Solid Modules

1. Motivation (Part I)

The theory of solid modules does not clearly represent a classical object / construction. But at least the motivation behind it does! $\rightsquigarrow$ completion

I. Cond(Ab) $\rightsquigarrow$ Excellent Categorical Properties
$\rightsquigarrow$ Computation of Hom, Ext with "good" results

But things break when forming tensor products
What cond. ab. gp structure shall we equip $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ with?

Note: This is a defect already present in the classical theory, and part of the motivation for considering completions of tensor products of topological rings.

II. Eye towards analytic geometry
$\rightsquigarrow$ Need to consider rings of conv. power series

Ex $A = \mathbb{Z}[\mathbb{N} \cup \{\infty\}]/([\infty]=0)$. The underlying ring of A is equipped with an element $T \in A$ with $T, T^2, \dots, T^n, \dots \longrightarrow 0$ $\rightsquigarrow$ polynomial algebra $\mathbb{Z}[T]$

As we are used to by now, whenever such difficulties occur, there exists a construction with excellent categorical properties to resolve them.

Motivation (Part II) $\rightsquigarrow$ structure of free cond. ab. gp.

Proposition: Let $S = \varprojlim_i \overset{\text{finite}}{S_i}$ be a profinite set. $\forall n$, let $\mathbb{Z}[S_i]_{\leq n} \subset \mathbb{Z}[S_i]$ consist of the formal sums $\sum_{n \in S_i} n_s [s]$ s.t. $\sum |n_s| \leq n$.

Note: $\mathbb{Z}[S_i]_{\leq n}$ is a finite set, the transition maps $\mathbb{Z}[S_i] \longrightarrow \mathbb{Z}[S_j]$ preserve these subsets.

There is a natural isomorphism of cond. abelian groups

$$\mathbb{Z}[S] \cong \bigcup_n \varprojlim_i \mathbb{Z}[S_i]_{\leq n} \subset \varprojlim_i \mathbb{Z}[S_i]$$

Remark: $\mathbb{Z}[S]$ is a countable union of profinite sets. $\bigcup_n \varprojlim \mathbb{Z}[S_i]_{\leq n}$ is a subgroup of $\varprojlim_i \mathbb{Z}[S_i]$

$S = \varprojlim S_i$

$\rightarrow \varprojlim \mathbb{Z}[S_i]$

Proof Sketch: Step 1: Show that $\mathbb{Z}[S] \longrightarrow \varprojlim_i \mathbb{Z}[S_i]$ is an injection.

Step 2: Show it factors through a surjection $\mathbb{Z}[S] \longrightarrow \bigcup_n \varprojlim_i \mathbb{Z}[S_i]_{\leq n}$

- The map of underlying abelian groups is injective.

For any finite formal sum $\sum_{j=1}^{k} n_j [s_j]$ with $s_j \in S$, $n_j \neq 0$, there exists some S_i where the images of the s_j are all distinct.

Then, $\sum_{j=1}^{k} n_j [s_j]$ projects to a nonzero element of $\mathbb{Z}[S_i]$.

- Now assume $\underset{\mathbb{Z}[C(T,S)]}{\overset{f}{\cup\!\!\!\!\,\shortmid}} \longmapsto 0 \in \varprojlim_i \mathbb{Z}[S_i](T)$

Then $\forall t \in T$, $f(t)$ in $\mathbb{Z}[S]$ is 0 by inj. of underlying ab. gps.

By the sheaf condition, It suffices to show that the preimage of f vanishes on $\mathbb{Z}[C(T_m, S)]$ for some cover of T by profinite sets T_m.

$f = \sum_{j=1}^{k} n_j [g_j]$, $g_j : T \to S$ distinct, continuous.

Choose $T_{jj'} \subset T$ be the closed subset of T where $g_j = g_{j'}$.

It is a cover, otherwise for some $t \in T$, the $g_j(t)$'s are pairwise distinct

$\Rightarrow \sum n_j [g_j(t)] \in \mathbb{Z}[S]$ non-trivial.

