Analytic Rings

Ring = "animated commutative ring" = simpl. ring / w.eq.

R $\quad$ $\mathrm{Mod}_R = \mathcal{D}_{\geq 0}(R)$ = connective modules

Defn An analytic ring is a pair

① R a condensed ring

② $\mathcal{M}$ is a full subcategory of Mod_R $\rightsquigarrow$ "completed modules"

s.t. ① $R \in \mathcal{M}$ $\quad$ ② $\mathcal{M}$ closed under all limits and colimits

③ $M \in \mathcal{M}$, $N \in \mathrm{Mod}_R$ then $\underline{R\mathrm{Hom}}(N, M) \in \mathcal{M}$

④ $M \in \mathcal{M}$ and module over $R \otimes_{\mathbb{Z}} \mathbb{F}_p$ then $\square \in \mathcal{M}$

$$\begin{tikzcd} \square \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & M \arrow[d] \\ R\otimes_{\mathbb{Z}} \mathbb{F}_p \arrow[r, "\mathrm{Frob}"'] & R\otimes_{\mathbb{Z}} \mathbb{F}_p \end{tikzcd}$$

⑤ $\exists$ left-adjoint $L : \mathrm{Mod}_R \to \mathcal{M}$ to inclusion.

Remarks:

- $\mathcal{M}$ = "complete modules"
- $R \longmapsto \mathrm{Spa}(R)$ $\quad$ think: $\mathrm{Spa}(R, \mathcal{M}) \subseteq \mathrm{Spa}(R)$ "feels affine"
- ⑤ automatic modulo set-theory.
- ②+③ $\Rightarrow$ generated by compact projectives $L(R[S])$
- colimits $\Rightarrow$ $\mathcal{M}$ behaves like $\mathrm{Mod}_{R[F^{-1}]} \subset \mathrm{Mod}_R$
- ③ $\Rightarrow$ $\exists \otimes$ on $\mathcal{M}$ s.t. L is sym. monoidal
- ④ $\Rightarrow$ Can define Sym^i for all $i \in \mathbb{N}$

Notation $\mathcal{R} = (R, \mathcal{M})$ write $\mathrm{Mod}_{\mathcal{R}} := \mathcal{M}$

Def A map of analytic rings $\mathcal{R} \to \mathcal{R}'$ is a map of condensed rings $R \to R'$ s.t. $\forall M' \in \mathcal{M}od_{\mathcal{R}'}$ $\mathrm{res}_R M' \in \mathcal{M}od_{\mathcal{R}}$

Equiv.: $M \in \mathcal{M}$ with $L(M) = 0$ then $L'(R' \otimes_R M) = 0$

Ex 1 R condensed ring $\mathcal{M} = \mathrm{Mod}_R$: initial analytic ring str.

2 $\exists$ terminal analytic ring structure?

If you have R and you specify "$L(R[S])$" : $ED \to \mathrm{Mod}_R$ s.t. if $M \in \mathrm{Mod}_R$ can be resolved by direct sums of $\bigoplus L(R[S])$ then $\underline{R\mathrm{Hom}}(L(R[S]), M) \xrightarrow{\sim} \underline{R\mathrm{Hom}}(R[S], M)$

$\to$ get analytic ring structure with $\mathcal{M} =$ such Ms

Ex Solid abelian groups = analytic ring str on $\mathbb{Z}$

$\hookrightarrow$ $\boxed{\text{This is terminal}}$ where $L(\mathbb{Z}[S]) = \varprojlim \mathbb{Z}[S_i]$

Ex p-liquid $\mathbb{R}$-modules = analytic ring str on $\mathbb{R}$

Have a candidate for terminal str. $L(R[S]) = \varprojlim R[S_i]$

$\subset$ "Ultrasolid theory" doesn't exist a lot of times...

