

ABELIAN AND NONABELIAN HODGE THEORY

An Overview

Lukas Brantner

*As every mathematician knows,
nothing is more fruitful than these obscure analogies,
these indistinct reflections of one theory into another,
these furtive caresses,
these inexplicable disagreements;
also nothing gives the researcher greater pleasure...
The day dawns when the illusion vanishes;
intuition turns to certitude;
the twin theories reveal their common source before disappearing;
as the Gita teaches us,
knowledge and indifference are attained at the same moment.
Metaphysics has become mathematics,
ready to form the material for a treatise whose icy beauty no longer
has the power to move us.*

André Weil

Contents

Chapter 1. Introduction	3
Overview	4
Chapter 2. Monodromy	6
2.1. Monodromy via Covering Spaces	6
2.2. Monodromy via Sheaves	8
2.3. Monodromy via Connections	9
2.4. Factors of Automorphy	13
Chapter 3. The Twisted Abelian Hodge Theorem	15
3.1. Elliptic Operator Theory	15
3.2. Hodge Isomorphism for de Rham cohomology	18
3.3. Hodge Isomorphism for Dolbeault cohomology	20
3.4. The Hodge-Decomposition for Kähler-manifolds	21
3.5. Hodge theorem for unitary representations	23
3.6. First Applications	24
3.7. Group cohomological considerations	26
Chapter 4. The Unitary Nonabelian Hodge Theorem	29
4.1. Motivation: The classification of holomorphic bundles	30
4.2. Rank and Degree of Coherent Torsion-Free Sheaves	30
4.3. Bundles on Curves	34
4.4. An Instructive Example	38
4.5. From Narasimhan-Seshadri to Kobayashi-Hitchin	40
4.6. The Unitary Nonabelian Hodge Theorem	44
Chapter 5. The General Nonabelian Hodge Theorem	46
5.1. Inspiration from the Abelian Case	46
5.2. Metricity Defect and Absolute Metrisation	47
5.3. The Nonabelian Hodge Correspondence	50
5.4. Further Discussion	55
Acknowledgement	57
Appendix A. Background in Complex Geometry	58
A.1. The three perspectives on complex manifolds	58
A.2. A Reformulation of Holomorphic Bundles	61
A.3. A Reminder on Kähler manifolds	61
A.4. The Hodge-Star-Operator	62
Appendix B. GAGA	64

B.1. Analytic Varieties	64
B.2. Coherent Analytic and Algebraic Sheaves	65
B.3. The Analytification functor	66
B.4. The statement of GAGA	67
B.5. Applications	67
Bibliography	69
Index	71

CHAPTER 1

Introduction

Smooth complex projective varieties are certain zero sets of homogeneous polynomials and inherit a natural topology from $\mathbb{P}^n(\mathbb{C})$. One of the most immediate questions that arise is:

What are the possible shapes of smooth complex projective varieties?

A first guess might be that any homeomorphism type of any oriented manifold can arise, but we will see that this is far from true. Since homeomorphism types are way too hard to handle, we will be more modest, restrict attention to certain topological invariants such as homotopy or cohomology, and try to find out which invariants can occur for smooth projective varieties.

The restrictive theory we will develop in this essay will mostly apply to compact Kähler manifolds, which are more general objects than projective varieties as a recent theorem of Voisin shows (see [45]):

THEOREM 1.1. (*Voisin*) *In any dimension bigger than 3, there exists a compact Kähler manifold which does not have the homotopy type of a smooth complex projective variety.*

In this essay, we will focus on particularly interesting invariant: $\pi_1(X)$

DEFINITION 1.2. A group is called projective (resp. Kähler) if it appears as the fundamental group of some smooth compact complex projective variety (resp. compact Kähler manifold).

A central, at this stage wide open problem is the so-called *Kähler problem*, which asks for a complete classification of all Kähler groups. A more modest unsolved classical question is:

Does the class of Kähler groups agree with the class of projective groups?

The rigid behaviour of algebraic varieties prohibits an application of several standard techniques we often use in algebraic topology to construct examples. However, several positive results are known - a very prominent such result is the following theorem of Serre (see [33]):

THEOREM 1.3. (*Serre*) *Any finite group is projective.*

This article gives an exposition of some of the Hodge-theoretic techniques and correspondences used to obtain restrictive results. We will see that almost all results are naturally concerned with *complex representations of the fundamental group*, which are the central objects of our studies.

Our didactic aim is to avoid sudden jumps, and rather present the theory as a chain of answers to natural questions the reader might have asked himself after having understood the preceding material.

Overview

Besides reminding the reader of some necessary background material in complex geometry, the main aim of Appendix A is to standardise notation and hence to minimise confusion in other chapters. This is crucial since various definitions in complex geometry appear in several different forms in the literature. This Appendix should be consulted if the reader is in doubt about notation or the precise form of basic complex-geometric notions. Appendix B will give a concise introduction to Serre's GAGA, sketches several nice applications and thereby introduces coherent analytic and algebraic sheaves.

We start the essay with Chapter 2 on monodromy, in which we shed light on monodromy from various different perspectives and hence establish several equivalences of categories. We will introduce connections as gadgets fixing the deficiency of a certain functor, and show that this deficiency is acquired as a trade off for using the natural rather than the discrete topology on \mathbb{C} .

We then give a brief review of factors of automorphy in 2.4, which allow us to express holomorphic line bundles in terms of group cohomology and will help us in 4.4 to understand one of our examples.

Chapter 3 will start off by giving fairly detailed proofs of the abelian Hodge theorems for de Rham and Dolbeault cohomology (assuming elliptic operator theory). We combine these two theorems to obtain the Hodge decomposition, which we phrase in its natural setting, namely for twisted cohomology of unitary representations.

After giving several standard corollaries of this theorem (Nontrivial free groups and the modular group are not Kähler, the Iwasawa manifold is not a Kähler manifold, classification of abelian Kähler groups), we turn to group-cohomological techniques. We use these to see that the integral Heisenberg group is not Kähler. In the particular case of aspherical Kähler manifolds, we deduce a Hodge-decomposition for the group cohomology of self-dual unitary representations. We then use a classical result of Lyndon (see [24]) to classify aspherical one-relator subgroups, thereby proving a very easy special case of a recent theorem of Biswas and Mj (see [10]) that classifies one-relator Kähler groups completely.

In Chapter 4, we leave the path of abelian Hodge theory temporarily and examine the classification of holomorphic vector bundles on smooth projective varieties. For this, we need several basic invariants for coherent sheaves on Kähler manifolds, which we motivate and define.

After reviewing Grothendieck's and Atiyah's classification of vector bundles on curves of genus 0 and 1, we will study the higher-genus case with representation-theoretic means. We introduce the Atiyah class, and use it to outline Atiyah's proof of Weil's theorem, answering the question which holomorphic bundles are induced by representations via monodromy. We then carry out an explicit computation using factors of automorphy on curves of genus 1, which naturally leads us to the theorem of Narasimhan-Seshadri for elliptic curves and illustrates the relation between certain Moduli spaces.

We then state this theorem over general curves, and describe Donaldson's reduction of its proof to the existence-question of Hermite-Einstein metrics. Generalising from here brings us to the Kobayashi-Hitchin correspondence, which allows us to

prove the unitary nonabelian Hodge theorem, which one might also call higher-dimensional theorem of Narasimhan-Seshadri.

Our final Chapter 5 then merges several previous streams of thought. We first use the abelian Hodge theorem to give us inspiration as to where to look for the right type of correspondence. We then introduce Higgs-fields as natural objects measuring how much a (semisimple) representation fails to be unitary, harmonic metrics as “most compatible metrics”, and Higgs bundles as generalised harmonic bundles remembering an error-term.

A theorem of Corlette allows us to define a fully faithful functor from semisimple representations to Higgs bundles. In order to determine the image of this functor, we use a generalisation of the Kobayashi-Hitchin correspondence to Higgs bundles.

This finally yields one of the deepest and most powerful theorems in complex geometry: The nonabelian Hodge theorem.

We close the essay by sketching several applications of this result.

CHAPTER 2

Monodromy

Various strong results like Burnside’s $p^a q^b$ -theorem, Frobenius’ theorem or Haboush’s theorem impressively demonstrate the close ties between (various classes of) groups and their representations. It is therefore evident that in order to solve the aforementioned Kähler-problem, a profound understanding of the representation theory of the relevant fundamental groups is desirable.

The theory of Kähler manifolds gains its richness from the variety of angles from which one can observe complex manifolds (as topological spaces, or as smooth, complex, or Hermitian manifolds). It is often enlightening to make clear which structures are used for which arguments, and we therefore decided to present most of the content of this chapter from a top-down perspective. Notice that the abstract language we use in the exposition of monodromy increases conceptual clarity and conciseness without adding any mathematical content to the classic case of permutation representations, which can be found in any first text on Algebraic Topology. For this chapter, we fix a sufficiently “nice”¹ pointed connected topological space (X, x) and a category \mathcal{C} of algebraic structures (i.e. models of an algebraic theory like **Set**, **Group** or **Vect_k**). We have a natural notion of \mathcal{C} -sheaves (compare chapter 6 of the stacks project). The constant sheaf associated to an object $D \in \mathcal{C}$ maps an open set $U \subset X$ to the set of locally constant functions from U to D , which is naturally an object of \mathcal{C} itself.

Recall that the category of \mathcal{C} -representations of a group G is nothing but the functor category $[G, \mathcal{C}]$ whose morphisms are natural transformations, i.e. homomorphism which intertwine the relevant representations. Here we think of a group as a category with one object in which all morphisms are invertible.

2.1. Monodromy via Covering Spaces

In this and the next paragraph, we will first show that the algebraically defined category of \mathcal{C} -representations of $\pi_1(X, x)$ is equivalent to topological and sheaf-theoretic counterparts. In the third paragraph, we will then specialise to the case where $\mathcal{C} = \mathbf{Vect}_{\mathbb{C}}^{fd}$ and X is a manifold, where we can also define a differential-geometric analogue. The complete proofs of the many equivalences of categories we will present here are long-winded and contain many easy checks. We therefore decided to just present the main steps and leave the easy checks to the reader.

There is a straightforward generalisation of ordinary covering spaces which will allow us to interpret \mathcal{C} -representations topologically, and hence forms the bridge between representations and local systems:

¹This means that X is path-connected, locally path-connected and semi-locally simply connected.

DEFINITION 2.1. A \mathcal{C} -covering space on X is a surjective continuous map $p : Y \rightarrow X$ such that we can find an open cover $\{U_\alpha\}_\alpha$ of X , elements $\{D_\alpha\}_\alpha$ of \mathcal{C} (equipped with the discrete topology) and homeomorphisms $\{\phi_\alpha : p^{-1}(U_\alpha) \rightarrow U_\alpha \times D_\alpha\}_\alpha$ with the following two properties:

- $\pi_1 \circ \phi_\alpha = p$
- Each transition function $\phi_\beta \circ \phi_\alpha^{-1}$ is a \mathcal{C} -isomorphism on each fibre.

Notice that all the fibres inherit the structure of elements in \mathcal{C} , and that transition functions are locally constant.

A morphism of \mathcal{C} -covering spaces $f : Y_1 \rightarrow Y_2$ over X is a fibre-preserving continuous map restricting to \mathcal{C} -morphisms on fibres.

A basic result from algebraic topology says that covering spaces on X and permutation representations of $\pi_1(X, x)$ are equivalent categories. We now extend this correspondence from **Set** to general categories \mathcal{C} of algebraic structures:

THEOREM 2.2. *There is an equivalence of categories*

$$\{\mathcal{C}\text{-covering spaces on } X\} \rightleftarrows \{\mathcal{C}\text{-representations of } \pi_1(X, x)\}$$

PROOF. For a \mathcal{C} -covering space $p : Y \rightarrow X$, we have an associated \mathcal{C} -representation ρ_p of $\pi_1(X, x)$ on the fibre $p^{-1}(x)$, called the *Monodromy representation*: Given $g \in \pi_1(X, x)$ and $y \in p^{-1}(x)$, we pick a representative loop γ of g , find a lift $\tilde{\gamma}$ starting at y , and then define gy to be the endpoint of this lift. By chopping up the unit interval $[0, 1]$, lifting locally, and using that transition functions are fibrewise isomorphisms, it is easy to check that this gives well-defined \mathcal{C} -representation ρ_p .

Conversely, we can build a covering space $p_\rho : Y \rightarrow X$ out of a given \mathcal{C} -representation $\rho : \pi_1(X, x) \rightarrow \text{Aut}(D)$ as follows. By our assumptions on X , we have a universal (**Set**-)cover $\pi : \tilde{X} \rightarrow X$, equipped with a right group action of $\pi_1(X, x)$ via Deck transformations. We can combine this free and properly discontinuous action with the representation ρ and obtain a new such action on $\tilde{X} \times D$ given by

$$g(z, d) = (g^{-1} \cdot z, \rho(g)(d))$$

We will now use the local triviality of \tilde{X} to prove that the projection

$$p : \tilde{X} \times D / \pi_1(X, x) \rightarrow X$$

on the first factor is a \mathcal{C} -covering with monodromy ρ .

Find a cover $X = \bigcup U_i$ of path-connected open sets trivialising \tilde{X} via $\psi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times E$. We fix an arbitrary element $e_0 \in E$ once and for all.

Restrict attention to some particular α . The action of $\pi_1(X, x)$ on $U_\alpha \times E$ which is given by

$$\gamma \cdot (z, e) = \phi_\alpha(\gamma^{-1} \cdot \phi_\alpha^{-1}(z, e))$$

fixes the first component. Since E is discrete and U_α is path-connected, we can find a right action of the fundamental group on the fibre E_α such that for all $z \in U_\alpha$:

$$\gamma(z, e) = (z, \gamma^{-1} \cdot e)$$

Finally, we combine this with ρ to obtain a (left) action of $\pi_1(X, x)$ on $U_\alpha \times E \times D$.