Now, by passing to the cover, we reduce the number of the g_j's

$\Rightarrow$ proceed by induction.

Note: There is a map $S = \varprojlim_i S_i \longrightarrow \bigcup_n \varprojlim_i \mathbb{Z}[S_i]_{\leq n}$,

from which we get a map $\mathbb{Z}[S] \longrightarrow \bigcup_n \varprojlim_i \mathbb{Z}[S_i]_{\leq n}$.

Suffices to show it is a surjection.

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2. Definition

We now have a canditate for how "completion" should behave on the free cond. ab. groups $\mathbb{Z}[S]$'s:

$$\underset{\text{free solid abelian group}}{\mathbb{Z}[S]^{\blacksquare}} := \lim_{\underset{i}{\leftarrow}} \mathbb{Z}[S_i]$$

This is already promising since it addresses part of the motivation:

looking at underlying space

$$\mathbb{Z}[\mathbb{N}\cup\{\infty\}]^{\blacksquare}/([\infty]=0) = \lim_{\underset{n}{\leftarrow}} \mathbb{Z}[\{0,1,\dots,n-1,\infty\}]/([\infty]=0) =$$

$$= \lim_{\underset{n}{\leftarrow}} \mathbb{Z}[T]/T^n = \mathbb{Z}[[T]] \rightsquigarrow \text{power series algebra.}$$

We will now, counterintuitively, use the desired universal property of the $\mathbb{Z}[S]^{\blacksquare}$ to define solid abelian groups in general:

Definition: (i) A solid abelian group is a condensed abelian group A such that for all profinite sets S and all maps $f: S \to A$, there is a unique map

$$\tilde{f}: \mathbb{Z}[S]^{\blacksquare} \longrightarrow A$$

extending f.

3. Free solid abelian groups are solid abelian groups

Note: $\mathbb{Z}[S]^{\blacksquare} = \varprojlim_i \underbrace{\mathbb{Z}[S_i]}_{\cong\, \bigoplus_{|S_i|} \mathbb{Z}} = \varprojlim_i \underbrace{\underline{\mathrm{Hom}}(C(S_i,\mathbb{Z}),\mathbb{Z})}_{\cong\, \prod_{|S_i|} \mathbb{Z}} =$

$= \underline{\mathrm{Hom}}(C(S,\mathbb{Z}),\mathbb{Z})$

The underlying abelian group of $\mathbb{Z}[S]^{\blacksquare}$ is the space of all $\mathbb{Z}$-valued measures on the profinite set S:

$$\mathcal{M}(S,\mathbb{Z}) := \mathrm{Hom}(C(S,\mathbb{Z}),\mathbb{Z})$$

Remark: For a solid abelian group A, $f: S \to A$, $\mu \in \mathcal{M}(S,\mathbb{Z})$, we can define

$$\int f\mu \in A$$

by evaluating $\tilde{f}: \mathbb{Z}[S]^{\blacksquare} \longrightarrow A$. (careful with interpr.)

Before saying anything meaningful about $\mathbb{Z}[S]^{\blacksquare}$, we thus need to first control $C(S,\mathbb{Z})$.

Theorem (Nöbeling, Specker)
For any profinite set S, the abelian group $C(S,\mathbb{Z})$ of continuous maps from S to $\mathbb{Z}$ is a free abelian group.

$\rightsquigarrow$ obvious for finite sets

$\rightsquigarrow$ quite surprising that $C(S,\mathbb{Z}) \cong \bigoplus_I \mathbb{Z}$, with $|I| \leq 2^{|S|}$

Corollary: $\mathbb{Z}[S]^{\blacksquare} \cong \prod_I \mathbb{Z}$.

Proof: $\mathbb{Z}[S]^{\blacksquare} = \underline{\mathrm{Hom}}(C(S,\mathbb{Z}),\mathbb{Z}) \cong \underline{\mathrm{Hom}}(\bigoplus_I \mathbb{Z},\mathbb{Z}) \cong \prod_I \mathbb{Z}$

("same" behavior as finite sets)

Proposition: For any profinite set S, $\mathbb{Z}[S]^{\blacksquare}$ is solid.

