RHom and Locally Compact Abelian Groups

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1 The enriched Hom functor

Definition 1.1. The category $\mathrm{Cond}(\mathrm{Ab})$ is symmetric monoidal and closed, with the usual sheaf tensor product and internal hom

$$\underline{\mathrm{Hom}}(M,N)(S) = \mathrm{Hom}(M \otimes \mathbb{Z}[\underline{S}], N)$$

The above can be difficult to compute but the following theorem allows us to compute somewhat easily the enriched hom of many condensed abelian groups. A proof can be found in [Sch19].

Theorem 1.2. Let A, B be Hausdorff abelian groups with A compactly generated. Then,

$$\underline{\mathrm{Hom}}(\underline{A}, \underline{B}) = \underline{\mathrm{Hom}(A, B)}$$

where $\mathrm{Hom}(A, B)$ is given the compact-open topology.

Definition 1.3.

$$R\underline{\mathrm{Hom}}(-,-) : D(\mathrm{Cond}(\mathrm{Ab}))^op \times D(\mathrm{Cond}(\mathrm{Ab})) \to D(\mathrm{Cond}(\mathrm{Ab}))$$

is the derived functor of

$$\mathrm{Hom}^\bullet(A^\bullet, B^\bullet) = \mathrm{Tot}^{\Pi}(\{\underline{\mathrm{Hom}}(A^{-p}, B^q)\})$$

Remark 1.4. To make any computation, we will use projective resolutions since the category of condensed abelian groups does not have any non-zero injectives (this result is nowhere published to the best of my knowledge but a discussion by Scholze can be found in a Math Overflow thread).

We have the following usual properties of $R\underline{\mathrm{Hom}}$.

1. $R\underline{\mathrm{Hom}}(\bigoplus_I A_i, B) = \prod_I R\underline{\mathrm{Hom}}(A_i, B)$

2. $R\underline{\mathrm{Hom}}(A, \prod_J B_j) = \prod_J R\underline{\mathrm{Hom}}(A, B_j)$

We have the following result for abelian groups, proved in [Sch19] (Theorem 4.5).

Theorem 1.5. Let A be an abelian group. Then it has a free resolution of the form

$$\cdots \to \bigoplus_{j=1}^{n_i} \mathbb{Z}[A^{r_{i,j}}] \to \cdots \to \mathbb{Z}[A] \to A \to 0$$

that is functorial in A and the $r_{i,j}$ are independent of A.

We are going to obtain a similar result for condensed abelian groups.

Theorem 1.6. Let A be a condensed abelian group. Then it has a free resolution of the form

$$\cdots \to \bigoplus_{j=1}^{n_i} \mathbb{Z}[A^{r_{i,j}}] \to \cdots \to \mathbb{Z}[A] \to A \to 0$$

Proof. Let $Z[A]$ be the presheaf $Z[A](S) = \mathbb{Z}[A(S)]$. Note that section-wise we may construct a resolution for extremally disconnected S

$$\cdots \to \bigoplus_{j=1}^{n_i} Z[A^{r_{i,j}}](S) \to \cdots \to Z[A](S) \to A(S) \to 0$$

Functoriality on A ensure that this induces a resolution of presheafs

$$\cdots \to \bigoplus_{j=1}^{n_i} Z[A^{r_{i,j}}] \to \cdots \to Z[A] \to A \to 0$$

Sheafification is an exact functor so we may now sheafify and obtain the desired resolution. □

Remark 1.7. Note that this is not necessarily a projective resolution. $\mathbb{Z}[\underline{S}]$ is projective whenever S is extremally disconnected ([Sch19][Theorem 2.2]) but the functor of free condensed abelian groups does not always give projective objects.

2 RHom in Locally Compact Abelian Groups

Locally compact abelian groups (refered to as LCA groups from now on) have a particularly convenient structure that will allow us to somewhat easily compute $R\underline{\mathrm{Hom}}$.

Definition 2.1. Let A be a topological group. Then A is a *locally compact abelian group* if A is Hausdorff and every point of A is contained in a compact neighbourhood.

We have the following results that simplify the study of LCA groups. A proof can be found in [HS07].

