Condensed Functional Analysis

<u>Defn</u> A topological vector space $V \in \mathrm{TVect}$ is called Banach if there is a continuous function $\|\_\|: V \longrightarrow \mathbb{R}_{\geq 0}$ satisfying:

① $\|v\| = 0 \Leftrightarrow v = 0$ ② $\|av\| = |a|\,\|v\|$ for $a \in \mathbb{R}$ ③ $\|v+w\| \leq \|v\| + \|w\|$

④ $\{v \in V \mid \|v\| < \varepsilon\}$ form a neighbourhood basis around 0

⑤ Any Cauchy sequence v_n $(\|v_n - v_m\| \longrightarrow 0)$ converges $(\|v_n - w\| \to 0)$

<u>Key-Example:</u> For $1 \leq p \leq \infty$ $\ell^p = \{(x_1, x_2, x_3, \ldots) \in \mathbb{R}^{\mathbb{N}} \mid \sum |x_i|^p < \infty\}$
with the norm $\|x\|_p = (\sum |x_i|^p)^{1/p}$.

<u>Note:</u> this is not Banach for $0 < p < 1$ because $\|\_\|_p$ fails ③.

<u>Observation:</u> The functor $\mathrm{Banach} \subset \mathrm{TVect} \longrightarrow \mathrm{Cond}(\mathbb{R})$ is fully faithful

$$V \longmapsto \underline{V}(S) = C(S, V)$$

because V is Hausdorff and compactly generated.

To understand the condensed vector space $\underline{V}$ we need to study $S \to V$.

Defn $V \in \mathrm{Vect}$ then $K \subset V$ is called <u>absolutely convex</u> if:
$\forall x, y \in K \ \forall a, b \in \mathbb{R} : \ |a| + |b| \le 1 \implies ax + by \in K$.

Proposition For S profinite and V Banach any continuous $f : S \to V$ factors through an absolutely convex and compact $K \subset V$.

Idea: take the absolutely convex hull of $f(S)$: $\{ \sum_{i=1}^{n} a_i f(s_i) \mid \sum |a_i| \le 1 \}$
and then let K be the topological closure.

In fact this suggests something better:

$$\begin{tikzcd}
S \arrow[d, hook] \arrow[rd, "f"] & \\
\mathbb{R}[S]_{\le 1} \arrow[r] \arrow[d, hook] & V \\
\mathcal{M}(S)_{\le 1} \arrow[ru, dashed, "\exists !"'] &
\end{tikzcd}$$

Defn Let $\mathbb{R}[S]_{\le c} = \{ \sum a_s [s] \in \bigoplus_S \mathbb{R} \mid \sum |a_i| \le c \}$

$S = \varprojlim S_i \qquad \mathcal{M}(S)_{\le c} := \varprojlim \mathbb{R}[S_i]_{\le c}$

and $\mathcal{M}(S) := \bigcup_{c > 0} \mathcal{M}(S)_{\le c} \in \mathrm{Cond}(\mathbb{R})$

Proposition S profinite, V Banach then any continuous $f : S \to V$ extends uniquely to $\tilde{f} : \underbrace{\mathcal{M}(S) \to \underline{V}}_{\text{morphism in } Cond(\mathbb{R})}$ and $f(\mathcal{M}(S)_{\leq 1})$ is absolutely convex and cpt. Hausdorff.

We can think of $\mathcal{M}(S)$ as signed Radon measures on S and of $\tilde{f}$ as $\mu \mapsto \int f \, d\mu$

Proof idea Let's assume $S = \varprojlim_{\mathbb{N}} S_n$ and pick sections $S \overset{t_n}{\underset{\pi_n}{\rightleftarrows}} S_n$

For $\mu \in \mathcal{M}(S) \rightsquigarrow (\mu_n \in \mathcal{M}(S_n))$ let $v_n := \sum_{s \in S_n} f(t_n(s)) \cdot \mu_n(s)$.