<u>Proof</u> We will prove sth stronger, namely for all profinite sets T, we have

$$R\mathrm{Hom}(\mathbb{Z}[T], \mathbb{Z}[S]^{\blacksquare}) = R\mathrm{Hom}(\mathbb{Z}[T]^{\blacksquare}, \mathbb{Z}[S]^{\blacksquare}).$$

- $\mathbb{Z}[S]^{\blacksquare} \cong \prod_I \mathbb{Z}$ for some I, so suffices to show

$$R\mathrm{Hom}(\mathbb{Z}[T],\mathbb{Z}) = R\mathrm{Hom}(\mathbb{Z}[T]^{\blacksquare},\mathbb{Z})$$

LHS: $\mathrm{Ext}^i(\mathbb{Z}[T],\mathbb{Z}) = H^i(T,\mathbb{Z}) = \begin{cases} 0 & \text{for } i>0 \\ C(T,\mathbb{Z}) \cong \bigoplus_J \mathbb{Z} & \text{for } i=0 \end{cases}$

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RHS: We will use the computations from Rodrigo's talk:

$$R\mathrm{Hom}(\prod_J \mathbb{Z}, \mathbb{Z}) = \bigoplus_J \mathbb{Z} = C(S,\mathbb{Z})$$

Starting with the short exact sequence

$$0 \longrightarrow \prod_J \mathbb{Z} \longrightarrow \prod_J \mathbb{R} \longrightarrow \prod_J \mathbb{R}/\mathbb{Z} \longrightarrow 0$$

we note:

- $R\mathrm{Hom}(\prod_J \mathbb{R}/\mathbb{Z}, \mathbb{Z}) = \bigoplus_J \mathbb{Z}[-1]$
- $R\mathrm{Hom}(\prod_J \mathbb{R}, \mathbb{Z}) = 0$

Recall that we had seen $R\underline{\mathrm{Hom}}(\mathbb{R},\mathbb{Z}) = 0 \Rightarrow$
$\Rightarrow R\mathrm{Hom}(\mathbb{R}\otimes^L M, \mathbb{Z}) = 0 \quad \forall$ cond. ab. gps M.
Take $M = \prod_J \mathbb{R} \rightsquigarrow \mathbb{R}$-module in $\mathrm{Cond}(\mathrm{Ab})$

In particular $\prod_J \mathbb{R}$ is a retract of $\mathbb{R}\otimes^L \prod_J \mathbb{R}$ and the conclusion follows.

$$\Longrightarrow R\mathrm{Hom}(\prod_J \mathbb{Z}, \mathbb{Z}) = \bigoplus_J \mathbb{Z} = C(S,\mathbb{Z}) \quad \blacksquare$$

4. Main Theorem

The category $\mathrm{Solid} \subset \mathrm{Cond(Ab)}$ of solid abelian groups is an abelian subcategory stable under all limits, colimits and extensions.

The objects $\prod_I \mathbb{Z} \in \mathrm{Solid}$, where I is any set, form a family of compact projective generators.

The inclusion $\mathrm{Solid} \subset \mathrm{Cond(Ab)}$ admits a left adjoint

$$M \longmapsto M^{\blacksquare} : \mathrm{Cond(Ab)} \longrightarrow \mathrm{Solid}$$

that is the unique colimit-preserving extension of $\mathbb{Z}[S] \longrightarrow \mathbb{Z}[S]^{\blacksquare}$.