In the commutative diagram below, the horizontal maps on the top row are equivariant, and hence descend to maps on the quotients. The embedding

$$U_\alpha \times D \rightarrow U_\alpha \times E \times D$$

obtained from e_0 gives rise to a homeomorphism with the indicated quotient.

$$\begin{array}{ccccc}
 & & U_\alpha \times E \times D & \xrightarrow{\psi_\alpha^{-1} \times \text{id}} & \pi^{-1}(U_\alpha) \times D \\
 & & \downarrow & \xleftarrow[\cong]{\psi_\alpha \times \text{id}} & \downarrow \\
 U_\alpha \times D & \xrightarrow{\cong} & U_\alpha \times E \times D / \pi_1(X, x) & \xrightarrow[\cong]{\text{---}} & \pi^{-1}(U_\alpha) \times D / \pi_1(X, x) \\
 & \searrow & \downarrow & & \downarrow \\
 & & U_\alpha & &
 \end{array}$$

$(z, d) \mapsto (z, e_0, d)$

If $z \in U_\alpha \cap U_\beta$, we can find $\gamma \in \pi_1(X, x)$ such that

$$\psi_\beta \psi_\alpha^{-1}(z, e_0) = (z, \gamma^{-1} \cdot e_0)$$

From here, one checks the transition function on the fibre of z is given by $\rho(\gamma)$, so a \mathcal{C} -isomorphism. Notice that the trivialisation functions are also homeomorphisms, hence we have indeed constructed a \mathcal{C} -covering. One checks that it has monodromy ρ .

It is now straightforward to give the definition of the above functors on morphisms, and one can then check that they define an equivalence of categories. \square

2.2. Monodromy via Sheaves

Since \mathcal{C} -coverings are in particular étalé spaces (i.e. local homeomorphisms), it should come as no surprise that they have a neat sheaf-theoretic description. The following construction will be particularly useful later, since it will allow us to define “cohomology of a representation of $\pi_1(X, x)$ ” as the cohomology of the corresponding sheaf. We need the following notion:

DEFINITION 2.3. A \mathcal{C} -local system on X is a locally constant sheaf \mathcal{F} on X taking values in the category \mathcal{C} .

The aforementioned correspondence then has the form:

THEOREM 2.4. *There is an equivalence of categories*

$$\{\mathcal{C}\text{-valued local systems on } X\} \simeq \{\mathcal{C}\text{-covering spaces on } X\}$$

PROOF. Given a \mathcal{C} -covering space $p : Y \rightarrow X$, the functor F associates the sheaf $FC = \mathcal{F}$ of sections defined by:

$$\mathcal{F}(U) = \{f : U \rightarrow Y \mid p \circ f = \text{id} \text{ and } f \text{ continuous}\}$$

One can then check that F is *full* and *faithful*.

To see that F is *essentially surjective*, assume we are given a \mathcal{C} -valued local system \mathcal{F} on X . We can take an open path-connected cover $X = \bigcup U_\alpha$, together with sheaf-isomorphisms

$$\phi_\alpha : \mathcal{F}|_{U_\alpha} \rightarrow \underline{D}|_{U_\alpha}$$

Here D is some fixed object in \mathcal{C} , which is the same everywhere by path-connectedness of X , and $\underline{D}|_{U_\alpha}$ is the corresponding constant sheaf on U_α .

If $z \in U_\alpha \cap U_\beta$, let W_z be the (open) path-connected component of z in this intersection. Then $\psi_\beta(W_z)\psi_\alpha^{-1}(W_z)$ defines an honest automorphism $\theta_{\alpha\beta}(z)$ of D , and the function $\theta_{\alpha\beta}$ is locally constant.

We can use these maps to produce a \mathcal{C} -covering $p : Y \rightarrow X$ by gluing. Set

$$Y = \coprod_{\alpha} U_\alpha \times \{\alpha\} \times D / \sim$$

where D carries the discrete topology and \sim is the equivalence relation defined by claiming that for all indices α, β , all $z \in U_\alpha \cap U_\beta$, and all $d \in D$, we have

$$(x, \alpha, d) \sim (x, \beta, \theta_{\alpha\beta}(z)(d))$$

Straightforward checks now prove that this defines a \mathcal{C} -covering over X , whose sheaf of continuous sections is isomorphic to the local system \mathcal{F} we started with.

The fact that F is full, faithful and essentially surjective implies that it is part of an equivalence of categories (using a sufficiently powerful version of the axiom of choice). \square

2.3. Monodromy via Connections

The two equivalent viewpoints on representations of $\pi_1(X, x)$ we introduced thus far hold for arbitrary topological spaces admitting universal covers and arbitrary categories of algebraic structures \mathcal{C} . However, the main results of this essay apply to a much more specific situation, namely the case $\mathcal{C} = \mathbf{Vect}_{\mathbb{C}}^{fd}$ of finite-dimensional complex representations. We will now present a third differential-geometric formulation of monodromy. In order to have the relevant tools available, we need to assume from now on that (X, x) is a fixed pointed smooth manifold (admitting a universal cover).

2.3.1. Problem: Discrete Bundles contain more Information than Smooth Bundles. When we work with finite-dimensional real or complex vector spaces, we are very used to the fact that these spaces carry a preferred topology induced by norms for which we have a good intuition. However, this result is highly dependent on finite-dimensionality and the topological structure of the ground field, and therefore does not generalise to arbitrary vector spaces over arbitrary fields. General vector spaces should therefore be considered as purely algebraic objects which do not come equipped with any particularly nice topology.

But we always have two trivial topologies at our disposal: the discrete and the indiscrete one. It is unsurprising that the discrete topology is more suited to topological studies because it can detect when points are distinct. For every field k , the procedure of putting the discrete topology on a vector space defines a fully faithful functor $F : \mathbf{Vect}_k \rightarrow \mathbf{TVect}_k$ to the category of topological vector spaces.

It was precisely this philosophy of “endowing algebraic structures with discrete topologies” which motivated our definition 2.1 of \mathcal{C} -covering spaces. In the special case where $\mathcal{C} = \mathbf{Vect}_k$, this deserves a new name:

DEFINITION 2.5. A *discrete k -vector bundle* is a \mathbf{Vect}_k -covering space. Write $\mathbf{DiscBun}_k$ for the category of such bundles on X .

Our previous theorem 2.2 specialises to an equivalence of categories between discrete k -bundles and representations on k -vector spaces.

However, differential geometry studies the category $\mathbf{Bun}_{\mathbb{K}}^{fd}$ of smooth (finite-dimensional) vector bundles on X whose fibres are vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We fix one of the two fields. The definition of \mathbb{K} -vector bundles differs from the one for discrete bundles in two points: First, we require the transition functions to vary smoothly, and secondly, the copies of \mathbb{K} appearing in the trivialisations carry their standard topology rather than the discrete one. The aim of the remainder of this paragraph is to study the relation between smooth and discrete bundles.

Given any vector space V in the category $\mathbf{TVect}_{\mathbb{K}}^{fd, disc}$ of discrete finite-dimensional \mathbb{K} -vector spaces, we can replace its topology by the one induced by norms. Since all linear maps are continuous for the discrete and the norm-topology, this procedure defines a fully faithful functor

$$S_V : \mathbf{TVect}_{\mathbb{K}}^{fd, disc} \rightarrow \mathbf{TVect}_{\mathbb{K}}^{fd}$$

which we will call the “smear-functor on vector spaces”. We can extend this construction to a “smear functor on bundles”

$$S_B : \mathbf{DiscBun}_{\mathbb{K}}^{fd} \rightarrow \mathbf{Bun}_{\mathbb{K}}^{fd}$$

as follows: Pick any trivialisaton $\{\phi_\alpha : p^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{K}_{disc}^n\}$ for our discrete bundle $p : Y \rightarrow X$. Since the transition functions are locally constant, they vary smoothly. We can now topologise Y in a way which makes the functions

$$\{\psi_\alpha : p^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{K}^n\}$$

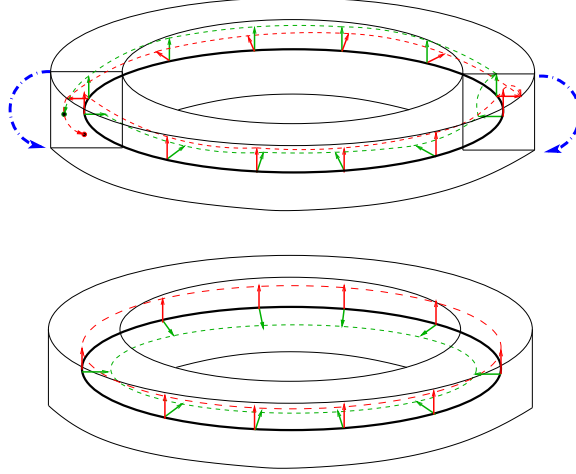
into homeomorphisms (here \mathbb{K}^n carries the norm topology). Easy checks then show that $p : Y \rightarrow X$ is a smooth bundle, that it is independent of the chosen trivialisaton, and that each morphism of discrete bundles yields a morphism of the corresponding smooth bundles.

But is the smear functor S on bundles also fully faithful as in the case of vector spaces? The key feature that made S_V full was that on the discrete side of topological vector spaces, all spaces are (unsurprisingly) actual discrete topological spaces, which means that all maps between them are automatically continuous.

In the case of vector bundles, the situation is more subtle: Locally, our spaces on the discrete side are products of discrete topological vector spaces and manifolds. In general, these are of course no longer discrete topological spaces but rather covering spaces of X . In this case, the discrete topology of the fibres puts nontrivial restrictions on the morphisms of discrete bundles which do not occur in the smooth case: they have to be “locally constant”. More precisely, if $f : Y \rightarrow Z$ is a morphism of discrete bundles, ϕ_α a chart of Z , then $\phi_\alpha \circ f$ has to be locally constant where defined. The following example will show that (as now expected) the smear-functor on bundles is not full, and that it even maps nonisomorphic discrete bundles to isomorphic smooth ones.

EXAMPLE 2.6. The next picture shows two discrete real plane bundles over S^1 . It is obvious by path-lifting that the first case corresponds to the monodromy representation $\rho_1 : \mathbb{Z} = \pi_1(S^1) \rightarrow Gl_2(\mathbb{R})$ mapping 1 to $-id$ whereas the monodromy of the second situation maps 1 to $+id$. These two representations are clearly not isomorphic. Since taking monodromy is functorial, this implies that our two discrete bundles are not isomorphic either.

However, the image of both of these discrete bundles under the smear functor is the trivial (real) plane bundle over S_1 . This is clear in the second case as the sketched global frame trivialises our bundle. We notice that the exact same global frame also trivialises the smeared version of the first bundle.



We can now state the problem we face when trying to encode representations of $\pi_1(X, x)$ by smooth bundles:

COROLLARY 2.7. *The smear-functor $S_B : \mathbf{DiscBun}_{\mathbb{K}}^{fd} \rightarrow \mathbf{Bun}_{\mathbb{K}}^{fd}$ is not full, and hence in particular **not** an equivalence of categories.*

2.3.2. Solution: Flat Connections. After a brief review of connections, we will show how we can use them to fix the smear functor and thus turn it into an equivalence. Fix $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and a \mathbb{K} -vector bundle E over X . Write $\mathcal{A}^p(E) := \mathcal{A}_{X, \mathbb{K}}^p(E)$ for the corresponding sheaf of smooth E -valued p -forms. Recall that a (smooth) \mathbb{K} -connection $\nabla : \mathcal{A}^0(E) \rightarrow \mathcal{A}^1(E)$ is a \mathbb{K} -linear sheaf homomorphism satisfying the Leibniz-rule. An easy exercise shows that there is a unique extension $\nabla : \mathcal{A}^k(E) \rightarrow \mathcal{A}^{k+1}(E)$ satisfying the (graded) Leibniz-rule. The curvature of ∇ is then defined as the unique bundle-homomorphism R_{∇} inducing the sheaf-homomorphism $\nabla \circ \nabla : \mathcal{A}^0(E) \rightarrow \mathcal{A}^2(E)$, and the connection is said to be flat if R_{∇} vanishes.

For the following proof, we need to understand how these concepts behave locally: If e_1, \dots, e_n is a local frame for E , we can write $\nabla(e_i) = \sum a_{ij} \nabla(e_j)$ where $A = (a_{ij})_{i,j}$ is the connection-matrix of sections in \mathcal{A}^1 , i.e. honest 1-forms.

By the Leibniz rule, this implies for a general section $e = \sum_i s_i e_i$ that

$$\nabla(e) = \sum_i d(s_i) e_i + \sum_{i,j} s_i a_{ij} e_j. \text{ Using obvious conventions we can therefore write}$$

$$\nabla = d + A. \text{ We obtain by an easy computation: } R_{\nabla} = d(A) - A \wedge A$$

We can now define the crucial category:

DEFINITION 2.8. Let $\mathbf{FlatBun}_{\mathbb{K}}^{fd}$ denote the category whose objects (E, ∇_E) are (finite-dimensional \mathbb{K} -)vector bundles with flat connections and whose morphisms are connection-preserving vector bundle morphisms.

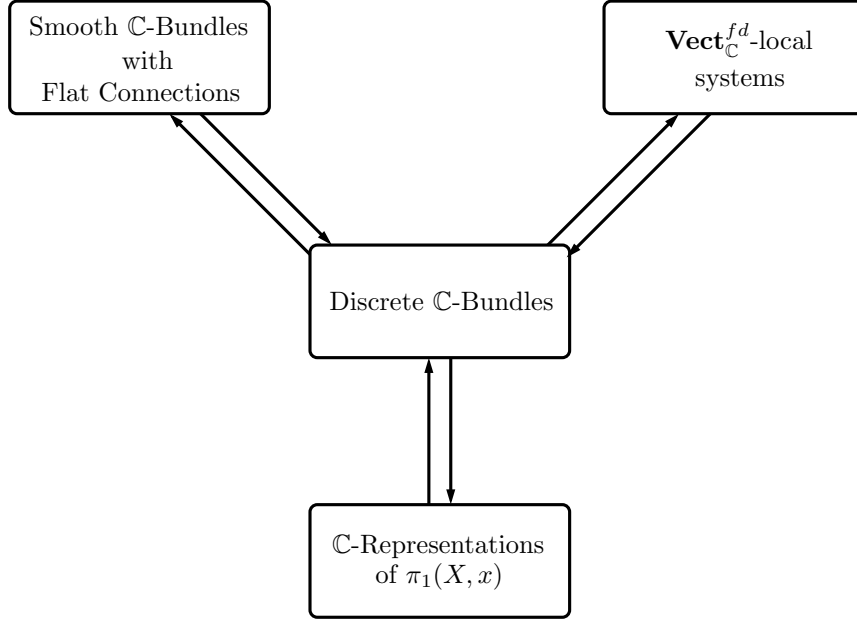
In order to fix the shortcoming of the smear functor S_B , we will extend it to a functor S_B^+ to the category of bundles with flat connections and hence keep track of more data as follows. Start with a discrete bundle $(p : Y \rightarrow X) \in \mathbf{DiscBun}_{\mathbb{K}}^{fd}$ and pick a trivialisation $\{\phi_\alpha : p^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{K}_{disc}^n\}$. Then the associated smeared bundle E has the trivialisation $\{\psi_\alpha : p^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{K}^n\}$.