This defines a Cauchy sequence and $v := \lim_{n \to \infty} v_n$ is independent of the t_n. $\square$

Example: $S = \mathbb{N} \cup \{\infty\}$, consider $z_n := \sum_{i=1}^{n} \frac{1}{n} [i] \in \mathcal{M}(\mathbb{N} \cup \{\infty\})_{\leq 1}$, $z_\infty = [\infty]$

$$\begin{array}{ccc} \mathbb{N} \cup \{\infty\} & \xrightarrow{z_\cdot} & \mathcal{M}(\mathbb{N} \cup \{\infty\})_{\leq 1} \\ n & & \downarrow \\ \downarrow & & \\ \frac{1}{n}([1] + \ldots + [N-1]) + \frac{n-N}{n}[\infty] & & \mathcal{M}(\{1, 2, \ldots, N-1, \infty\})_{\leq 1} \end{array}$$

Defn $V \in \mathrm{Cond}(\mathbb{R})$ is __$\mathcal{M}$-complete__ if* for any profinite S and $f: S \to V$ cont. there is a unique extension $\hat{f}: \mathcal{M}(S) \to V$ in $\mathrm{Cond}(\mathbb{R})$.

* This is equivalent to $\underline{\mathrm{Hom}}_{\mathbb{R}}(\mathcal{M}(S), V) \xrightarrow{\text{restr.}} C(S, V)$ being an iso.
* One can equivalently ask this for S extremally disconnected or S CHaus.

Analogy with solidification: $\mathbb{Z}[S]^{\blacksquare} = \varprojlim \mathbb{Z}[S_i] = \underline{\mathrm{Hom}}(C(S,\mathbb{Z}), \mathbb{Z})$

For $K \in$ CHaus the following are isomorphic to the $\mathcal{M}$-completion of $\mathbb{R}[K]$:

1. the topological vector space of signed Radon measures on K.
2. $\mathrm{coeq}(\mathcal{M}(S \times_K S) \rightrightarrows \mathcal{M}(S))$ for S profinite and $S \twoheadrightarrow K$.
3. $\underline{\mathrm{Hom}}_{\mathbb{R}}(C(K, \mathbb{R}), \mathbb{R})$

Thm If $V \in$ TVect is locally convex and complete then $\underline{V}$ is $\mathcal{M}$-complete.

Thm There is a left adjoint $(-)^{\mathcal{M}}$ to the inclusion $\mathrm{Cond}(\mathbb{R})^{\mathcal{M}\text{-cpl}} \hookrightarrow \mathrm{Cond}(\mathbb{R})$

Examples

* Any Banach space: $C(K,\mathbb{R})$, ℓ^p for $1\le p\le\infty$, $C^n(M,\mathbb{R})$ M compact
* Any Smith space: $\mathcal{M}(K,\mathbb{R})$, $W_\infty = \bigcup_{c>0} \prod_{\mathbb{N}} [-c,c]$
* Any Fréchet space: $C^\infty(M,\mathbb{R})$ M any manifold

Non-example: $\ell^{1/2}$ with topology induced by $\|x\|_{1/2} = \sum_{n=1}^{\infty} |x_n|^{1/2}$

$$f(n) := (0,\dots,0,1/k,0,\dots) \in \ell^{1/2} \qquad f(\infty)=0$$

$$\tilde{f}(z_n) = \sum_{k=1}^{n} \frac{1}{n} f(k)$$

$$= \frac{1}{n}(1, 1/2, \dots, 1/n, 0, 0, \dots) \in \ell^{1/2}$$

$$\begin{tikzcd}
\mathbb{N}\cup\{\infty\} \arrow[d, hook] \arrow[dr, "f"] & \\
\mathcal{M}(\mathbb{N}\cup\{\infty\}) \arrow[r, dashed] & \ell^{1/2} \\
\mathbb{N}\cup\{\infty\} \arrow[u, "z_\bullet"'] &
\end{tikzcd}$$

$$\|\tilde{f}(z_n)\|_{1/2} = \sum_{k=1}^{n} \left(\frac{1}{n}\cdot\frac{1}{k}\right)^{1/2} \geq \sum_{k=1}^{n} \left(\frac{1}{n^2}\right)^{1/2} = 1$$

What are the building blocks of this category?

<u>Def</u> $V \in \mathrm{Cond}(\mathbb{R})$ is a Smith space if $V = \bigcup_{c>0} cK$ for some $K \subset V$ CHaus.

<u>Ex</u>: $\mathcal{M}(S)$ with $K = \mathcal{M}(S)_{\leq 1}$

<u>Theorem</u> $V \in \mathrm{TVect}$ complete & locally convex, then $\underline{V} = \operatorname{colim}_{\substack{W \subset V \\ \text{is Smith}}} \underline{W}$

<u>Theorem</u> Any $\mathcal{M}$-complete $V \in \mathrm{Cond}(\mathbb{R})$ is the filtered colimit of Smith spaces.