Proof strategy:
- Isolate key categorical property (Main Lemma)
- Show how main Lemma applies in our setup

5. Main Lemma

Let A be an abelian category with all colimits
$\rightsquigarrow$ to be identified later with $\mathrm{Cond}(\mathrm{Ab})$
that admits a subcategory A^{cp} of comp. proj. gen.
Assume that $F: A^{cp} \to A$ is a functor
$\rightsquigarrow$ to be identified later with $\mathbb{Z}[S] \mapsto \mathbb{Z}[S]^{\blacksquare}$
equipped with a natural transformation $X \to F(X)$
$\rightsquigarrow$ injection $\mathbb{Z}[S] \longrightarrow \mathbb{Z}[S]^{\blacksquare}$
such that:

<u>Condition</u>: For any $X \in A^{cp}$, any $Y = \bigoplus_i F(P_i)$, $Z = \bigoplus_j F(Q_j)$ and any $f: Y \to Z$ with kernel $K \in A$, the map

$$R\mathrm{Hom}(F(X), K) \longrightarrow R\mathrm{Hom}(X, K)$$

is an isomorphism.

Let $A_F \subset A$ be the full subcategory of all $Y \in A$ s.t. for all $X \in A^{cp}$, the map

$$\mathrm{Hom}(F(X), Y) \longrightarrow \mathrm{Hom}(X, Y)$$

is an isomorphism $\rightsquigarrow$ to be identified with Solid.

Then, $A_F \subset A$ is an abelian subcategory stable under all limits, colimits and extensions, and the objects $F(X)$, $X \in A^{cp}$ are compact proj. generators. The inclusion $A_F \subset A$ admits a left adjoint $L: A \to A_F$ that is the unique colimit-preserving extension of $F: A^{cp} \to A_F$.

Main idea behind proof

Defining condition on $\mathcal{A}_F \rightsquigarrow$ stable under kernels and all limits.

$$\begin{tikzcd} C \arrow[r] \arrow[dr] & Y \\ & F(C) \arrow[u, dashed, "\exists!"'] \end{tikzcd}$$

For cokernels: Let $f: Y \to Z$ in $\mathcal{A}_F$. WTS $\operatorname{coker} f \in \mathcal{A}_F$.

Choose $\bigoplus_i P_i \twoheadrightarrow Z$ by compact projectives

By def of $\mathcal{A}_F$ this extends to $\bigoplus_i F(P_i) \twoheadrightarrow Z$

$$\begin{tikzcd} \bigoplus_j F(Q_j) \arrow[r, two heads] & Z^* \arrow[r] \arrow[d] \arrow[d, phantom, "\in \mathcal{A}_F"', shift right=4] & \bigoplus_i F(P_i) \arrow[d] \\ & Y \arrow[r] & Z \end{tikzcd}$$

We can thus assume that Y, Z are sums of objects in the image of F.

$$0 \longrightarrow K \longrightarrow Y \longrightarrow Z \longrightarrow Q \longrightarrow 0$$

$R\mathrm{Hom}(F(X), K) = R\mathrm{Hom}(X, K)$

$R\mathrm{Hom}(F(X), Y) = R\mathrm{Hom}(X, Y)$ $\quad \Rightarrow \quad R\mathrm{Hom}(F(X), Q) = R\mathrm{Hom}(X, Q)$

$R\mathrm{Hom}(F(X), Z) = R\mathrm{Hom}(X, Z)$

We showed that every cokernel of a map between direct sums of objects in the image of F lies in A_F.

Conversely, let $Y \in A_F$.

$$\begin{tikzcd}
& \bigoplus_j Q_j \arrow[dl] \arrow[dd] & \bigoplus_i P_i \arrow[r, two heads] \arrow[dd] & Y \in A_F \\
\bigoplus_j F(Q_j) \arrow[dr, two heads] & & & \\
& K \in A_F \arrow[r] & \bigoplus_i F(P_i) \in A_F \arrow[uur] &
\end{tikzcd}$$

$\Rightarrow$ Y can be written as $\operatorname{coker}\left(\bigoplus_j F(Q_j) \longrightarrow \bigoplus_i F(P_i)\right)$

$\Rightarrow$ A_F is stable under direct sums and under extensions.

The theorem follows formally after getting this description of A_F.

implies previous condition by taking a resolution of K by comp. proj.