Write $f_i(x)^\alpha = \psi^{-1}(x, e_i)$ for the corresponding smooth local frames. For a local section $\sum s_i f_i$ with $s_i \in \mathcal{A}^0$, we can define $\nabla_E(\sum s_i f_i) = \sum ds_i \otimes f_i$. Since the transition functions of our bundle are locally constant, this definition is independent of the chosen frame. As $A = 0$ in the above notation, this connection is flat.

To see that S_B^+ is essentially surjective, we start with an arbitrary bundle $p : E \rightarrow X$ together with a flat connection ∇ . An application of Frobenius' theorem (see Lemma 9.12 [44] for details) shows that around every point $P \in X$, we can choose a horizontal frame f_1, \dots, f_n . Let $\phi_P : p^{-1}(U_P) \rightarrow U_P \times \mathbb{K}_{disc}^n$ be given by $\phi_P^{-1}(x, e_i) = f_i$. Since all transition functions $\phi_Q \phi_P^{-1}$ are homeomorphisms and linear on fibres, we can equip E with a unique topology making the ϕ_P 's into homeomorphisms. We obtain a discrete bundle E^{disc} with $S_B^+(E^{disc}) \cong E$. A final check reveals that S_B^+ is full and faithful, and we have therefore proved:

THEOREM 2.9. *The improved smear-functor $S_B^+ : \mathbf{DiscBun}_{\mathbb{K}}^{fd} \rightarrow \mathbf{FlatBun}_{\mathbb{K}}^{fd}$ defines an equivalence of categories.*

We can summarise this paragraph in the following picture, which indicates the four mentioned (equivalent) categories for a pointed smooth manifold (X, x) :



2.3.3. The Holomorphic Structure of Flat Complex Bundles on Complex Manifolds. Assume now that the manifold X we examine is not just smooth but complex.

Let (E, ∇) be a flat complex bundle on X . By A.2, the $(0, 1)$ -part $\bar{\partial}_E$ of our connection gives E the structure of a holomorphic bundle such that ∇ is *compatible with the holomorphic structure* (i.e. the $(0, 1)$ part is given by $\bar{\partial}_E$).

Notice that there is a different notion with a similar name:

DEFINITION 2.10. A *holomorphic connection* on a holomorphic bundle E on a complex manifold X is a \mathbb{C} -linear sheaf homomorphism $\nabla : \Omega^0(E) \rightarrow \Omega^1(E)$ such that for all holomorphic functions f and all holomorphic sections e , we have

$$\nabla(f \cdot e) = \partial(f) \otimes e + f\nabla(e)$$

Its curvature is ∇^2 , where we again consider the extension of ∇ satisfying the graded ∂ -Leibniz rule.

Notice that if ∇ is a holomorphic connection, then $\nabla + \bar{\partial}_E$ is an honest connection compatible with the holomorphic structure. The converse is far from true, i.e. the $(1, 0)$ part of a connection which is compatible with the holomorphic structure is not necessarily holomorphic.

However, if $\nabla = \nabla^{1,0} + \bar{\partial}_E$ is *flat* and compatible with the holomorphic structure, we can use $\nabla^2 = 0$ to conclude that for s a holomorphic section, we have $\bar{\partial}_E \circ \nabla^{1,0}(s) = -\nabla^{1,0} \circ \bar{\partial}_E(s) = 0$ and so $\nabla^{1,0}(s) \in \Omega^1(E)$. We therefore see that $\nabla^{1,0}$ is a holomorphic connection.

Hence given any smooth complex bundle with a flat connection, we can equally well think of it as a holomorphic bundle with a flat connection ∇ induced by a holomorphic connection $\nabla^{1,0}$.

It should be mentioned at this point that the property of admitting a holomorphic connection puts very strong and not completely understood constraints on the topology our bundle can have. As an example, we mention the following (according to [8]) long-standing open conjecture:

CONJECTURE 2.11. *If a holomorphic vector bundle E on a compact complex manifold admits a holomorphic connection, then it also admits a flat holomorphic connection.*

We will need this machinery in Chapter 4.

2.4. Factors of Automorphy

We shall briefly outline the theory of factors of automorphy of line bundles, which allows us to relate the Picard group of a complex manifold to a certain first group cohomology and will turn out very useful in later *examples*. We will follow the treatment of [9].

Let X be a complex manifold with a universal cover \tilde{X} . We saw how representations $\rho : \pi_1(X) \rightarrow \mathbb{C}^*$ define an action of this group on $\tilde{X} \times \mathbb{C}$, and how dividing out by this action defines the holomorphic vector bundle associated to the representation. Every line bundle arising in this way has locally constant transition functions.

To obtain more general holomorphic line bundles, we allow our transition function map $\rho(\gamma)$ to vary from point to point of \tilde{X} in a holomorphic manner and carry out a construction analogous to the one introduced last paragraphs. Writing $H_{\tilde{X}}^*$

for the multiplicative group of nowhere zero holomorphic functions on \tilde{X} , such ρ are then maps $\rho : \pi_1(X) \rightarrow H_{\tilde{X}}^*$ (using the currying convention). In order for the previous construction to go through smoothly, we certainly need that

$$\gamma(x, z) = (\gamma^{-1} \cdot x, \rho(\gamma)(x) \cdot z)$$

defines a group action of $\pi_1(X)$ on $\tilde{X} \times \mathbb{C}$, i.e. respects composition.

Topologically, the action by γ corresponds to lifting γ^{-1} . The above rule then defines an action if and only if the transition function $\rho(\gamma_1\gamma_2)(x) \in \mathbb{C}^*$ we obtain for a composite path at a point x agrees with the product $\rho(\gamma_2)(x) \cdot \rho(\gamma_1)(\gamma_2^{-1} \cdot x)$ we obtain by firstly walking along γ_2^{-1} and then along γ_1^{-1} . (\star)

The group $H_{\tilde{X}}^*$ is naturally a $\pi_1(X)$ -module under $\gamma \cdot f(x) = f(x \cdot \gamma^{-1})$, hence we can compute *group cohomology* using standard differential complex

$$0 \rightarrow C^0(\pi_1(X), H_{\tilde{X}}^*) \rightarrow C^1(\pi_1(X), H_{\tilde{X}}^*) \rightarrow \dots$$

Here $C^n(\pi_1(X), H_{\tilde{X}}^*)$ is the group of functions $(\pi_1(X))^n \rightarrow M$ and the differential is given by the usual rather unintuitive formula. For $\rho \in C^1(\pi_1(X), H_{\tilde{X}}^*)$, it turns out that this formula has geometric content since

$$d\rho(\gamma_2, \gamma_1)(x) = \rho(\gamma_2)(x) \cdot \rho(\gamma_1)(\gamma_2^{-1} \cdot x) \cdot \rho(\gamma_1\gamma_2)(x)^{-1}$$

equals 1 if and only if our transition function ρ satisfies (\star) . Write $Z^1(\pi_1(X), H_{\tilde{X}}^*)$ for the set of 1-cycles. The 1-boundaries $B^1(\pi_1(X), H_{\tilde{X}}^*)$ are then given by functions of the form $df(\gamma) = (\gamma \cdot f)f^{-1}$ for $f \in H_{\tilde{X}}^*$ and $\gamma \in \pi_1(X)$.

A similar check as before show that for such 1-cycles ρ , we can indeed define a vector bundle $E_\rho = \tilde{X} \times \mathbb{C} / \pi_1(X)$. In fact, this yields a group homomorphism

$$\phi : Z^1(\pi_1(X), H_{\tilde{X}}^*) \rightarrow H^1(X, \mathcal{H}_X^*)$$

to the holomorphic Picard group.

We will only state the following theorem, whose proof is easy but technical:

THEOREM 2.12. *The above group homomorphism $\phi : Z^1(\pi_1(X), H_{\tilde{X}}^*) \rightarrow H^1(X, \mathcal{H}_X^*)$ has kernel exactly equal to $B^1(X, \mathcal{H}_X^*)$.*

The induced homomorphism $\Phi : H^1(\pi_1(X), H_{\tilde{X}}^) \rightarrow H^1(X, \mathcal{H}_X^*)$ is therefore injective, and its image precisely consists of the holomorphic line bundles which pull back to trivial line bundles on the universal cover.*

REMARK 2.13. Not that if we are given a representation $\rho : \pi_1(X) \rightarrow \mathbb{C}^*$, then $\rho \in Z^1(X, H_{\tilde{X}}^*)$ is the function which maps $g \in \pi_1(X)$ to the constant function $\rho(g)$. We note that the notion of “line bundle associated to ρ ” we defined here agrees with the one in the previous paragraph.

The Twisted Abelian Hodge Theorem

Assume we are given a topological space X and a sheaf \mathcal{F} on X . When we compute sheaf cohomology, there are usually two tasks:

- (1) **Resolution-problem** Firstly, we have to find acyclic resolution ¹

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{G}_0 \xrightarrow{d_0} \mathcal{G}_1 \xrightarrow{d_1} \mathcal{G}_2 \xrightarrow{d_2} \dots$$

of our sheaf \mathcal{F} such that the associated sequence of global sections

$$0 \rightarrow G_0 \xrightarrow{d_0} G_1 \xrightarrow{d_1} G_2 \xrightarrow{d_2} \dots$$

is computationally tractable and consists of groups we understand well. This task is nontrivial since the canonical acyclic (in fact flasque) Godement resolution is unfortunately not suited to practical computations.

- (2) **Representative-problem** Assume we have solved this first task and found such a resolution, hence $H^n(X, \mathcal{F}) = \ker(d_n)/\text{im}(d_{n-1})$. If we want to understand what our cohomology classes “*really are*”, it is natural to ask if we can find *canonical representatives* for them, i.e. if given some $\alpha \in \ker(d_n)/\text{im}(d_{n-1})$, there is a distinguished representative in the (by assumption well-understood) group G_n .

Abelian Hodge theory gives an answer to the representation-problem for certain nice sheaves on certain nice manifolds, which will always be assumed to be connected. In this chapter, we will present several instances of this vague meta theorem, and show how one can combine two different such instances to obtain the (analytic) proof of the twisted abelian Hodge decomposition for compact Kähler manifolds X . We will show that this theorem can be naturally interpreted as a theorem for unitary representations of $\pi_1(X)$. Main references for this material are [44] and [7]. As before, our aim is to state clearly which structures are needed at which point.

3.1. Elliptic Operator Theory

The analytic proof of the abelian Hodge theorem we will present requires (besides many other more conceptual insights) a strong result from functional analysis. We will only state this so-called “*Finiteness Theorem*” since its proof required techniques of a flavour which is very distinct from the rest of this essay.

¹ By an acyclic resolution, we mean a resolution of our sheaf such that all components \mathcal{G}_i satisfy $H^p(X, \mathcal{G}_i) = 0$ for all $p > 0$.

3.1.1. Definition of Elliptic Operators. As before, let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. For $U \subset \mathbb{K}^r$ with coordinates z_1, \dots, z_r , a *differential operator*

$$P : \mathcal{C}^\infty(U, \mathbb{K}^n) \rightarrow \mathcal{C}^\infty(U, \mathbb{K}^m)$$

is a \mathbb{K} -linear sheaf morphism which is given on global sections by

$$\mathbf{f} = (f_1, \dots, f_n) \mapsto \left(\sum_{\alpha, i} P_{1i}^\alpha \frac{\partial^{|\alpha|} f_i}{\partial z^\alpha}, \dots, \sum_{\alpha, i} P_{mi}^\alpha \frac{\partial^{|\alpha|} f_i}{\partial z^\alpha} \right) = \sum_{\alpha} \left(\mathbf{P}^\alpha \frac{\partial^{|\alpha|}}{\partial z^\alpha} \right) \cdot \mathbf{f}$$

where all sums are finite and $\mathbf{P}^\alpha \in \mathcal{C}^\infty(U, M(m \times n, \mathbb{K})) = \text{Hom}_U(K^n, K^m)$ are matrices with smoothly varying entries.

Its *degree* d is the largest number $d = |\alpha|$ for which not all \mathbf{P}^α vanish.

To each operator of the form $D = \frac{\partial^k}{\partial z_{i_1} \dots \partial z_{i_k}}$, we can associate a section

$\tau(D) = \frac{\partial}{\partial z_{i_1}} \dots \frac{\partial}{\partial z_{i_k}} \in S^k(T_{U, \mathbb{K}})$ of the symmetric power of the tangent bundle.

By only considering forms of highest power d , this allows us to define the *symbol* of P as

$$\sigma(P) = \sum_{|\alpha|=d} \mathbf{P}^\alpha \tau \left(\frac{\partial^{|\alpha|}}{\partial z^\alpha} \right) \in \text{Hom}_U(K^n, K^m) \otimes S^k(T_{U, \mathbb{K}})$$

This symbol governs the coarse behaviour of the functions on which our operator vanishes.

The above definitions cover the case of trivial bundles over submanifolds of the form $U \subset \mathbb{K}^r$, but can be easily extended to the case of general real or complex vector bundles over general smooth manifolds:

DEFINITION 3.1. Let E and F be two smooth \mathbb{K} -bundles of rank n and m over the smooth manifold X of dimension r . A \mathbb{K} -linear morphism of sheaves

$$P : \mathcal{A}^0(E) \rightarrow \mathcal{A}^0(F)$$

is a *differential operator* if for every chart $\phi : U \rightarrow \phi(U) \subset \mathbb{K}^d$ and any two trivialisations $\psi_1 : E|_U \rightarrow U \times \mathbb{K}^n$, $\psi_2 : F|_U \rightarrow U \times \mathbb{K}^m$, the induced operator $\tilde{P} : \mathcal{C}^\infty(\phi(U), \mathbb{K}^n) \rightarrow \mathcal{C}^\infty(\phi(U), \mathbb{K}^m)$ is a differential operator in the above sense.

The *degree* k is the maximal degree occurring in any such trivialisation.