Ex $\mathcal{R}$ analytic ring $R \to R'$ $\rightsquigarrow$ induced analytic ring str

$= (R', M \in \mathcal{M}' \text{ if } \mathrm{res}_R M \in \mathcal{M})$

Ex $\mathbb{R} \longrightarrow \mathbb{R}[T] = \bigoplus_{*\geq 0} \mathbb{R}$ $\qquad$ $\mathrm{Spa}(\mathbb{R}[T]) = \mathbb{A}^1_{\mathrm{Spa}(\mathbb{R})}$

Ex The category of all analytic rings has all colimits

Filtered colimits are computed as $\varinjlim (R_i, \mathcal{M}_i) = (\varinjlim R_i, \ldots)$

Pushouts can be quite subtle.

$(R_1, \mathcal{M}_1) \otimes_{(R,\mathcal{M})} (R_2, \mathcal{M}_2) \rightsquigarrow$ unnormalized $(R_1 \otimes_R R_2$, those $\mathcal{M}$ st. $\mathrm{res}_1 \mathcal{M}$ $\mathrm{res}_2 \mathcal{M}$ are complete$)$

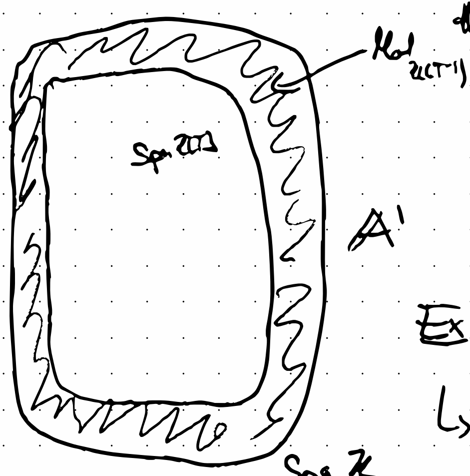
Restrict to solid analytic rings

i.e underlying condensed ring is solid.

Thm ① The ultrasolid theory exists for $\mathbb{Z}[T]$. $\qquad \prod_I (\mathbb{Z}[T]) \supsetneq (\prod_I \mathbb{Z})[T]$

Note: this is not the induced analytic ring structure.

② $\mathrm{Ker}\, L = \mathrm{Mod}_{\mathbb{Z}((T^{-1}))}$ $\qquad$ $\mathbb{Z}((T^{-1}))$ is example of solid $\mathbb{Z}$-module that is not a solid $\mathbb{Z}[T]$-module.



$\mathrm{Spa}(\mathbb{Z}[T]^{\blacksquare})$ = solid closed unit disk

Ex $\mathbb{Q}_p \otimes_{\mathbb{Z}^{\blacksquare}} \mathbb{Z}[T]^{\blacksquare}$ = closed unit disk over $\mathbb{Q}_p$

$\hookrightarrow$ underlying ring $\mathbb{Z}[T]^{\wedge}_p[\frac{1}{p}]$

= ring of rigid-analytic fcts on closed unit disc

Given any solid ring R and any subset $S \subset R(*)$

we can universally force "$|f| \leq 1$" $\forall f \in S$

by taking the initial analytic ring structure

$$\begin{tikzcd} \mathbb{Z}[T] \arrow[r, "T \mapsto f"] \arrow[d] & R \\ \mathbb{Z}[T]^{\blacksquare} \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

↳ Yields unnormalised analytic ring str. on R.

We can also force any $T \subset R(*)$ to be invertible. $\mathbb{Z}[T^{\pm 1}] \otimes_{\mathbb{Z}[T]} R$

$\underline{\text{Ex}}$ Rigid analytic geometry

$\exists$ analytic ring str. on any affinoid st. these ~~are~~ $\hookrightarrow$ usual ones + $\underline{\text{more}}$

more: eg. $\mathbb{Z}_p[[T]][\frac{1}{p}]$

$\underline{\text{Ex}}$ R discrete ring $\Rightarrow$ analytic ring str. s.t all $f \in R$ are ≤ 1

$= \text{Solid}_R$ (= UltraSolid iff R fingen)

$\underline{\text{Ex}}$ $\exists$ correspondence to Huber's theory $(R, M) \leftrightarrow (R, R^+)$