<u>Theorem</u> $\underline{\mathrm{Hom}}_{\mathbb{R}}(-, \mathbb{R}) : \mathrm{Cond}(\mathbb{R})^{op} \longrightarrow \mathrm{Cond}(\mathbb{R})$ restricts to an equivalence

$$\mathrm{Banach}^{op} \xleftrightarrow{\sim} \mathrm{Smith}$$
$$C(K, \mathbb{R}) \longleftrightarrow \mathcal{M}(K)$$

<u>Note</u>: We can recover the injective and projective tensor product of Banach spaces from a single canonical tensor product $\otimes^{\mathcal{M}}$ on $\mathrm{Cond}(\mathbb{R})^{\mathcal{M}\text{-cpl}}$.

But there are problems:

* $\mathcal{M}(S)$ is not usually projective for S profinite. (Unlike $\mathbb{Z}[S]^{\blacksquare} \in \mathrm{Solid}$)
* $\mathrm{Cond}(\mathbb{R})^{\mathcal{M}\text{-cpl}} \hookrightarrow \mathrm{Cond}(\mathbb{R})$ is not closed under cokernels or extensions
  ↳ No well-behaved derived theory as for Solid!

The counterexample is based on the Ribe-extension $\mathbb{R} \to \tilde{\ell}_1 \to \ell_1$.

Consider the Smith spaces

$$W_1 = \mathcal{M}(\mathbb{N} \cup \{\infty\}) / \mathbb{R}[\infty] = \bigcup_{c>0} \{ (x_s) \in \prod_{\mathbb{N}} [-c,c] \mid \sum |x_s| \leq c \}$$

$$\tilde{W}_1 = \bigcup_{c>0} \{ (x_s, y_s) \in \prod_{\mathbb{N}} [-c,c] \times \prod_{\mathbb{N}} [-c,c] \mid \sum_n |x_n| + |y_n - x_n \cdot \log|x_n|| \leq c \}$$

Then we have an extension:

$$\begin{array}{ccccc} W_1 & \longrightarrow & \tilde{W}_1 & \longrightarrow & W_1 \\ (y_s) & \longmapsto & (0, y_s) & & \\ & & (x_s, y_s) & \longmapsto & (x_s) \end{array}$$

But $\tilde{W}_1$ is not $\mathcal{M}$-complete.

$f: \mathbb{N} \cup \{\infty\} \to \tilde{W}_1$

$n \longmapsto x_n = 1$, other values 0

$\infty \longmapsto 0$

$\tilde{f}(z_n) = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}, 0, \ldots)$

$\| \tilde{f}(z_n) \| = \sum_n \frac{1}{n} + |\frac{1}{n} \cdot \log(\frac{1}{n})|$

$= 1 + \log n$ ↯

This problem is well-known (to Banach space experts).

An extension of Banach spaces is not always Banach, but always p-Banach!

<u>Defn</u> A topological vector space $V \in \mathrm{TVect}$ is called p-Banach if there is a continuous function $\|\_\|: V \longrightarrow \mathbb{R}_{\geq 0}$ satisfying:

① $\|v\| = 0 \iff v = 0$ ② $\|av\| = |a|^p \|v\|$ for $a \in \mathbb{R}$ ③ $\|v+w\| \leq \|v\| + \|w\|$

④ $\{v \in V \mid \|v\| < \varepsilon\}$ form a neighbourhood basis around 0

⑤ Any Cauchy sequence v_n ($\|v_n - v_m\| \longrightarrow 0$) converges ($\|v_n - w\| \to 0$)

<u>Note</u> $p' \leq p \Rightarrow p'\text{-Banach} \supseteq p\text{-Banach}$ by $\|\_\| \rightsquigarrow \|\_\|^{p/p'}$

<u>Fact</u> Any extension of two p-Banachs is p'-Banach for all $p' < p$.