A standard check shows that the symbols $\sigma(\tilde{P})$ associated to the operators induced by trivialisations glue to give a global section $\sigma(P)$ of the bundle

$$\text{Hom}_X(E, F) \otimes S^k(T_{X, \mathbb{K}})$$

Notice that for any vector space V with basis z_1, \dots, z_r , we have an obvious map

$$\bullet : S^k V \times V^* \rightarrow \mathbb{K}$$

obtained by extending

$$\left(\frac{\partial}{\partial z_{i_1}} \dots \frac{\partial}{\partial z_{i_k}}, \sum_j f_j dz_j \right) \mapsto \sum_j f_j^k \left(dz_j \left(\frac{\partial}{\partial z_{i_1}} \right) \right) \dots \left(dz_j \left(\frac{\partial}{\partial z_{i_k}} \right) \right)$$

linearly (in the first entry).

If we are given a symbol $\sigma = \sigma(P) \in \text{Hom}_X(E_1, E_2) \otimes S^k(T_{X, \mathbb{K}})$ and a point Q in X , we can first evaluate to obtain

$$\sigma_Q = \sum f_i \otimes s_i \in \text{Hom}((E_1)_Q, (E_2)_Q) \otimes S^k(T_{X, \mathbb{K}})_Q$$

Sending $\alpha \in (T_{X,\mathbb{K}}^*)_Q$ to $\sum_i f_i(s_i \bullet \alpha) \in \text{Hom}((E_1)_Q, (E_2)_Q)$, we have defined a linear map $\sigma_Q(\alpha)$ (and the dependence on α is homogeneous of degree d).

We can now define the nice class of operators to which the finiteness theorem will apply later:

DEFINITION 3.2. Given bundles E, F as above, a differential operator

$$P : \mathcal{A}^0(E_1) \rightarrow \mathcal{A}^0(E_2)$$

with symbol σ is called *elliptic* if for all points $Q \in X$ and all *nonzero* cotangent vectors $\alpha \in (T_{X,\mathbb{K}}^*)_Q$, the linear map $\sigma_Q(\alpha)$ is injective.

EXAMPLE 3.3. Consider the smooth manifold $X = \mathbb{R}^n$ together with two copies $E = F = X \times \mathbb{R}$ of the trivial bundles. Then $\mathcal{A}(E) = \mathcal{C}^\infty(X)$ and the map

$$f \in \mathcal{C}^\infty(X) \mapsto \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}$$

defines a differential operator P of rank 2 with symbol $\sum_{i=1}^n \left(\frac{\partial}{\partial x_i} \right)^2$.

If we fix a point $Q \in X$ and a cotangent vector $\alpha = \sum_{i=1}^n a_i dx_i|_Q$ at Q , we have

$$\sigma_Q(\alpha) = \sum_{i=1}^n a_i^2 \in \text{Hom}(\mathbb{R}, \mathbb{R})$$

This is an injective homomorphism for nonzero α and hence P is elliptic.

3.1.2. Formal Adjoints of Differential Operators. Let (X, dV) be a compact smooth manifold together with a volume form, i.e. a nowhere vanishing form of top degree. Fix two Hermitian bundles $(E, \langle \cdot, \cdot \rangle_E), (F, \langle \cdot, \cdot \rangle_F)$ over X .

We can then define an inner product on the space $\mathcal{A}^0(E)$ of global smooth sections by

$$\langle \langle u, v \rangle \rangle_E := \int_M \langle u(x), v(x) \rangle_{E_x} dV(x)$$

An analogous construction holds over F .

DEFINITION 3.4. In the described situation, a formal adjoint for a differential operator $P : \mathcal{A}^0(E) \rightarrow \mathcal{A}^0(F)$ is a differential operator $P^* : \mathcal{A}^0(F) \rightarrow \mathcal{A}^0(E)$ such that for all global sections $u \in \mathcal{A}^0(E), v \in \mathcal{A}^0(F)$ we have:

$$\langle \langle Pu, v \rangle \rangle_F = \langle \langle u, P^*v \rangle \rangle_E$$

Using bump-functions, we notice that such an adjoint, if it exists, must be unique.

We are now faced with the problem of constructing P^* (see [44] for details). It is enough to construct adjoints locally on an open cover of X , and once we have done this, these sheaf-homomorphisms will glue by uniqueness. We can therefore assume that our bundles are trivial \mathbb{K} -bundles on an open connected subset $X \subset \mathbb{K}^r$.

Moser's theorem states that for any two volume forms dV_1, dV_2 on a compact connected oriented manifold X giving X the same volume, there is a diffeomorphism $f : X \rightarrow X$ with $dV_1 = f^*dV_2$.

Applying this to X (having euclidean coordinates dz_1, \dots, dz_r), we may therefore assume without restriction that $dV = dz_1 \wedge \dots \wedge dz_r$ is the Euclidean volume form.

From here, an application of Stokes' theorem allows us to explicitly construct the formal adjoint operator.

We just outlined the proof of the following theorem:

THEOREM 3.5. *Given a compact smooth manifold X with volume form dV and a differential operator $P : \mathcal{A}^0(E) \rightarrow \mathcal{A}^0(F)$ on Hermitian bundles E and F on X . Then there is a unique formal adjoint $P^* : \mathcal{A}^0(F) \rightarrow \mathcal{A}^0(E)$ for P .*

3.1.3. Finiteness Theorem. We finally have all necessary ingredients at hand to state the finiteness theorem, which will later fuel our proof of the abelian Hodge theorem:

THEOREM 3.6. (*Finiteness Theorem*)

Let E, F be two Hermitian bundles of same rank on some compact oriented manifold X with volume form dV . Assume $P : \mathcal{A}^0(E) \rightarrow \mathcal{A}^0(F)$ is an elliptic differential operator. Then:

- $\ker(P(X))$ is a finite-dimensional subspace of $A^0(E) = \mathcal{A}^0(E)(X)$.
- $\text{im}(P(X))$ is a closed subspace of $A^0(F)$ of finite codimension.
- We have a $\langle \cdot, \cdot \rangle_E$ -orthogonal decomposition of the smooth global sections of E into a direct sum

$$A^0(E) = \ker(P(X)) \oplus \text{im}(P^*(X))$$

Here P^* denotes the formal adjoint of P .

3.2. Hodge Isomorphism for de Rham cohomology

In this paragraph, we will answer the resolution- and the representation-problem mentioned initially in the following setting:

- Nice space: A compact oriented Riemannian manifold (X, g) of dimension m with natural volume form dV given locally by $dV = \sqrt{|g|} dx_1 \wedge \dots \wedge dx_r$.
- Nice sheaf: A local systems \mathcal{E}^∇ given as sheaf of horizontal sections of some flat Hermitian bundle $(E, \langle \cdot, \cdot \rangle, \nabla)$, i.e. a (complex) Hermitian bundle with a flat connection ∇ such that $d\langle u, v \rangle = \langle \nabla u, v \rangle_E + \langle u, \nabla v \rangle_E$.

3.2.1. Solution of the resolution problem. We write $\mathcal{A}^k(E)$ for the sheaf $\mathcal{A}_{X, \mathbb{C}}^k(E)$ of smooth E -valued complex k -forms. First, we find an acyclic resolution of our sheaf \mathcal{E}^∇ . If $U \subset X$ open, then $\mathcal{E}^\nabla(U) = \{f \in \mathcal{A}^0(E) \mid \nabla(U)(f) = 0\}$.

Hence we have a short exact

$$0 \longrightarrow \mathcal{E}^\nabla \xrightarrow{\iota} \mathcal{A}^0(E) \xrightarrow{\nabla} \mathcal{A}^1(E)$$

This is the point where the condition of ∇ being flat helps us, since it implies that the following complex of the canonical extensions of ∇ is a *differential complex*:

$$\mathcal{A}^0(E) \xrightarrow{\nabla} \mathcal{A}^1(E) \xrightarrow{\nabla} \mathcal{A}^2(E) \xrightarrow{\nabla} \mathcal{A}^3(E) \xrightarrow{\nabla} \dots$$

Locally on some open set U , we may pick horizontal sections e_i for which ∇ takes the simple form $\nabla(\sum \alpha_i e_i) = \sum d(\alpha_i) e_i$. A coordinatewise application of the Poincaré Lemma shows that ∇ -closed forms are locally ∇ -exact, and thus that the above resolution of sheaves is exact.

One can then show that since we have smooth partitions of unity, all sheaves $\mathcal{A}^k(E)$ are *fine* and hence acyclic.

Hence we have solved the resolution problem with the acyclic resolution

$$0 \longrightarrow \mathcal{E}^\nabla \xrightarrow{\iota} \mathcal{A}^0(E) \xrightarrow{\nabla} \mathcal{A}^1(E) \xrightarrow{\nabla} \mathcal{A}^2(E) \xrightarrow{\nabla} \mathcal{A}^3(E) \xrightarrow{\nabla} \dots$$

which shows that $H^k(X, \mathcal{E}^\nabla) \cong \ker(A^k(E) \xrightarrow{\nabla} A^{k+1}(E)) / \text{im}(A^{k-1}(E) \xrightarrow{\nabla} A^k(E))$.

3.2.2. Solution of the Representative Problem. Our aim is to choose exactly one differential form $\alpha \in A^k(E)$ in each cohomology class. The property which will distinguish this form from others is that it has “*minimal length*”, in a sense that we will specify now.

The Length of Differential Forms: We fix the degree k of our forms. The Riemannian metric g on the real tangent bundle $T_{X,\mathbb{R}}$ gives rise to a Hermitian metric h on the complex tangent bundle $T_{X,\mathbb{C}} = T_{X,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ in the obvious way.

This Hermitian metric in turn defines a canonical metric $\langle \cdot, \cdot \rangle$ on the complex cotangent bundle $T_{X,\mathbb{C}}^*$ in a way that makes the dual basis of an orthonormal basis is again orthonormal.

At each point $x \in X$, we have an oriented inner product space

$$((T_{X,\mathbb{C}}^*)_x, \langle \cdot, \cdot \rangle_x, dV(x))$$

and a hermitian space $(E_x, \langle \cdot, \cdot \rangle_{E,x})$. Applying the procedure described in A.4 on each fibre, we can obtain a Hermitian metric \prec, \succ on the complex bundle $\Lambda^k T_{X,\mathbb{C}}^* \otimes E$, a wedge-operator $\wedge : (\Lambda^k T_{X,\mathbb{C}}^* \otimes E) \times (\Lambda^l T_{X,\mathbb{C}}^* \otimes E^*) \rightarrow \Lambda^{k+l} T_{X,\mathbb{C}}^*$, and a Hodge- \star -operator

$$\star : \Lambda^k T_{X,\mathbb{C}}^* \otimes E \rightarrow \Lambda^{n-k} T_{X,\mathbb{C}}^* \otimes E^*$$

This allows us to define an honest Hermitian inner product $\langle\langle \cdot, \cdot \rangle\rangle$ on the space $A^k(E)$ of global E -valued smooth k -forms by:

$$\langle\langle u, v \rangle\rangle := \int_M \prec u(x), v(x) \succ dV(x) = \int_M u \wedge \star v$$

Hence we have given every E -valued form a real length, and it therefore makes sense to ask:

Does any cohomology class contain a unique form of minimal length?

A standard exercise in differential geometry shows that $\nabla = \nabla_E$ gives rise to a dual connection (respecting the dual metric) ∇_{E^*} on E^* . From the formal properties of the Hodge- \star -operator, it then follows that

$$\nabla^* = (-1)^{mk+1} \star_{E^*} \nabla_{E^*} \star_E$$

is a formal adjoint of ∇ for $\langle\langle \cdot, \cdot \rangle\rangle$.

We are now in a position to prove an equivalent condition for being of minimal length:

LEMMA 3.7. *A global closed form $\alpha \in A^k(E)$ is of minimal length in its cohomology class if and only if $\nabla\alpha = \nabla^*\alpha = 0$.*

PROOF. Let $\alpha \in A^k(E)$ be any closed form. Then a general element in the cohomology class of α can be written as $\alpha + \nabla\beta$ for some $\beta \in A^{k+1}(E)$. Then

$$\|\alpha + \nabla\beta\|^2 = \|\alpha\|^2 + \|\nabla\beta\|^2 + 2\text{Re}\langle\langle \nabla^*\alpha, \beta \rangle\rangle$$

If $\nabla^*\alpha = 0$, then it clearly has minimal length.

Conversely if α is of minimal length and β is any form, we expand $\|\alpha + t\nabla\beta\|^2$ and $\|\alpha + it\nabla\beta\|^2$ as above to conclude by minimality at $t = 0$ that $\langle\langle \nabla^*\alpha, \beta \rangle\rangle$ has vanishing real and imaginary part for all β , and hence that $\nabla^*\alpha = 0$. \square

We can encode the vanishing of ∇ and ∇^* in one single operator

$$\Delta = \nabla\nabla^* + \nabla^*\nabla$$

which is called the *Laplace-Beltrami Operator*. A standard calculation (see [44] for details) shows that Δ is an *elliptic operator*, and it is clear from the above that Δ is self-adjoint.

LEMMA 3.8. *For $\alpha \in A^k(E)$, we have:*

$$\Delta\alpha = 0 \iff \nabla\alpha = \nabla^*\alpha = 0$$

PROOF. One direction is clear. For the converse, simply observe that

$$0 = \langle\langle \Delta\alpha, \alpha \rangle\rangle = \langle\langle \nabla\nabla^*\alpha, \alpha \rangle\rangle + \langle\langle \nabla^*\nabla\alpha, \alpha \rangle\rangle = \|\nabla\alpha\|^2 + \|\nabla^*\alpha\|^2$$

□

We now give these forms of minimal length a new name:

DEFINITION 3.9. Forms $\alpha \in A^k(E)$ with $\Delta\alpha = 0$ are called *harmonic*, and we write $\mathcal{H}^k(E)$ for the subspace of harmonic forms.

We are now finally ready to state and prove the abelian Hodge theorem for Riemannian manifolds:

THEOREM 3.10. *Let (X, g) be a compact oriented Riemannian manifold and $(E, \langle \cdot, \cdot \rangle, \nabla)$ be a flat Hermitian bundle. Writing \mathcal{E}^∇ for the associated local system, we have an isomorphism of vector spaces*

$$H^k(X, \mathcal{E}^\nabla) \cong \mathcal{H}^k(E)$$

Moreover, this space is finite-dimensional.