Note: We can replace the condition of the lemma with the following:

For all $X \in A^{cp}$ and any complex

$$C: \dots \to C_i \to \dots \to C_1 \to C_0 \to 0,$$

where all C_i are direct sums of objects in the image of F,

$$R\mathrm{Hom}(F(X), C) \longrightarrow R\mathrm{Hom}(X, C)$$

is iso.

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6. "Proof" of Main Theorem

We apply the Lemma above to $\mathcal{A} = \mathrm{Cond}(\mathrm{Ab})$,
$\mathcal{A}^{cp}$ the subcategory of $\mathbb{Z}[S]$ for S extremally disconnected,
$F : \mathcal{A}^{cp} \longrightarrow \mathcal{A}$ taking $\mathbb{Z}[S] \longmapsto \mathbb{Z}[S]^{■}$.

Why is F a functor? $\mathbb{Z}[S]^{■} = \underline{\mathrm{Hom}}(C(S,\mathbb{Z}),\mathbb{Z}) =$
$= \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(\mathbb{Z}[S],\mathbb{Z}),\mathbb{Z})$

functorial in $\mathbb{Z}[S]$

Our goal is to show

$$R\mathrm{Hom}(\mathbb{Z}[S]^{■}, C) = R\mathrm{Hom}(\mathbb{Z}[S], C)$$

$C_j = \bigoplus_i \mathbb{Z}[T_i]^{■}$, T_i extremally disconnected

In fact more is true:

$$R\underline{\mathrm{Hom}}(\mathbb{Z}[S]^{■}, C)(S') = R\mathrm{Hom}(\mathbb{Z}[S\times S'], C)$$

$C_j = \bigoplus_I \prod_J \mathbb{Z}$, S, S' profinite.

The key ingredient of the proof is to use the exact seq.

$$0 \longrightarrow \mathcal{M}(S,\mathbb{Z}) \longrightarrow \mathcal{M}(S,\mathbb{R}) \longrightarrow \mathcal{M}(S,\mathbb{R}/\mathbb{Z}) \longrightarrow 0$$

$$\mathbb{Z}[S]^{■} \rightsquigarrow \mathcal{M}(S,\mathbb{Z}) \cong \underline{\mathrm{Hom}}(C(S,\mathbb{Z}),\mathbb{Z}), \qquad \mathcal{M}(S,\mathbb{R}) \cong \underline{\mathrm{Hom}}(C(S,\mathbb{Z}),\mathbb{R})$$

$\mathcal{M}(S,\mathbb{R}) \simeq \prod_I \mathbb{R}$, $\mathcal{M}(S,\mathbb{R}/\mathbb{Z}) \cong \prod_I \mathbb{R}/\mathbb{Z}$

then show that

$$R\underline{\mathrm{Hom}}(\mathcal{M}(S,\mathbb{R}/\mathbb{Z}), C)(S') = R\mathrm{Hom}(\mathbb{Z}[S\times S'], C)[-1]$$

for all profinite sets S, S'.

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Remark / Warning on $\mathbb{R}^{\blacksquare}$

As a condensed abelian group, $\mathbb{R}$ is pseudocoherent. Recall this is characterized by the following equiv. conditions (for p.c. M)

(1) $\mathrm{Ext}^i(M,-)$ commutes with filtered colimits for all i.

(2) M admits a proj. resolution by compact projective condensed abelian groups.

(pseudocoherence of $\mathbb{R}$ follows from $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$)

We have seen that

$$\mathbb{Z}[S]^{\blacksquare} = \underline{\mathrm{Hom}}(C(S,\mathbb{Z}),\mathbb{Z}) \simeq \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(\mathbb{Z}[S],\mathbb{Z}),\mathbb{Z})$$

Thus, (2) implies that for all pseudocoherent cond. ab. groups M,

$$M^{\blacksquare} = \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(M,\mathbb{Z}),\mathbb{Z}).$$

$\Rightarrow \mathbb{R}^{\blacksquare} = 0.$

$\rightsquigarrow$ motivating input for liquid theory?

