED}_\kappa \to \mathrm{ED}_{\kappa'}$ induces a functor $\mathit{Cond}_{\kappa'} \to \mathit{Cond}_\kappa$, whose left Kan extension $T \mapsto T_{\kappa'}$ is fully faithful. Further, $T \mapsto T_{\kappa'}$ commutes with all $\mathit{cof}(\kappa)$-small limits.

Definition

The category of condensed sets, denoted *Cond*, is the filtered colimit of the categories of κ-condensed sets along the filtered poset of all uncountable strong limit cardinals κ.

Remark

By transfinite induction, one can define a sequence κ_α of strong limit cardinals of strictly increasing cofinality. The category of condensed sets can be identified with the corresponding filtered colimit: this shows that *Cond* has limits.

Topological Spaces vs Condensed Sets I

Caution

Let X be a topological space. The functor $\underline{X} := Cont(-, X) : \mathrm{ED} \to Set$ is not necessarily a condensed set. For instance, take X to be a two point set with exactly one closed point.

Proposition

Let X be a T_1 topological space, then $\underline{X}$ is a condensed set.

Theorem

The functor $X \mapsto \underline{X}$ induces an equivalence between the category of compact Hausdorff spaces and qcqs condensed sets.

Proof

Proof

Let X be a compact Hausdorff space. Clearly, $\underline{X}$ is quasicompact. Now, every compact Hausdorff space is the quotient of some profinite set S under a closed equivalence relation R. Thus, R is compact Hausdroff, and as $\underline{X} = \underline{S}/\underline{R}$, we find that $\underline{X}$ is quasiseparated.

Conversely, let T be a qcqs condensed set, and let $S \to T$ be a surjection from a profinite set. Then $R := S \times_T S \subset S \times S$ is a quasicompact sub-condensed set, and thus is a closed subset. Now, R defines a closed equivalence relation on S, so $X = S/R$ is a compact Hausdorff space, and by definition we have $\underline{X} \cong T$.

Remark

More generally, a compactly generated space X is weak Hausdorff if and only if $\underline{X}$ is quasiseparated.

Cohomology on Compact Hausdorff Spaces

Let S be a compact Hausdorff space. There are several classical ways of defining the cohomology of S:

1. The singular cohomology groups $H^i_{sing}(S, \mathbb{Z})$.
2. The Čech cohomology groups $H^i_{Cech}(S, \mathbb{Z})$.
3. The sheaf cohomology groups $H^i_{sheaf}(S, \mathbb{Z})$.

Comparison

It is well-known that there is a natural isomorphism

$$H^i_{Cech}(S, \mathbb{Z}) \to H^i_{sheaf}(S, \mathbb{Z}).$$

It is also known that if S is a finite CW complex, then they also agree with $H^i_{sing}(S, \mathbb{Z})$.

Čech Cohomology of Profinite Sets

Lemma

Let $S_j, j \in J$ be a cofiltered system of compact Hausdorff spaces. Then, the natural map

$$\text{colim}\, H^*_{Cech}(S_j, \mathbb{Z}) \to H^*_{Cech}(\lim S_j, \mathbb{Z})$$

is an isomorphism of graded rings.

Example

If S is a profinite set, then $H^0_{Cech}(S, \mathbb{Z}) = Cont(S, \mathbb{Z})$ and the higher groups vanish. However, $H^0_{sing}(S, \mathbb{Z}) = Set(S, \mathbb{Z})$.

Example

Let I be a set. Then for $i \geq 1$, we have:

$$H^i_{Cech}(\prod_I S^1, \mathbb{Z}) = \bigwedge^i (\bigoplus_I \mathbb{Z}).$$

Cohomology of Condensed Sets

Definition

Let S be a compact Hausdorff space, we denote by $H^i_{cond}(S, \mathbb{Z})$ the cohomology of $\mathbb{Z}$ (condensed abelian group) over S in the "site" of compact Hausdorff spaces with finite families of jointly surjective maps.

Explicitly, as $H^i_{cond}(S, \mathbb{Z}) \cong Ext^i_{Cond(Ab)}(\mathbb{Z}[S], \mathbb{Z})$, it is enough to find a projective resolution of $\mathbb{Z}[S]$. To this end, take a simplicial hypercover $S_\bullet \to S$ by **extremally disconnected** spaces, then

$$\ldots \to \mathbb{Z}[S_1] \to \mathbb{Z}[S_0]$$

is a projective resolution of $\mathbb{Z}[S]$. Now, the homology groups of the complex

$$0 \to \Gamma(S_0, \mathbb{Z}) \to \Gamma(S_1, \mathbb{Z}) \to \ldots$$

are the condensed cohomology groups $H^i(S, \mathbb{Z})$.