PROOF. Since Δ is elliptic and self-adjoint, the finiteness theorem 3.1.3 gives a $\langle\langle \cdot, \cdot \rangle\rangle$ orthogonal decomposition:

$$A^k(E) = \mathcal{H}^k(E) \oplus \text{im}(\Delta)$$

Here we suppress the domains of the operators Δ which are clear from the context. Clearly $\text{im}(\nabla)$ and $\text{im}(\nabla^*)$ are orthogonal spaces since ∇ is a flat connection, and $\text{im}(\Delta) \subset \text{im}(\nabla) \oplus \text{im}(\nabla^*)$. But the spaces $\text{im}(\nabla)$ and $\text{im}(\nabla^*)$ are also perpendicular to $\mathcal{H}^k(E)$ and therefore contained in $\text{im}(\Delta)$. Hence $\text{im}(\Delta) = \text{im}(\nabla) \oplus \text{im}(\nabla^*)$ and we obtain a new orthogonal decomposition:

$$A^k(E) = \mathcal{H}^k(E) \oplus \text{im}(\nabla) \oplus \text{im}(\nabla^*)$$

We have $\ker(\nabla) = (\text{im}(\nabla^*))^\perp$. Taking the quotient of closed by exact k -forms therefore gives the space of harmonic forms, which is finite-dimensional by the finiteness theorem. □

3.3. Hodge Isomorphism for Dolbeault cohomology

We will now apply similar techniques as before to a different sheaf. Since we have gone into details in the previous section, we will only give the main steps.

- Nice space: A compact Hermitian manifold (X, h) with its natural volume form $dV = \frac{\omega^n}{n!}$ associated to the corresponding Riemannian metric g (notation as in A.3).
- Nice sheaf: A sheaf $\Omega^p(E)$ of holomorphic p -forms with values in some Hermitian holomorphic bundle $(E, \bar{\partial}_E, \langle \cdot, \cdot \rangle)$ (recall A.2). Here p is assumed to be fixed.

3.3.1. Solution of the Resolution Problem. The operator

$$\bar{\partial}_E : \mathcal{A}^{0,0}(E) \rightarrow \mathcal{A}^{0,1}(E)$$

has a natural extension to an operator

$$\bar{\partial}_E : \mathcal{A}^{p,q}(E) \rightarrow \mathcal{A}^{p,q+1}(E)$$

for all p, q by simply defining $\bar{\partial}_E(\sum_i \alpha_i \otimes e_i) = \sum_i (\bar{\partial}\alpha_i) \otimes e_i$.

An E -valued $(p, 0)$ form β is holomorphic if and only if $\bar{\partial}_E\beta = 0$. We therefore get a differential complex:

$$0 \xrightarrow{\bar{\partial}_E} \Omega^p(E) \xrightarrow{\bar{\partial}_E} \mathcal{A}^{p,0}(E) \xrightarrow{\bar{\partial}_E} \mathcal{A}^{p,1}(E) \xrightarrow{\bar{\partial}_E} \mathcal{A}^{p,2}(E) \xrightarrow{\bar{\partial}_E} \dots$$

There is a version of the Poincaré Lemma (see [21]) which implies as before that this complex of sheaves is in fact exact. Since the involved sheaves are fine, it therefore computes cohomology.

3.3.2. Solution of the Representation Problem. We apply the same techniques as before to the operator $\bar{\partial}_E$ and again aim to find forms of minimal length in each cohomology class (with respect to the inner product given by integration). The Hodge- \star -operator (defined using the hermitian metric h on $T_{X,\mathbb{C}}$) maps (p, q) -forms to $(n-p, n-q)$ -forms and allows us to define a formal adjoint $\bar{\partial}_E^* = -\star_E \bar{\partial}_E \star_E$ (here $\bar{\partial}_E^*$ is the analogue operator on the dual bundle). The new Laplace-operator is then $\bar{\Delta} = \bar{\partial}_E^* \bar{\partial}_E + \bar{\partial}_E \bar{\partial}_E^*$, and one can prove that it is self-adjoint and elliptic. The E -valued (p, q) -forms on which $\bar{\Delta}$ vanishes are called $\bar{\Delta}$ -harmonic and denoted by $\mathcal{H}_{\bar{\Delta}}^{p,q}(E)$.

The exact same proof as before then yields:

THEOREM 3.11. *Let (X, h) be a compact Hermitian complex manifold (X, h) and $(E, \bar{\partial}_E, \langle \cdot, \cdot \rangle)$ be a Hermitian holomorphic bundle. Then*

$$H^q(X, \Omega^p(E)) = \mathcal{H}_{\bar{\Delta}}^{p,q}(E)$$

and moreover, this space is finite-dimensional.

3.4. The Hodge-Decomposition for Kähler-manifolds

The two isomorphisms established in the previous sections are important results in their own right. However, their true strength only appears when they can be combined.

Assume again that we are given a compact complex manifold X with Kähler metric h (in the sense of A.3) on the complex tangent bundle $T_{X,\mathbb{C}}$, with an associated Riemannian metric g with natural volume form dV . Moreover, assume that we are also given a flat Hermitian bundle $(E, \nabla, \langle \cdot, \cdot \rangle)$ on X in the sense of 3.2. As ∇ is flat, can then take the $(0, 1)$ part $\bar{\partial}_E$ of ∇ to give us a holomorphic structure on E by A.2. Notice that ∇ is then the Chern-connection of (E, h) .

The Hodge- \star -operators defined in 3.2 for the Riemannian manifold (X, g) and in 3.3 for the Hermitian manifold (X, h) agree by definition. We obtain Laplacians Δ associated to ∇ and $\bar{\Delta}$ associated to $\bar{\partial}_E$.

Write $\mathcal{H}_\Delta^k(E)$ ($\mathcal{H}_\Delta^{p,q}(E)$) for the Δ -harmonic k -forms ((p,q) -forms), and $\mathcal{H}_{\bar{\Delta}}^{p,q}$ for the $\bar{\Delta}$ -harmonic (p,q) -forms.

By 3.2, we have an isomorphism $H^k(X, \mathcal{E}^\nabla) \cong \mathcal{H}_\Delta^k(E)$ for each integer k .

By 3.3, the inclusion also gives the following isomorphism for all p and q :

$$H^q(X, \Omega^p(E)) \cong \mathcal{H}_{\bar{\Delta}}^{p,q}(E)$$

The obvious question arises: How do these two decompositions relate? This is the point when the magic condition of h being Kähler comes in: Using the Kähler identities (which we shall not describe here, the very interesting details can be found in [7]), we can prove that

$$\Delta = 2\bar{\Delta}$$

This relation has two tremendous consequences:

- We have $\mathcal{H}_\Delta^k(E) = \bigoplus_{p+q=k} \mathcal{H}_\Delta^{p,q}(E)$. This follows from the fact that since $\bar{\Delta}$ preserves bidegrees, so does Δ , which shows that all components in the (p,q) -decomposition of a Δ -harmonic form must be Δ -harmonic.
- It is immediate that $\mathcal{H}_\Delta^{p,q}(E) = \mathcal{H}_{\bar{\Delta}}^{p,q}(E)$.

Write $H^{p,q}(X, \mathcal{E}^\nabla)$ for the image of $\mathcal{H}_\Delta^{p,q}(E)$ under the Hodge isomorphism for de Rham cohomology. We obtain a decomposition $H^k(X, \mathcal{E}^\nabla) = \bigoplus_{p+q=k} H^{p,q}(X, \mathcal{E}^\nabla)$.

One final worry enters at this stage: How does this decomposition change if we choose different metrics on X and E ?

The surprisingly nice answer is: not at all (as long as our metric on E is compatible with ∇). One can show with our tools by a few easy manipulations that $H^{p,q}(X, \mathcal{E}^\nabla)$ contains exactly the cohomology classes in $H^{p+q}(X, \mathcal{E}^\nabla)$ which can be represented by ∇ -closed forms of type (p,q) . This condition is independent of the chosen metrics.

We therefore obtain the abelian Hodge theorem:

THEOREM 3.12. *Let X be a compact Kähler manifold and (E, ∇) a flat bundle for which a compatible hermitian metric can be chosen. Then we have a decomposition (independent of the chosen metrics on X and E) of finite-dimensional vector spaces given by*

$$H^k(X, \mathcal{E}^\nabla) = \bigoplus_{p+q=k} H^{p,q}(X, \mathcal{E}^\nabla)$$

where $H^{p,q}(X, \mathcal{E}^\nabla)$ is the set of classes which can be represented by closed forms of type (p,q) .

Moreover, we have an isomorphisms of vector spaces

$$H^{p,q}(X, \mathcal{E}^\nabla) \cong H^q(X, \Omega^p(E))$$

Notice that, for a given metric on E , the map

$$\alpha \otimes e \mapsto \bar{\alpha} \otimes \langle \cdot, e \rangle$$

defines an anti-isomorphism $A^{p,q}(E) \rightarrow A^{q,p}(E^*)$. By checking that this anti-isomorphism maps ∇ -harmonic to ∇^* -harmonic forms, we can prove the *Hodge-symmetry*

$$\overline{H^{p,q}(X, \mathcal{E}^\nabla)} \cong H^{q,p}(X, \mathcal{E}^{*\nabla^*})$$

The Hodge theorem has tremendous consequences for the homology of compact Kähler manifolds which we shall describe in section B.5. But first, we will give a representation-theoretic interpretation.

3.5. Hodge theorem for unitary representations

We now want to apply the abelian Hodge theorem to representations of the fundamental group of some pointed Kähler manifold (X, x) . Recall from 2.3 that giving a representation $\rho : \pi_1(X, x) \rightarrow Gl_n(\mathbb{C})$ is equivalent to specifying a flat bundle (E^ρ, ∇^ρ) (which carries a canonical compatible holomorphic structure, compare 2.3.3.) We are therefore almost in a situation where we can apply the abelian Hodge theorem 3.12. The question whose answer determines if we are lucky is:

Can we find a hermitian metric on E^ρ which is compatible
with the connection ∇^ρ ?

We will now give a purely representation theoretic condition on ρ which determines the answer to this question:

LEMMA 3.13. *Let $\rho : \pi_1(X, x) \rightarrow Gl_n(\mathbb{C})$ be a representation. We can choose a hermitian metric $\langle \cdot, \cdot \rangle$ on E^ρ compatible with ∇^ρ if and only if ρ is conjugate to a unitary representation $\pi_1(X, x) \rightarrow U(n)$.*

PROOF. Assume first that we are given a compatible metric $\langle \cdot, \cdot \rangle$ on E . Given a point $P \in X$, we can find a unitary basis $e_1|_P, \dots, e_n|_P$ of $E|_P$. By Lemma 9.12 in [44], we can extend this basis to a local ∇ -horizontal frame e_1, \dots, e_n on an open connected neighbourhood of P . Then

$$d\langle e_i, e_j \rangle = \langle \nabla e_i, e_j \rangle + \langle e_i, \nabla e_j \rangle = 0$$

and hence the frame is unitary on the entire neighbourhood.

We thus trivialise our manifold in a way which makes the transition functions unitary. By the way in which we constructed our representation by path-lifting, this implies that the monodromy representation τ of this flat bundle preserves an inner product and is therefore conjugate to a unitary representation.

Conversely after replacing the representation by an isomorphic one, we may assume that ρ is unitary. Then the associated discrete bundle has unitary transition functions, and hence so does the associated flat bundle (E^ρ, ∇^ρ) . For each point P , we pick a trivialisation $\phi : E|_U \rightarrow U \times \mathbb{C}^n$ around P . For $v, w \in E|_P$, we then set

$$\langle v, w \rangle_{E_x} = \langle \phi(v), \phi(w) \rangle_{\mathbb{C}^n}$$

This definition is independent of the chosen trivialisation since the transition functions are unitary, and one checks easily that $\langle \cdot, \cdot \rangle_E$ defines a hermitian metric on E . Finally, it follows directly from the way in which we defined the associated connection ∇^ρ from the standard connection on the trivial bundles that $\langle \cdot, \cdot \rangle_E$ is compatible with ∇ . \square

We can therefore conclude the representation-theoretic Hodge theorem:

THEOREM 3.14. *Let $\rho : \pi_1(X, x) \rightarrow U(n)$ be a unitary representation of the fundamental group of a compact Kähler manifold. Then we have a decomposition of finite-dimensional vector spaces*

$$H^k(X, (\mathcal{E}^\rho)^{\nabla^\rho}) = \bigoplus_{p+q=k} H^{p,q}(X, (\mathcal{E}^\rho)^{\nabla^\rho})$$

where $H^{p,q}(X, (\mathcal{E}^\rho)^{\nabla^\rho})$ is the set of classes which can be represented by closed forms of type (p, q) .

Moreover we have isomorphisms of vector spaces

$$H^{p,q}(X, (\mathcal{E}^\rho)^{\nabla^\rho}) \cong H^q(X, \Omega^p(\mathcal{E}^\rho))$$

where the holomorphic structure on \mathcal{E}^ρ is given by $(\nabla^\rho)^{1,0}$.

We can also apply the Hodge-symmetry to see that, since the monodromy functor preserves duals, we have

$$\overline{H^{p,q}(X, (\mathcal{E}^\rho)^{\nabla^\rho})} = H^{q,p}(X, (\mathcal{E}^{\rho^*})^{\nabla^{\rho^*}})$$

3.6. First Applications

The most accessible applications all use the case of the trivial line bundle $E = \mathbb{C}$ on a compact Kähler manifold X . The Hodge theorem then specialises to a decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^q(X, \Omega^p)$$

and Hodge-symmetry becomes $\overline{H^{p,q}(X, \mathbb{C})} = H^{q,p}(X, \mathbb{C})$.

Compact smooth (and even topological) manifolds are well-known to have finitely generated fundamental groups (see chapter 5 in [18]). We can use the Hodge-symmetry and the initially mentioned theorem 1.3 of Serre to classify abelianisations of Kähler groups completely:

THEOREM 3.15. *An abelian group H is the abelianisation of some Kähler group G if and only if it is of the form $C \times \mathbb{Z}^{2n}$ for some finite abelian group C .*

PROOF. Take any complex elliptic curve inside $\mathbb{P}^n(\mathbb{C})$ which is automatically Kähler. Topologically, it is a 2-torus and hence has fundamental group \mathbb{Z}^2 . By Serre's theorem, every finite abelian group C is Kähler. Since the category of Kähler manifolds is closed under finite direct products and the functor π_1 respects these, we obtain that $C \times \mathbb{Z}^{2n}$ is a Kähler group for all n .