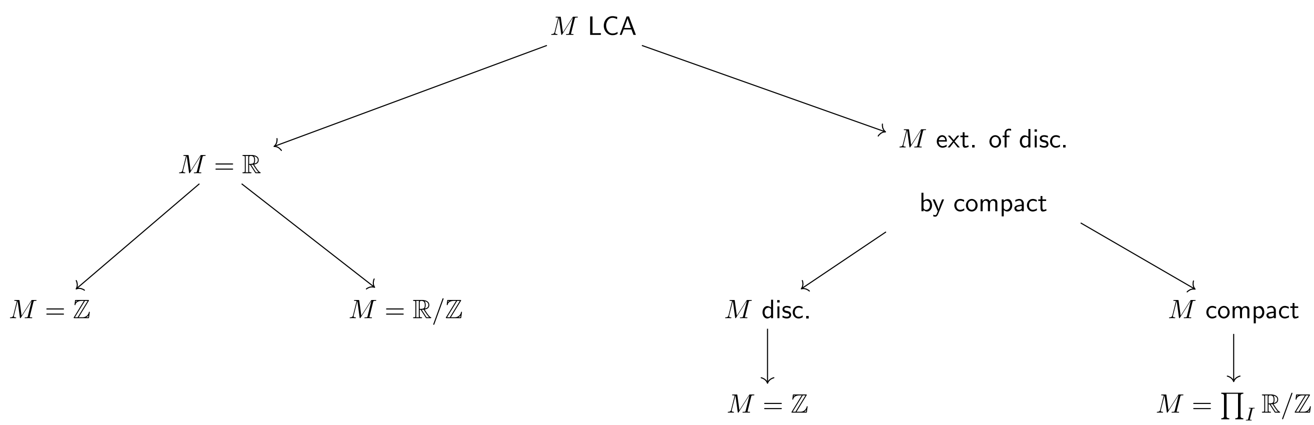
Theorem 2.2. Let A be a LCA group. Then

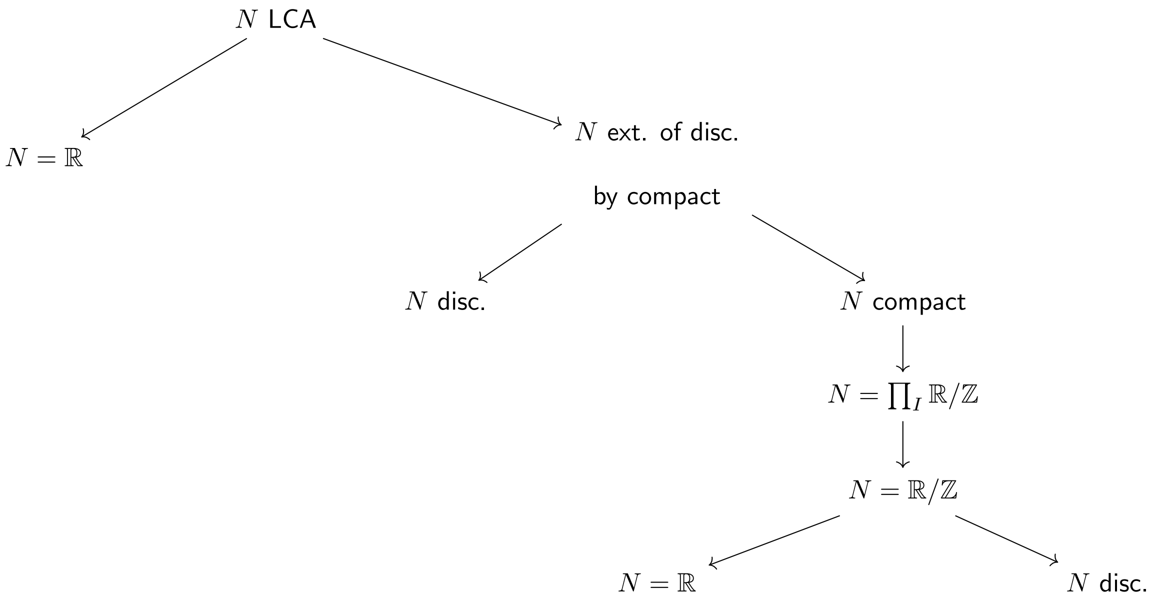
1. $A \cong \mathbb{R}^n \times A'$ where $n \geq 0$ and A' is an extension of a discrete group by a compact group.

2. We have the Pontryagin biduality functor $\hat{A} = \mathrm{Hom}(A, \mathbb{R}/\mathbb{Z})$, where Hom is given the compact-open topology and $\hat{\hat{A}} \cong A$.

3. If A is compact, $\hat{A}$ is discrete.

The following diagrams explain how one can simplify the computation of $R\underline{\mathrm{Hom}}(\underline{M}, \underline{N})$ for condensed locally compact abelian groups. All of the simplifications are done using cofibered sequences (induced by short exact sequences), the fact that finite product factor out of $R\underline{\mathrm{Hom}}$ and Pontryagin duality (the dual of a compact group is discrete so one can find a free resolution).





Hence, all of the derived mapping spaces $R\underline{\mathrm{Hom}}(\underline{M}, \underline{N})$ can be calculated by assuming $M = \mathbb{Z}$ or $M = \prod_I \mathbb{R}/\mathbb{Z}$ and $N = \mathbb{R}$ or N discrete.

This is dealt with by the following theorems.