Condensed Cohomology vs Sheaf Cohomolgoy

Theorem

For every compact Hausdorff space S, there is a natural isomoprhism

$$H^i_{sheaf}(S, \mathbb{Z}) \cong H^i_{cond}(S, \mathbb{Z}).$$

Notation

We write $H^i(S, \mathbb{Z})$ for any of $H^i_{Cech}(S, \mathbb{Z})$, $H^i_{sheaf}(S, \mathbb{Z})$, or $H^i_{cond}(S, \mathbb{Z})$.

proof

First, assume that S is a profinite set, and write $S = \lim_j S^j$ a cofiltered limit of finite sets. By definition, $H^0_{cond}(S, \mathbb{Z}) = Cont(S, \mathbb{Z})$. Thus, it is enough to prove that $H^i_{cond}(S, \mathbb{Z}) = 0$ for $i > 0$.

Continuation of the proof

Pick a hypercover by extremally disconnected sets $S_\bullet \to S$, and for each S^j choose a hypercover by finite sets $S^j_\bullet \to S^j$ such that $S_\bullet = \lim_j S^j_\bullet$. As S^j is extremally disconnected, the following sequence is exact

$$0 \to \Gamma(S^j, \mathbb{Z}) \to \Gamma(S^j_0, \mathbb{Z}) \to \Gamma(S^j_1, \mathbb{Z}) \to \ldots.$$

Taking the filtered colimit over j shows the exactness of

$$0 \to \Gamma(S, \mathbb{Z}) \to \Gamma(S_0, \mathbb{Z}) \to \Gamma(S_1, \mathbb{Z}) \to \ldots.$$

Thus $H^i_{cond}(S, \mathbb{Z}) = 0$ for $i \geq 1$. Note that this implies that the condensed abelian groups $\mathbb{Z}[S]$ with S profinite are acyclic for $\Gamma_{cond}(-, \mathbb{Z})$.

A map of topoi

Now, let S be any compact Hausdorff space. There is a morphism of topoi

$$\alpha : \{\text{sheaves on } \mathsf{CH}/S\} \to \{\text{sheaves on } S\}$$

Explicitly, the right adjoint α_* is given on a sheaf $\mathscr{F}$ over CH/S by

$$\alpha_* \mathscr{F}(U) := \lim_{U \supseteq V \text{ closed}} \mathscr{F}(V \hookrightarrow S).$$

Observe that $\Gamma_{sheaf}(S, \alpha_*(-)) = \Gamma_{cond}(S, -)$. In particular, we have

$$H^*_{cond}(S, \mathbb{Z}) = H^*(R\Gamma_{cond}(S, \mathbb{Z})) = H^*(R\Gamma_{sheaf}(S, R\alpha_* \mathbb{Z})).$$

Thus, it is enough to prove that $R\alpha_* \mathbb{Z} \cong \mathbb{Z}$ in the derived category of abelian sheaves on S.

Back to the proof

Note that we have $H^0(R\alpha_*\mathbb{Z}) \cong \alpha_*\mathbb{Z}$, whence $\Gamma_{sheaf}(S, H^0(R\alpha_*\mathbb{Z})) \cong \Gamma_{cond}(S, \mathbb{Z})$, and there is a map of complexes of abelian sheaves

$$\mathbb{Z}[0] \to R\alpha_*\mathbb{Z}.$$

In order to prove that this is an isomorphism, it is enough to check on stalks. Fix $s \in S$, we have

$$(R\alpha_*\mathbb{Z})_s = \operatorname*{colim}_{s\in U \text{ open}} R\Gamma_{sheaf}(U, R\alpha_*\mathbb{Z}) = \operatorname*{colim}_{s\in V \text{ closed nbhd}} R\Gamma_{cond}(V, \mathbb{Z})$$

using the fact that compact Hausdorff spaces are normal. For each closed V, the pullback $S_\bullet \times_S V$ is a hypercover of V by profinite sets. By what we have proven already, applying $\Gamma_{cond}(-, \mathbb{Z})$ and taking homology gives us the condensed cohomology groups of V.

End of the proof

Passing to the colimit, we get isomorphisms of complexes of abelian groups

$$
\begin{aligned}
0 \to \operatorname*{colim}_{s \in V} \Gamma_{cond}(S_0 \times_S V, \mathbb{Z}) \to \operatorname*{colim}_{s \in V} \Gamma_{cond}(S_1 \times_S V, \mathbb{Z}) \to \dots \\
\cong 0 \to \Gamma_{cond}(S_0 \times_S \{s\}, \mathbb{Z}) \to \Gamma_{cond}(S_1 \times_S \{s\}, \mathbb{Z}) \to \dots \\
\cong R\Gamma_{cond}(\{s\}, \mathbb{Z}) \cong \mathbb{Z}.
\end{aligned}
$$

NB: The first isomorphism uses the fact that $\mathbb{Z}$ is discrete! In fact, this is the only property of the condensed abelian group $\mathbb{Z}$ that we have used.

Cohomology of $\mathbb{R}$

Theorem

For any compact Hausdorff space S, we have

$$H^i_{cond}(S, \mathbb{R}) = \begin{cases} Cont(S, \mathbb{R}), & \text{if } i = 0 \\ 0, & \text{if } i \geq 1. \end{cases}$$

proof

See the original notes by P. Scholze or a somewhat expanded version in the slides by F. Schremmer.