Conversely let X be a compact Kähler manifold with fundamental group G whose abelianisation is $H_1(X)$. Since fundamental groups of compact manifolds are finitely generated, the structure theorem tells us that $H_1(X)$ must have the form $H_1(X) = C_{d_1} \times \dots \times C_{d_n} \times \mathbb{Z}^r$

By the Universal Coefficient Theorem, we have a short exact sequence

$$0 \rightarrow \text{Ext}^1(H_0(X), \mathbb{C}) \rightarrow H^1(C, \mathbb{C}) \rightarrow \text{Hom}(H_1(X), \mathbb{C}) \rightarrow 0$$

Since $H_0(X)$ is a free group, it is also projective, hence $\text{Ext}^1(H_0(X), \mathbb{C}) = 0$.

Now we can use Hurewicz's theorem and the universal property of the abelianisation to conclude:

$$\mathbb{C}^r = \text{Hom}(H_1(X), \mathbb{C}) = H^1(X, \mathbb{C})$$

By Hodge symmetry, $H^1(X, \mathbb{C}) = H^{1,0}(X, \mathbb{C}) \oplus H^{0,1}(X, \mathbb{C})$ is the sum of two anti-isomorphic spaces, and hence r is even. \square

This theorem rules out, colloquially speaking, half of all groups as Kähler groups. In particular, an abelian group is Kähler if and only if it has even rank.

EXAMPLE 3.16. We see immediately that the free group F_k cannot be such a group for k odd.

But we can improve this result: Since finite étale coverings of compact Kähler manifolds inherit the Kähler structure, finite index subgroups of a Kähler group are Kähler. Hence:

COROLLARY 3.17. *Every finite index subgroup of a Kähler group has an abelianisation of even rank.*

EXAMPLE 3.18. It is not hard to see that for $n > 0$, F_n contains a finite index free subgroup of odd rank. Hence no nontrivial free group is Kähler.

EXAMPLE 3.19. More impressively, one can use this corollary to prove that the modular group $SL_2(\mathbb{Z})$ is not Kähler since it contains a finite index subgroup isomorphic to F_2 . Indeed, one can show with the Ping-Pong lemma that the so-called Sanov subgroup

$$\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$$

is free of rank 2 and has finite index.

Unfortunately, this example doesn't generalise to $n > 2$, and one needs the much more advanced technology of nonabelian Hodge theory to prove that these groups are not Kähler.

It is well-known that any finitely generated group is the fundamental group of some smooth compact orientable 4-manifold. This in particular shows with the above theorem that not every smooth compact orientable manifold supports a Kähler metric. We will use the following easy corollary of the general theory (see [7]) to construct an explicit example of such a manifold.

COROLLARY 3.20. *Any holomorphic differential form on a compact Kähler manifold X is d -closed.*

PROOF. If $\alpha \in \Omega_X^p$ is holomorphic, then $\bar{\partial}\alpha = 0$ is immediate. $\bar{\partial}^*\alpha$ has bidegree $(p, -1)$ and hence vanishes. Therefore $\Delta\alpha = \bar{\Delta}\alpha = 0$ by the Kähler identities. Since harmonic forms are closed, the claim follows. \square

EXAMPLE 3.21. We can now give an explicit example of a manifold which does not support a Kähler metric. Let $H_{\mathbb{C}}$ be the complex 3×3 Heisenberg group and $K \subset H$ be the group of all matrices whose entries are Gaussian integers. It is clear that K acts on H holomorphically and properly discontinuously, and therefore the quotient $X = K/H$ inherits the structure of a complex manifold of (complex) dimension 3. This manifold is called the *Iwasawa manifold*.

Consider the coordinates

$$(x, y, z) \mapsto \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

One verifies that $dz - xdy$ is a left invariant form on K and hence descends to a form on X which is not d -closed. Therefore X is not a Kähler manifold.

Coming back to the classification of Kähler groups, a natural question arises:

Is the condition of not having any finite index subgroups with odd rank
abelianisation sufficient?

Unfortunately, this is far too optimistic as we will show now. We recall several basic facts about group cohomology:

3.7. Group cohomological considerations

Given a group G , we have a left-exact functor from the abelian category of G -modules (which has enough injectives) to the category of abelian groups obtained by mapping a G -module M to the subgroup of all vectors fixed under the action of G . The associated right derived functors are called *group cohomology groups* and denoted by $H^k(G, M)$. We have an explicit way of computing these groups via the so-called *bar complex*, which we shall not describe further here.

Given a (discrete) group G , a classifying space for it is a connected CW-complex BG with $\pi_1(BG) = G$ and a contractible universal cover. Using Whitehead's theorem, one can show that the second condition is equivalent to BG being *aspherical*, i.e. having vanishing higher homotopy groups. One can prove that all groups have classifying spaces.

A G -module is the same as a representation of G in the category of abelian groups, and thus by our chapter on monodromy gives rise to a system on BG with values in this category. We quote the following theorem from [12]:

THEOREM 3.22. *Let M be a module over some group G with classifying space BG and corresponding local system \mathcal{M} . Then there is a natural isomorphism*

$$H^k(M, G) \cong H^p(BG, \mathcal{M})$$

If X is a finite path-connected CW complex with $G = \pi_1(X)$, we can build its classifying space by attaching cells and hence obtain a tautological continuous map $X \rightarrow BG$. One can then show that this map induces a homomorphism of cohomology algebras

$$H^*(G, \mathbb{Z}) \rightarrow H^*(X, \mathbb{Z})$$

which is an isomorphism in degree 0, 1 and injective in higher degrees (compare example 1.20 in [5]).

In [22], Johnson and Rees use Poincaré Duality and the Hard Lefschetz theorem to prove:

THEOREM 3.23. *Let G be a Kähler group acting trivially on \mathbb{R} such that $H^1(G, \mathbb{R})$ is nonzero. Then the cup product yields a nonzero map*

$$\Lambda^2 H^1(G, \mathbb{R}) \rightarrow H^2(G, \mathbb{R})$$

EXAMPLE 3.24. Let H be the (integral 3×3) Heisenberg group. It can be shown that every finite index subgroup has even rank. However, the cup-product map $\Lambda^2 H^1(G, \mathbb{R}) \rightarrow H^2(G, \mathbb{R})$ vanishes identically, and thus H is not Kähler.

3.7.1. On aspherical Kähler groups. Every compact connected Kähler manifold is canonically a Riemannian manifold, and also by compactness (geodesically) complete.

Such Riemannian manifolds fall into three main classes according to their sectional curvature: indefinitely curved, positively curved and nonpositively curved.

Myers theorem tells us that positively curved complete Riemannian manifolds have finite fundamental groups. Since Serre proved that any finite group is Kähler, this case is uninteresting for the Kähler-problem.

In the case of non-positive sectional curvature, we quote a central theorem from Riemannian geometry:

THEOREM 3.25. (*Cartan-Hadamard*) *The universal cover of connected (geodesically) complete Riemannian manifold X with non-positive sectional curvature is diffeomorphic to \mathbb{R}^n and thus contractible. Therefore X is aspherical.*

Hence rather than attempting to classify all Kähler groups, a more modest aim would be to classify all *aspherical* Kähler groups, i.e. fundamental groups of compact aspherical Kähler manifolds. In this case, the sheaf-cohomological constraints for local systems translate into group cohomological ones.

EXAMPLE 3.26. All compact Riemann surfaces of positive genus are aspherical since their universal cover is either the real or the hyperbolic plane, so contractible. Similarly, all higher-dimensional complex tori are aspherical.

We can now prove the following corollary of the representation-theoretic Hodge theorem, giving group-cohomological restrictions on aspherical Kähler groups:

THEOREM 3.27. *Let X be a compact connected aspherical Kähler manifold with fundamental group G and $\rho : G \rightarrow \mathbb{C}^n$ a self-dual (i.e. $\rho \cong \rho^*$) unitary representation giving \mathbb{C}^n the structure of a G -module. Then all groups $H^k(G, \mathbb{C}^n)$ are finite-dimensional and for k odd, $\dim^k(G, \mathbb{C}^n)$ is even.*

PROOF. Write \mathcal{E}^ρ for the corresponding local system and ∇^ρ for the induced flat connection on the bundle corresponding to the local system. Then the Hodge-theorem for representations gives a decomposition of finite-dimensional spaces

$$H^k(X, (\mathcal{E}^\rho)^{\nabla^\rho}) = \bigoplus_{p+q=k} H^{p,q}(X, (\mathcal{E}^\rho)^{\nabla^\rho})$$

and the Hodge-symmetry by self-duality becomes

$$\overline{H^{p,q}(X, (\mathcal{E}^\rho)^{\nabla^\rho})} = H^{q,p}(X, (\mathcal{E}^{\rho^*})^{\nabla^{\rho^*}}) = H^{q,p}(X, (\mathcal{E}^\rho)^{\nabla^\rho})$$

This implies the claim. \square

It is not clear to the author how valuable this corollary is in its full generality. For ρ the trivial representation, it just specialises to say that the rank of G is even.

The simplest type of nonabelian groups after free groups are in some sense *one-relator* groups, i.e. finitely presented group with one single relation. One can combine diagonalisation of idempotent complex matrices with Lyndon's theorem (see [24]) to see that $H^k(G, \mathbb{C}^n) = 0$ for such groups G , $k > 2$, and any representation of G on \mathbb{C}^n . By Poincaré Duality applied to the trivial representation on \mathbb{C} , this implies in particular that an aspherical manifold can only have such a *one-relator* group as fundamental group if its real dimension is at most 2. We therefore obtain (counting the point \mathbb{C}^0 as a Kähler manifold):

LEMMA 3.28. *The aspherical Kähler groups which are one-relator groups are precisely of the form*

$$\left\langle a_1, b_1, \dots, a_g, b_g \mid \prod_{i=1}^n [a_i, b_i] \right\rangle$$

for $g \geq 0$.

PROOF. Any one-relator aspherical Kähler group must be the fundamental group of a point or a curve, and therefore of the claimed form.

Conversely, for any $g > 0$, there is a compact Riemann surface of genus g for all g obtained as a quotient of the contractible hyperbolic plane, and since all compact Riemann surfaces are projective, this curve supports a Kähler metric. This proves that the above groups are indeed all aspherical Kähler. \square

Biswas and Mj in fact very recently completely classified general one-relator Kähler groups in (see [10]) (dropping the condition of being aspherical).

THEOREM 3.29. *(Biswas, Mj) A one-relator group G is Kähler if and only if it is finite cyclic or has the form*

$$\left\langle a_1, b_1, \dots, a_g, b_g \mid \left(\prod_{i=1}^g [a_i, b_i] \right)^N \right\rangle$$

We state a complete classification of aspherical Kähler manifolds whose fundamental group is virtually solvable (see [6]):

THEOREM 3.30. *Let X be a compact aspherical Kähler manifold with virtually solvable fundamental group. Then X is biholomorphic to a finite quotient of a complex torus.*

There are many more restrictive results on Kähler groups of this flavour, but we shall move on at this point the techniques of nonabelian Hodge theory.

The Unitary Nonabelian Hodge Theorem

The nonabelian Hodge theorem a milestone in the theory of complex geometry which took over 30 years to develop. It establishes a correspondence between certain representations of the fundamental group and holomorphic bundles with extra structure. As it is somehow hard to appreciate the depth of the statement in its most general form without knowing the very concrete special cases it comprises, we choose to follow the mathematical history and first present the unitary nonabelian Hodge theorem as the peak of a series of generalizations.

From now on, all representations and vector spaces will be assumed to be finite-dimensional, and all manifolds are connected and come equipped with a base point. We will only properly describe the correspondences as identifications of sets. However, many of the involved identifications actually hold on the level of *moduli spaces*, i.e. topological spaces (in our case) with additional structure whose points represent the objects we examine. Since the construction of these spaces is a difficult task in its own right, we shall not peruse this approach. In our representation-theoretic formulation of the abelian Hodge theorem, we have seen two identifications (here \sim denotes identification up to isomorphism):

- The bijection

$$\{\mathbb{C} - \text{representations of } \pi_1(X)\} / \sim \xrightarrow{F} \{\mathbb{C} - \text{vector bundles with flat connections}\} / \sim,$$

given by monodromy identifies (up to isomorphism) unitary representations exactly with flat bundles for which a compatible hermitian metric can be chosen.

- The correspondence

$$\{\mathbb{C} - \text{vector bundles with connection } D \text{ such that } (D^{0,1})^2 = 0\} / \sim$$

$$\begin{array}{c} G \\ \downarrow \end{array}$$

$$\{\text{Holomorphic bundles with compatible connection}\} / \sim$$

which identifies bundles equipped with flat connections with holomorphic bundles equipped with flat holomorphic connections (compare 2.3.3).

Applying first F and then G , we associate a holomorphic bundle to every representation. Moreover, we obtained an identification between unitary representations and flat holomorphic bundles for which a compatible metric can be chosen. Coming from the smooth angle, the holomorphic structure seems rather irrelevant, since it is just an extra datum that is uniquely determined by the rest. However, this is in some sense the wrong perspective from an algebraic-geometric point of view as we will explain now.

4.1. Motivation: The classification of holomorphic bundles

One fundamental aim in algebraic geometry is to classify algebraic vector bundles on algebraic varieties. Their sections can be thought of as generalised regular functions, and they have connections to various other areas such as number theory (e.g. via the class group) or Physics (e.g. via the Penrose transform in Yang-Mills theory). In this section, we will restrict ourselves to bundles on complex manifolds.

By theorem B.15, the classification of algebraic bundles on a complex projective variety is equivalent to the classification of holomorphic bundles. Notice that the holomorphic situation is very different from the topological case, where several very strong results are known (for example, topological bundles on complex curves are classified by rank and degree (defined below), and bundles of degree $r > n$ on \mathbb{P}^n split as direct sums of one rank n bundle and a trivial rank $(r - n)$ -bundle). The task of classifying all bundles is extraordinarily difficult, and has only been completed in very few cases. Before being able to state the complete classification for the Riemann sphere and complex elliptic curves, we need to introduce the definition of the *rank* and *degree* of a torsion-free coherent sheaf. This definition will remain highly relevant in the rest of this essay. The discussion we give here is by no means exhaustive and a lot more work has been done on bundles on $\mathbb{P}^n(\mathbb{C})$ for $n > 1$, see [28] for details.

Recall that a holomorphic bundle is *indecomposable* if it cannot be written as a sum of two proper holomorphic subbundles. If X is a connected compact complex manifold, Atiyah proved the following result known as Krull-Schmidt theorem (see [1]):

THEOREM 4.1. *(Krull-Schmidt) Every holomorphic vector bundle E on X decomposes as a finite direct sum of indecomposable subbundles. Moreover this decomposition is unique up to reordering. Such a decomposition is called Remak decomposition.*

4.2. Rank and Degree of Coherent Torsion-Free Sheaves

We will see that coherent sheaves are just vector bundles with singularities and use this to define their rank. Afterwards, we will extend the definition of the first Chern class. All coherent sheaves here are analytic.