This condition encodes some kind of "local p-convexity"

<u>Def</u> $K \subset V$ is p-convex if

$\forall x_1, \dots x_n \in K \ \forall a_1, \dots a_n \in \mathbb{R} : \sum |a_i|^p \leq 1 \Rightarrow \sum a_i x_i \in K$

Defn $\mathbb{R}[S]_{\ell^p\le c} = \{ \sum a_s[s] \in \bigoplus_S \mathbb{R} \mid \sum |a_s|^p \le c \}$

$$\mathcal{M}_p(S) = \bigcup_{c>0} \varprojlim \mathbb{R}[S_i]_{\ell^p\le c} \qquad \mathcal{M}_{<p}(S) = \operatorname{colim}_{p'<p} \mathcal{M}_{p'}(S)$$

Thm $\mathrm{Liq}_p(\mathbb{R}) \subset \mathrm{Cond}(\mathbb{R})$ is defined as the full subcategory on the objects $V \in \mathrm{Cond}(\mathbb{R})$ satisfying the following equivalent conditions:

① $\forall p'<p$ V is $\mathcal{M}_{p'}$-complete

② V is $\mathcal{M}_{<p}$-complete

$$\begin{tikzcd} S \arrow[r] \arrow[d, phantom, "\cap"] & V \\ \mathcal{M}_{<p}(S) \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

③ V can be written as $\operatorname{coker}\left(\bigoplus_I \mathcal{M}_{<p}(S_i) \to \bigoplus_J \mathcal{M}_{<p}(S_j)\right)$

This subcategory satisfies the following:

* $\mathrm{Liq}_p(\mathbb{R}) \subset \mathrm{Cond}(\mathbb{R})$ is closed under all limits, colimits and extensions.
* this inclusion has a left-adjoint $(-)^{liq}_p$ and $\mathbb{Z}[S]^{liq}_p = \mathcal{M}_{<p}(S)$
* The $\mathcal{M}_{<p}(S)$ for S extr.d. form compact proj. generators
* There is a unique $\otimes^{liq}_p$ on $\mathrm{Liq}_p(\mathbb{R})$ making $(-)^{liq}_p$ sym. mon.

  and this agrees with $\otimes^{\mathcal{M}}$ on nuclear spaces.

The crucial step in proving this is ③ => ① which at it's core amounts to:

<u>Thm</u> For $0<p'<p\leq 1$, S profinite, V p-Banach $\mathrm{Ext}^i_{\mathrm{Cond(Ab)}}(\mathcal{M}_{p'}(S), V) = 0$ $i>0$

This is extremely difficult because $\mathcal{M}_{p'}(S)$ is not locally finite and hence difficult to resolve.

<u>Solution:</u> The condensed ring $\mathbb{Z}((T))_r$.

<u>Defn</u> $\mathbb{Z}((T))_{r,\leq c}(S) = \{ \sum_{n\gg -\infty} a_n T^n \mid a_n \in C(S,\mathbb{Z}) \text{ and } \forall s\in S: \sum_{n\gg-\infty} |a_n(s)|\cdot r^n \leq c \}$

$\mathbb{Z}((T))_r = \bigcup_{c>0} \mathbb{Z}((T))_{r,\leq c}$ $\qquad$ $\mathbb{Z}((T))_{>r} = \bigcup_{r'>r} \mathbb{Z}((T))_{r'}$

<u>Lemma:</u> $\mathbb{Z}((T))_r$ is a condensed ring and each $\mathbb{Z}((T))_{r,\leq c}$ is profinite.

<u>Proof</u> $\mathbb{Z}((T))_{r,\leq c} = \varprojlim_N \{ \sum_{n\gg -\infty}^{N} a_n T^n \mid \sum_{n\gg-\infty} |a_n|\cdot r^n \leq c \} \leftarrow$ these are finite.

<u>Theorem</u> For any $r'\leq r$ the map $\theta_{r'}: \mathbb{Z}((T))_{>r} \longrightarrow \mathbb{R}$ sending T to r':

* is surjective and idenfies $\mathbb{R}$ as a quotient $\mathbb{R} = \mathbb{Z}((T))_{>r}/\ker\theta_{r'}$
* $\ker \theta_{r'}$ is generated by a single $f_{r'} \in \mathbb{Z}((T))_{>r}$.
* $\mathcal{M}(S, \mathbb{Z}((T))_{>r})/(f_{r'}) \cong \mathcal{M}_p(S)$ where $r=(r')^p$

In the end one "only" has to show:

<u>Thm</u> For $1 > r' > r > 0$, V an r-normed $\mathbb{Z}[T^{\pm 1}]$-module and S profinite

$\llcorner$ the map $R\mathrm{Hom}_{\mathbb{Z}[T^{\pm 1}]}(\mathcal{M}(S, \mathbb{Z}((T))_{r'}), \hat{V}) \longrightarrow \hat{V}(S)$ is a quasi iso.