Theorem 2.3. Let N be a condensed abelian group. Then,

$$R\underline{\mathrm{Hom}}(\underline{\mathbb{Z}}, N) = N[0]$$

Proof. Note that $\underline{\mathbb{Z}} = \mathbb{Z}[\underline{*}]$ (the locally constant sheaf $\mathbb{Z}$). It follows that $\underline{\mathbb{Z}}$ is projective so

$$R\underline{\mathrm{Hom}}(\underline{\mathbb{Z}}, N) = \underline{\mathrm{Hom}}(\underline{\mathbb{Z}}, N)$$

Evaluating at extremally disconnected S,

$$\underline{\mathrm{Hom}}(\underline{\mathbb{Z}}, N) = \mathrm{Hom}(\underline{\mathbb{Z}} \otimes \mathbb{Z}[\underline{S}], N) = \mathrm{Hom}(\mathbb{Z}[\underline{S}], N) = N(S)$$

so the result follows. □

The following is proved in [Sch19] and we do not include a detailed argument here.

Theorem 2.4. Let $M = \prod_I \mathbb{R}/\mathbb{Z}$ for some set I and let N be a discrete group. Then,

1. $R\underline{\mathrm{Hom}}(\underline{M}, \underline{N}) = \bigoplus_I N[-1]$

2. $R\underline{\mathrm{Hom}}(\underline{M}, \underline{\mathbb{R}}) = 0$

This matches with known results in the literature. In [HS07], a derived category D^b(LCA) of the quasi-abelian category of locally compact abelian groups.

Theorem 2.5. The functor $D^b(\mathsf{LCA}) \to D(\mathrm{Cond}(\mathrm{Ab}))$ is fully faithful.

Proof. It suffices to prove that

$$R\,\mathrm{Hom}_{\mathsf{LCA}}(A, B) \cong R\,\mathrm{Hom}_{\mathrm{Cond}(\mathrm{Ab})}(\underline{A}, \underline{B})$$

But comparing with the work in [HS07], we see that all the derived mapping spaces match. □

3 The Spectral Sequence Argument

In this section we outline an argument often used in [Sch19] to prove the previous results that can be useful to approach derived mapping spaces.

Example 3.1. Let M and N be abelian groups. Then by Theorem 1.6 one can find a resolution of the form

$$\cdots \to \bigoplus_{j=1}^{n_i} \mathbb{Z}[\underline{M}^{r_{i,j}}] \to \cdots \to \mathbb{Z}[\underline{M}] \to \underline{M} \to 0$$

We can study $R\underline{\mathrm{Hom}}(\underline{M}, \underline{N})$ section-wise noting that since $\underline{\mathrm{Hom}}(\underline{M}, \underline{N})(S) = \mathrm{Hom}(\underline{M} \otimes \mathbb{Z}[\underline{S}], \underline{N})$ we must have

$$\underline{\mathrm{Ext}}^i(\underline{M}, \underline{N})(S) = \mathrm{Ext}^i(\underline{M} \otimes \mathbb{Z}[\underline{S}], \underline{N})$$

We can now tensor the above resolution by the projective $\mathbb{Z}[\underline{S}]$ to obtain a resolution

$$\cdots \to \bigoplus_{j=1}^{n_i} \mathbb{Z}[\underline{M}^{r_{i,j}} \times \underline{S}] \to \cdots \to \mathbb{Z}[\underline{M} \times \underline{S}] \to \underline{M} \otimes \mathbb{Z}[\underline{S}] \to 0$$

Now consider a Cartan-Eilenberg resolution of this complex. To obtain $R\underline{\mathrm{Hom}}$, we apply the functor $\underline{\mathrm{Hom}}(-, \underline{N})$ and calculate the total complex. To calculate the cohomology of the total complex, one can use spectral sequences in the usual way, where the groups in the first iteration will be the vertical cohomology of the total complex. Since the columns in a Cartan-Eilenberg resolution are projective resolutions, one has

$$E_1^{pq} = \prod_{j=1}^{n_p} \mathrm{Ext}^i(\mathbb{Z}[\underline{M}^{r_{pj}} \times \underline{S}, \underline{N}) = \prod_{j=1}^{n_p} H^i_{\mathsf{cond}}(M^{r_{pj}} \times S, N) \implies \underline{\mathrm{Ext}}^1(\underline{M}, \underline{N})(S)$$

where H^i_{cond} is condensed cohomology.

Then, to prove equality between derived mapping spaces one can look at condensed cohomology, using known results there.

References

[HS07] Norbert Hoffmann and Markus Spitzweck, *Homological algebra with locally compact abelian groups*, Advances in Mathematics **212** (2007), no. 2, 504 – 524.

[Sch19] Peter Scholze, *Lectures on condensed mathematics*, `https://www.math.uni-bonn.de/people/scholze/Condensed.pdf`, May 2019.