4.2.1. Rank of Coherent Sheaves. Given a general coherent sheaf \mathcal{F} , we can define the following set of “evil” points:

DEFINITION 4.2. For a coherent sheaf \mathcal{F} on a complex manifold X , its *singularity set* is the set of points $x \in X$ such that the stalk \mathcal{F}_x is *not* free.

One can check that \mathcal{F} is locally free outside its singularity set. We quote the following lemma from [23]:

LEMMA 4.3. *The singularity set $S(\mathcal{F})$ of a coherent sheaf \mathcal{F} is a closed analytic set of codimension at least one.*

If \mathcal{F} is a torsion-free sheaf, the codimension of its singularity set is at least 2.

Hence coherent sheaves should be thought of as vector bundles which degenerate on certain closed proper analytic subsets.

On curves, we obtain:

COROLLARY 4.4. *Torsion-free sheaves on Riemann surfaces are always locally free, i.e. sheaves of sections of vector bundles.*

We can now define the rank of a coherent sheaf:

DEFINITION 4.5. The *rank* of a coherent sheaf \mathcal{F} is the dimension of the $(\mathcal{H}_X)_x$ -vector space \mathcal{F}_x at any point x not lying in the singularity set.

One checks that the rank is independent of the point x we picked (see [23] for details).

4.2.2. Rank of Coherent Sheaves. We will assume familiarity with the definitions of Chern and Euler-classes. From this, we will obtain the general definition in a step-by-step process of generalisation.

Topological interpretation of the first Chern class of line bundles. Let L be a complex line bundle on a compact complex manifold X of real dimension $2n$. By a *generic section* $\sigma : X \rightarrow L$, we mean a smooth section of L which is transversal to the zero section. Its zero locus Z then defines a homology class $[Z] \in H_{2n-2}(X, \mathbb{Z})$. By Poincaré Duality, there is a unique cohomology class $e(L) \in H^2(X, \mathbb{Z})$ representing integration over this zero locus Z , i.e. such that for any form $\alpha \in H^{2n-2}$, we have

$$\int_X e(L) \wedge \alpha = \int_Z \alpha|_Z$$

We call this class the *Euler class* of the line bundle E , and one can in fact prove that it is independent of the chosen generic section. It can be shown that under the given assumptions on X , this class is equal to the first Chern class $c_1(L)$ of L , i.e. the integral cohomology class in $H^2(X, \mathbb{Z})$ associated to $\mathcal{L} \in H^1(X, (\mathcal{A}_{X, \mathbb{C}}^0)^*)$ by the connecting map of the exponential sequence (recall convention A.1).

Degree of line bundles on a Riemann surfaces. In the case where X is a Riemann surface, the zero locus Z of a generic section is just collection of oriented points, and integrating the class $e(L) = c_1(L)$ over X simply counts the number of zeros (with orientation) of a generic section. This integer therefore measures in a specific sense how “twisted” L is, and we shall call it the *degree of L* :

$$\deg(L) = \int_X c_1(L)$$

We see immediately that trivial line bundles over Riemann surfaces have degree 0 as we can pick our generic section to be nowhere zero.

Degree of line bundles on Kähler manifolds. For complex line bundles L over compact complex manifolds X of real dimension $2n > 2$, the situation is more delicate: Here $c_1(L) = e(L)$ defines an integral cohomology class of degree 2, and since this class is not top-dimensional, integrating over it does not make sense anymore. However, if we assume that we are also given a Kähler form ω on X , then we can define the degree of L with respect to ω by first wedging $c_1(L)$ with ω^{n-1} and then integrating over X :

$$\deg_\omega(L) = \int_X c_1(L) \wedge \omega^{n-1}$$

Notice that if Z is a generic section of our bundle, then

$$\deg_\omega(L) = \int_Z \omega^{n-1}|_Z$$

Colloquially speaking, the $(\omega-)$ degree measures the “size” of the Kähler form on the zero locus of a generic section of the line bundle.

We will now extend this definition quite drastically to torsion-free coherent sheaves on X , i.e. coherent analytic sheaves for which all stalks are torsion-free modules over the respective stalks of \mathcal{H}_X .

The Determinantal Bundle of Torsion-Free Coherent Sheaves. In order to proceed, we need to associate a line bundle to every torsion-free coherent sheaf, called the *determinantal bundle*. Details can be found in [23]. If E is a holomorphic vector bundle of rank r on X , we define

$$\det(E) = \Lambda^r E$$

Here Λ^r is the extension of the top-exterior-power-operator on vector spaces to bundles. Note that this defines a (complex) line bundle.

One checks that if

$$0 \longrightarrow E_k \longrightarrow \dots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow 0$$

is an exact sequence of bundles, then the alternating product of the determinantal line bundles

$$\bigotimes_{i=0}^k \det(E_i)^{(-1)^i}$$

vanishes in the Picard group, i.e. is the trivial bundle. In particular

$$\det(E_0) = \bigotimes_{i=1}^k \det(E_i)^{(-1)^{i-1}}$$

Finally, assume we are given a general torsion-free coherent sheaf \mathcal{F} . For U a sufficiently small open set, we can always resolve \mathcal{F} by locally free sheaves \mathcal{E}_i corresponding to holomorphic bundles E_i :

$$0 \longrightarrow \mathcal{E}_k \longrightarrow \dots \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

We define the sheaf $\det(\mathcal{F})|_U$ on U to be the line bundle

$$\det(\mathcal{F})|_U = \bigotimes_{i=0}^k \det(E_i)^{(-1)^i}$$

With some work, it can then be shown that these local definitions glue and yield a global locally free sheaf $\det(\mathcal{F})$ corresponding to a globally defined line bundle. One checks that on locally free sheaves, \det agrees with the previous definition.

Degree of torsion-free coherent sheaves on Kähler manifolds. Looking at a holomorphic vector bundle or, more generally, at a torsion-free coherent sheaf \mathcal{F} through the prism of the determinantal operator allows us to define the (real) first Chern class of \mathcal{F} by

$$c_1(\mathcal{F}) = c_1(\det(\mathcal{F}))$$

If the sheaf \mathcal{F} lives on a compact complex manifold with specified Kähler form ω , we use this to define the ω -degree as :

$$\deg_{\omega}(\mathcal{F}) = \deg_{\omega}(\det(\mathcal{F})) = \int_X c_1(\mathcal{F}) \wedge \omega^{n-1} \in \mathbb{R}$$

This indeed extends the previous definition and shows that the ω -degree measures the “size” of the Kähler form on a generic zero locus of the determinantal bundle of \mathcal{F} . Note that the degree of bundles on Riemann surfaces is defined without a choice of Kähler-metric.

REMARK 4.6. Recall that the *Grothendieck group* $K(\text{Coh}(X))$ of a Kähler manifold (X, ω) is the quotient of the free group generated by all coherent sheaves quotiented out by the rule that if

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is a short exact sequence of coherent sheaves, then $\mathcal{F} = \mathcal{F}' + \mathcal{F}''$.

Since both the rank and the (ω -)degree respect short exact sequences (see [23]), we obtain a homomorphism $(\deg_{\omega}, \text{rk}) : K(\text{Coh}(X)) \rightarrow \mathbb{R} \times \mathbb{Z}$.

REMARK 4.7. We digress briefly and discuss the formula of Riemann-Roch and its extensions.

Given a holomorphic vector bundle E with associated sheaf \mathcal{E} on a compact complex curve X with genus g , the Riemann-Roch formula expresses the Euler-Poincaré-characteristic¹ of \mathcal{E} as

$$\chi(\mathcal{E}) = \deg(E) + \text{rk}(E)(1 - g)$$

It is possible to extend the definition of the usual *Chern character* to make sense on all coherent sheaves (see [29]).

Also, there is some distinguished power series (universal for all X and E) $P(x_1, x_2, \dots)$ (whose precise form shall not be stated here) such that for each complex manifold,

$$\text{Todd}(X) := P(c_1(T_X^{1,0}), c_2(T_X^{1,0}), \dots)$$

is a finite sum and hence a well-defined mixed cohomology class. We call this class the *Todd class* of X . This class can be thought of as a universal correction factor for the lack of commutativity between the first Chern class map from the Grothendieck group to the first cohomology and the pushforward of morphisms on both sides. A nice motivation of the precise form of the Todd-class can be found in [19].

In this situation, there is a generalisation of the above Riemann-Roch formula for holomorphic bundles on curves: The Hirzebruch-Riemann-Roch theorem allows us to compute the Euler-Poincaré-characteristic of a coherent sheaf \mathcal{F} on a compact complex manifold X in terms of the Chern character $ch(\mathcal{F})$ of \mathcal{F} and the Todd class $\text{Todd}(X)$ of our manifold X as:

$$\chi(\mathcal{F}) = \int_X [\text{Todd}(X)ch(\mathcal{F})]_{2n}$$

In the case where \mathcal{F} is the sheaf of sections of a holomorphic vector bundle, this formula has a far-reaching generalisation: The Atiyah-Singer index theorem.

¹The Euler-characteristic of a sheaf is the alternating sums of the cohomology dimensions, i.e. $\chi(\mathcal{F}) = \sum_i (-1)^i \dim(H^i(X, \mathcal{F}))$

4.3. Bundles on Curves

4.3.1. Vector bundles on the Riemann Sphere $\mathbb{P}^1(\mathbb{C})$. Recall that isomorphism classes of holomorphic line bundles on some variety X with tensor product as multiplication form the so-called *Picard group* $Pic(X)$, which can be seen to be isomorphic to $H^1(X, \mathcal{H}_X^*)$ via the exponential sequence.

By associating to each point in $\mathbb{P}^1(\mathbb{C})$ the line in \mathbb{C}^2 it encodes, we can define a holomorphic line bundle called the *tautological line bundle* $\mathcal{H}(-1)$. We write $\mathcal{H}(-d)$ for its d^{th} multiple so that $\mathcal{H}(1)$ is Serre's twisting sheaf .

Relying heavily on the Birkhoff-factorization of matrix-valued functions whose components are given by Laurent-series, Grothendieck proved in 1957 in [17]:

THEOREM 4.8. (*Birkhoff-Grothendieck*) *For every holomorphic vector bundle E on $\mathbb{P}^1(\mathbb{C})$, there are unique integers $d_1 \geq \dots \geq d_n$ such that*

$$E = \mathcal{H}(d_1) \oplus \dots \oplus \mathcal{H}(d_n)$$

Hence the only indecomposable bundles are line bundles, which are determined uniquely by their degree.

In the end of the aforementioned paper [17], Grothendieck conjectures that the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ is in fact the only complex projective manifold for which every holomorphic line bundle decomposes as a sum of line bundles. He verifies this fact for smooth projective curves and higher projective spaces $\mathbb{P}^n(\mathbb{C})$, $n > 1$.

According to [36], van de Ven proved this conjecture in a talk in 1962 by proving that the restriction of the tangent bundle of $\mathbb{P}^n(\mathbb{C})$ to our manifold does not split into line bundles, unless our variety is isomorphic to the $\mathbb{P}^1(\mathbb{C})$.

4.3.2. Vector bundles on Elliptic Curves. In the same year as Grothendieck classified the bundles of the Riemann sphere, Atiyah achieved a classification of all vector bundles on elliptic curves (see [2]).

THEOREM 4.9. (*Atiyah*) *Let X be a complex elliptic curve, fix natural numbers r and d , and write $Bun(r, d)$ for the set of isomorphism classes of indecomposable holomorphic bundles over X of dimension r and degree d .*

There is a natural identification $\phi_{r,d} : Bun(r, d) \rightarrow X$.

Moreover, the following diagram commutes for all r and d :

$$\begin{array}{ccc} Bun(r, d) & \xrightarrow{\det} & Bun(1, d) \\ \phi_{r,d} \downarrow & & \phi_{1,d} \downarrow \\ X & \xrightarrow{\cdot \gcd(r, d)} & X \end{array}$$

Here, we use the \mathbb{Z} -module structure on X inherited by the group law, and the map which builds the determinantal bundle as defined in 4.2.2.

4.3.3. Higher genus curves. To the best of our knowledge, the problem of classifying *all* holomorphic vector bundles on higher genus curves is incomparably harder and has not been solved thus far. However, there are strong results for a class of certain nice bundles. In the following, by a *curve*, we mean a compact (connected) Riemann surface, or equivalently a complex projective algebraic curve.

The wish to invoke representation theory motivates the following seemingly innocent question, which marks the starting point of a chain of generalisations which,

after about 30 years, lead to the Nonabelian Hodge theorem:

*Which holomorphic bundles on a curve come from complex representations of the fundamental group via monodromy?*²

By our work on monodromy, we know that a holomorphic bundle

$$(E, \bar{\partial}_E : \mathcal{A}^{0,0}(E) \rightarrow \mathcal{A}^{0,1}(E))$$

(compare section A.2) comes from a complex representation if and only if we can complete the holomorphic structure with a complementary operator

$$\partial_E : \mathcal{A}_{X,\mathbb{C}}^{0,0}(E) \rightarrow \mathcal{A}_{X,\mathbb{C}}^{1,0}(E)$$

to a flat connection $\nabla = \partial_E + \bar{\partial}_E$. If that is possible, then we have seen in 2.3.3 that ∂_E is automatically a holomorphic connection.

The magic which will allow us to give a converse to this over curves comes from dimension considerations: Since $\dim_{\mathbb{C}} X = 1$, the cotangent space is locally spanned by one holomorphic and one antiholomorphic frame, and it is hence immediate that $\mathcal{A}^{2,0} = \mathcal{A}^{0,2} = 0$.

Therefore, given any holomorphic connection ∂_E as above, we have for the associated smooth connection $\nabla = \partial_E + \bar{\partial}_E$:

$$\nabla^2 = (\partial_E + \bar{\partial}_E)^2 = \bar{\partial}_E \circ \partial_E + \partial_E \circ \bar{\partial}_E$$

This expression vanishes on holomorphic sections of E , and since the curvature arises from an underlying bundle homomorphism, we can conclude $\nabla^2 = 0$.