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7. Completed tensor product

Theorem: There is a unique way to endow Solid with a symmetric monoidal tensor product $\otimes^{\blacksquare}$ such that the solidification functor

$$M \longmapsto M^{\blacksquare} : \mathrm{Cond}(\mathrm{Ab}) \longrightarrow \mathrm{Solid}$$

is symmetric monoidal.

Proof

symmetric monoidal => uniqueness since our only possible choice is $M \otimes^{\blacksquare} N = (M \otimes N)^{\blacksquare}$

This choice works: WTS $(M \otimes N)^{\blacksquare} \longrightarrow (M^{\blacksquare} \otimes N^{\blacksquare})^{\blacksquare}$

$$(M \otimes N)^{\blacksquare} \longrightarrow (M^{\blacksquare} \otimes N)^{\blacksquare} \longrightarrow (M^{\blacksquare} \otimes N^{\blacksquare})^{\blacksquare}$$

Assume $M = \mathbb{Z}[S]$, $N = \mathbb{Z}[T]$ $\rightsquigarrow$ all functors here preserve colimits

WTS $\mathbb{Z}[T \times S]^{\blacksquare} \longrightarrow (\mathbb{Z}[S]^{\blacksquare} \otimes \mathbb{Z}[T])^{\blacksquare}$ is iso

But we know

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(\mathbb{Z}[S]^{\blacksquare}, \overset{\rightsquigarrow\ \text{solid}}{A})(T) & \xrightarrow{\simeq} & \underline{\mathrm{Hom}}(\mathbb{Z}[S], A)(T) \\ \wr\Vert & & \wr\Vert \\ \mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare} \otimes \mathbb{Z}[T], A) & & \mathrm{Hom}(\mathbb{Z}[S \times T], A) \end{array}$$

8. Examples

To do any computations, note that $\otimes^{\blacksquare}$ commutes with filtered colimits in both variables (solidification is colimit preserving and symmetric monoidal)

After resolving with compact projectives, we need to control $\prod_I \mathbb{Z} \otimes^{\blacksquare} \prod_J \mathbb{Z}$.

Prop: $\prod_I \mathbb{Z} \otimes^{\blacksquare} \prod_J \mathbb{Z} \cong \prod_{I\times J} \mathbb{Z}$

Proof: $\prod_I \mathbb{Z} \otimes^{\blacksquare} \prod_J \mathbb{Z} = \underline{\mathrm{Hom}}(\bigoplus_I \mathbb{Z}, \mathbb{Z}) \otimes^{\blacksquare} \underline{\mathrm{Hom}}(\bigoplus_J \mathbb{Z}, \mathbb{Z}) \cong$

$\cong \underline{\mathrm{Hom}}(C(S,\mathbb{Z}), \mathbb{Z}) \otimes^{\blacksquare} \underline{\mathrm{Hom}}(C(T,\mathbb{Z}), \mathbb{Z}) =$

$= \mathbb{Z}[S]^{\blacksquare} \otimes^{\blacksquare} \mathbb{Z}[T]^{\blacksquare} = \mathbb{Z}[S\times T]^{\blacksquare} \cong \prod_{I\times J} \mathbb{Z}$

Examples

$\mathbb{Z}_p^{\blacksquare} \otimes^{\blacksquare} \mathbb{R}^{\blacksquare}$?

$\mathbb{Z}_p \otimes^{\blacksquare} \mathbb{R} = 0$ $\qquad 0 \to \mathbb{Z}[[T]] \xrightarrow{T-p} \mathbb{Z}[[T]] \to \mathbb{Z}_p \to 0$

$\mathbb{Z}[[U]] \otimes^{\blacksquare} \mathbb{Z}[[T]] = \mathbb{Z}[[U,T]]$

$\mathbb{Z}_p \otimes^{\blacksquare} \mathbb{Z}[[T]] = \mathbb{Z}_p[[T]]$

$\mathbb{Z}_p \otimes^{\blacksquare} \mathbb{Z}_p = \mathbb{Z}_p$

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