We have therefore proven:

LEMMA 4.10. *A holomorphic bundle $(E, \bar{\partial}_E)$ on a curve X comes from a representation if and only if E supports a holomorphic connection ∂_E .*

A naive guess would be that just like in the smooth case, holomorphic connections always exist. But as usual, the lack of holomorphic partitions of unity lets the usual proof break down. Therefore our task now is to find out which bundles on curves support holomorphic connections. Amazingly, there is a theory providing an answer to this question in a general setting.

The Atiyah Class. Let E be a holomorphic bundle over a compact complex manifold X . Following the treatment in [25] of the material found by Atiyah in [3], we will define a cohomology class which measures how much E fails to support a holomorphic connection. The ‘‘amazing idea’’ which allows us to do this is to reformulate the existence of a holomorphic section as a splitting of a certain short exact sequence.

DEFINITION 4.11. Given a vector bundle E with sheaf of sections \mathcal{E} on a complex manifold X . We define the *Jet bundle* to be the sheaf of abelian groups

$$\mathcal{J}(E) = \mathcal{E} \oplus \Omega_X^1(E)$$

and turn it into an \mathcal{H}_X -module by locally defining

$$f \cdot (e, \alpha \otimes e') = (fe, (f\alpha) \otimes e' + (\partial f) \otimes s')$$

²in the sense specified in the beginning of Chapter 4

We then have an obvious short exact sequence of \mathcal{H}_X -modules given by inclusion and projection, called the *Jet sequence* :

$$0 \longrightarrow \Omega_X^1(E) \longrightarrow \mathcal{J}(E) \longrightarrow \mathcal{E} \longrightarrow 0$$

A trivial check reveals that splittings $\phi : \mathcal{E} \longrightarrow \mathcal{J}(E)$ of this sequence are exactly morphisms of the form $(\text{id}_{\mathcal{E}}, \partial_E)$ for holomorphic connections ∂_E . Hence we can find a holomorphic connection if and only if the above extension is trivial, i.e. the extension-class $\alpha \in \text{Ext}^1(\mathcal{E}, \Omega_X^1(E))$ it defines is zero ³.

Straightforward homological algebra shows that for any locally free sheaf \mathcal{E} and any other sheaf \mathcal{F} of \mathcal{H}_X -modules, we have:

$$\text{Ext}^1(\mathcal{E}, \mathcal{F}) = H^1(X, \mathcal{F} \otimes \mathcal{E}^*)$$

Applying this to our specific situation, we obtain

$$\text{Ext}^1(\mathcal{E}, \Omega_X^1(E)) = \text{Ext}^1(\mathcal{E}, \Omega_X^1 \otimes \mathcal{E}) = H^1(X, \Omega_X^1 \otimes \mathcal{E} \otimes \mathcal{E}^*) = H^1(X, \Omega_X^1 \otimes \text{End}(\mathcal{E}))$$

DEFINITION 4.12. The *Atiyah class* of a holomorphic vector bundle E on a complex manifold X is the cohomology class $A(E) \in H^1(X, \Omega_X^1 \otimes \text{End}(\mathcal{E}))$ associated to the extension class α of the Jet sequence.

The following lemma is a tautology:

LEMMA 4.13. *A holomorphic bundle E on a complex manifold X admits a holomorphic connection if and only if its Atiyah-class $A(E)$ vanishes.*

On curves, we combine this with our previous discussion to obtain:

COROLLARY 4.14. *A holomorphic bundle E on a curve X comes from a representation if and only if its Atiyah-class $A(E)$ vanishes.*

This invariant seems utterly inaccessible at first glance. Amazingly, in the case where $E = L$ is a *line* bundle on a compact Kähler-manifold X , going through the trouble of spelling out the Čech cohomological description of

$$A(L) \in H^1(X, \Omega_X^1 \otimes \text{End}(\mathcal{L})) = H^1(X, \Omega_X^1) \subset H^2(X, \mathbb{C})$$

and comparing it with the first Chern class $c_1(L)$ gives the following surprising result (see [25] for details):

THEOREM 4.15. *If L is a line bundle on a compact Kähler-manifold X , then*

$$c_1(L) = \frac{i}{2\pi} A(L)$$

We close this digression on Atiyah classes with the following crucial theorem:

THEOREM 4.16. *Taking the Atiyah class $A(\cdot)$ is additive.*

PROOF. (Outline) An easy check reveals that the process of associating the jet-sequence to a given bundle E is in fact an exact functor, and therefore additive. This implies that $A(\cdot)$ is additive too. \square

Coming back to our initial question which bundles come from representations, we can use the additivity of $A(\cdot)$ to reduce our problem to the indecomposable case:

³Here Ext^i is the right derived functor of the functor $\text{Hom}(\mathcal{A}, \cdot)$, which associates abelian groups to \mathcal{H}_X -modules.

COROLLARY 4.17. *Let E be a vector bundle over a curve X with Remak decomposition $E \cong E_1 \oplus \dots \oplus E_k$. Then E comes from a representation if and only if all summands E_i do.*

PROOF. Combine additivity from theorem 4.16 with the Atiyah-class criterion 4.14. \square

We are now in a position to sketch Atiyah's proof of Weil's answer to our initial question:

THEOREM 4.18. (Atiyah-Weil) *A holomorphic vector bundle E on a curve X comes from a representation if and only if all the indecomposable components in its Remak decomposition have degree 0.*

PROOF. By the above remark, we may assume without restriction that E is indecomposable.

As a first step, Atiyah reformulates the condition of a bundle being indecomposable as a property of its endomorphism algebra:

LEMMA 4.19. *A holomorphic bundle F on a curve is indecomposable if and only if the \mathbb{C} -algebra $\text{End}(F)$ of bundle-endomorphisms is special, i.e. satisfies the following three conditions:*

- A has a unit element I
- The set of nilpotent elements N defines a subalgebra of A
- The algebra A splits as a \mathbb{C} -vector space as $A = \langle I \rangle \oplus N$

We will not spell out the proof of this Lemma which can be found in [3].

Hence the endomorphism algebra of our indecomposable bundle E is special.

Since the canonical bundle on a Riemann surface is just the cotangent bundle, Serre duality says after dualising a few times that for any holomorphic bundle F , we have

$$H^0(X, F^*)^* = H^1(X, \Omega_X^1 \otimes F)$$

and as $(\text{End}(E))^* \cong (E \otimes E^*)^* \cong E^* \otimes E^{**} \cong E^* \otimes E \cong \text{End}(E)$ is a canonical isomorphism, Serre duality gives an identification

$$\phi : H^1(X, \Omega_X^1 \otimes \text{End}(E)) \longrightarrow (H^0(X, \mathcal{E}nd(E)))^* = \text{End}(E)^*$$

Since $\text{End}(E)$ is special, a linear form on it vanishes if and only if it vanishes on the identity I and all nilpotent endomorphisms.

Atiyah concludes the proof by computing

$$\phi(A(E))(I) = -2\pi \deg(E)$$

and for $f : E \rightarrow E$ a nilpotent endomorphism, we have

$$\phi(A(E))(f) = 0$$

Therefore the Atiyah class vanishes if and only if $\deg(E) = 0$. \square

In particular, 1-dimensional representations induce exactly the group $\text{Pic}^0(X)$ of line bundles of degree 0.

We have just found a nice description of the essential image of the functor

$$HBun : \{\text{Complex representations of } \pi_1(X)\} \longrightarrow \{\text{Holomorphic bundles on } X\}$$

in the case where X is a Riemann surface. One question immediately arises:

Is $HBun$ “essentially injective”, i.e. does a holomorphic bundle in the image of M determine the representation it is induced by up to isomorphism?

It turns out that this is too optimistic as the following example shows:

4.4. An Instructive Example

We want to examine a complex torus $T = \mathbb{C}/\Lambda$ for some lattice $\Lambda = \langle 1, \lambda \rangle$. Similar arguments also hold over higher tori, but since the algebra gets harder, and we will focus on this case.

It is well-known that T has fundamental group $\pi_1(T) = \mathbb{Z}^2$, write a, b for the generators. Assume we are given two one-dimensional representations

$$\rho_1, \rho_2 : \mathbb{Z}^2 \rightarrow GL_1(\mathbb{C}) = \mathbb{C}^*$$

given by $\rho_i(a) = \alpha_i$ and $\rho_i(b) = \beta_i$.

We want to find a criterion that tells us when the two holomorphic line bundles E_1, E_2 induced by monodromy are isomorphic. Remembering our treatment of factors of automorphy in 2.4, we can consider the two representations ρ_1, ρ_2 as two cycles in $Z^1(\mathbb{Z}^2, H_{\mathbb{C}}^*)$, and they induce isomorphic bundles if and only if they differ by a boundary. This means that there is a holomorphic function $f : \mathbb{C} \rightarrow \mathbb{C}^*$ such that for all $\gamma = na + mb \in \mathbb{Z}^2$ and all $x \in \mathbb{C}^n$, we have:

$$\alpha_1^n \beta_1^m = \rho_1(\gamma) = \rho_2(\gamma) f(\gamma^{-1}x) \cdot f(x)^{-1} = \alpha_2^n \beta_2^m f(x - n - \tau m) \cdot f(x)^{-1}$$

Hence the bundles are isomorphic if and only if we can find a holomorphic function $f : \mathbb{C}^n \rightarrow \mathbb{C}^*$ satisfying the functional equation

$$f(x - n - \tau m) = f(x) \left(\frac{\alpha_1}{\alpha_2} \right)^n \left(\frac{\beta_1}{\beta_2} \right)^m$$

Assume f is such a function. Since \mathbb{C} is contractible, we have $H^1(\mathbb{C}, \mathbb{Z}) = 0$ and therefore the long exact sequence associated to the exponential sequence shows that we can write $f = \exp(2\pi i g)$ for some holomorphic function g . Choose s, t such that $\exp(2\pi i s) = \frac{\alpha_1}{\alpha_2}$ and $\exp(2\pi i t) = \frac{\beta_1}{\beta_2}$. For every fixed pair of integers n, m , the functional equation is satisfied if and only if

$$g(x - n - \tau m) - g(x) - sn - tm \in \mathbb{Z}$$

By varying x and using continuity, we see that we can write this difference as a fixed integer $K(n, m)$.

Hence $\frac{d}{dx}(g(x - n - \tau m)) = \frac{d}{dx}(g(x))$ everywhere, and so $\frac{d}{dx}(g(x))$ descends to a holomorphic function on the torus T . But any such function must be constant, so we can write $\frac{d}{dx}(g(x)) = -\lambda x$ for some $\lambda \in \mathbb{C}$. This implies that

$$g(x) = -\lambda x + c$$

for some constant $c \in \mathbb{C}$. From here, we immediately obtain:

$$\frac{\alpha_1}{\alpha_2} = \exp(2\pi i(g(-1) - g(0))) = \exp(2\pi i\lambda)$$

$$\frac{\beta_1}{\beta_2} = \exp(2\pi i(g(-\tau) - g(0))) = \exp(2\pi i\lambda\tau)$$

Conversely, it is clear that if $\frac{\alpha_1}{\alpha_2}$ and $\frac{\beta_1}{\beta_2}$ relate in this way, then their representations differ by a boundary and hence induce isomorphic bundles. We have proved:

THEOREM 4.20. *Let ρ_1, ρ_2 be two one-dimensional representations of the fundamental group $\mathbb{Z}^2 = \langle a, b \rangle$ of $T = \mathbb{C}/\langle 1, \tau \rangle$ with $\rho_i(a) = \alpha_i$ and $\rho_i(b) = \beta_i$.*

Then the associated holomorphic bundles E_{ρ_1} and E_{ρ_2} are isomorphic if and only if there is a number $\theta \in \mathbb{C}$ such that

$$\begin{aligned}\alpha_1 &= \alpha_2 \exp(\theta) \\ \beta_1 &= \beta_2 \exp(\tau\theta)\end{aligned}$$

Since one-dimensional representations are isomorphic if and only if they are equal, we have in particular found many negative examples answering the question raised immediately before this example:

COROLLARY 4.21. *The functor $HBun$ which associates a holomorphic bundle to a complex representation maps nonisomorphic representations to isomorphic bundles.*

Recall that we could fix the deficiency of the smear functor in our initial chapter on monodromy by keeping track of more data and hence turning it into an equivalence of categories. Using different methods, nonabelian Hodge theory will analogously fix the functor $HBun$ by keeping track of more data. But before we describe this general very beautiful answer, we examine our example further.

4.4.1. Comparison of Moduli Spaces. We will now indicate why this failure of injectivity should have been expected from a moduli-theoretic viewpoint. Since the general theory of these spaces is hard and technical, we will not give a proper treatment here. A detailed exposition can be found in [39] and [40].

It is easy to define the moduli space of complex 1–dimensional representations of a finitely presented group rigorously since no two distinct representations are isomorphic.

DEFINITION 4.22. Let $G = \langle a_1, \dots, a_n | R_1(a_1, \dots, a_n), \dots, R_n(a_1, \dots, a_n) \rangle$ be a finitely presented group. Then the set

$$\begin{aligned}\{z_1, \dots, z_n \in (\mathbb{C}^*)^n \mid \forall i : R_i(z_1, \dots, z_n) = 1\} \\ = \{z_1, \dots, z_n \in \mathbb{C}^n \mid \forall i : R_i(z_1, \dots, z_n), z_1 \cdot \dots \cdot z_n \neq 0\}\end{aligned}$$

is naturally a complex affine variety (as a hypersurface of a Zariski-closed subset of \mathbb{C}^n). Its points represent representations of G into \mathbb{C}^* , and distinct points represent distinct representations. This allows us to call this variety the *Moduli space of 1-dimensional representations*.

The philosophy of nonabelian Hodge theory is that equivalences of categories should give rise to homeomorphisms on the level of Moduli spaces.

We immediately see that the Moduli space of representations of $\pi_1(T) = \mathbb{Z}^2$ of a torus is homeomorphic (in the usual topology) to $\mathbb{C}^* \times \mathbb{C}^*$. As a smooth manifold, \mathbb{C}^* is isomorphic to $S^1 \times \mathbb{R}$ via the exponential map, and applying this to both copies, we see that the moduli space of representations is homeomorphic to $S^1 \times S^1 \times \mathbb{C}$.

On the other hand, Atiyah’s theorem in 4.3.2 suggests that the Moduli space of line bundles of degree zero should be the same as the torus itself, i.e. homeomorphic to $S^1 \times S^1$.

Hence intuitively, the above result should have been expected: The functor associating line bundles of degree zero to 1-dimensional complex representations has \mathbb{C} “as a fibre”.

4.4.2. The theorem of Narasimhan-Seshadri on Elliptic Curves. We have understood the image and the “fibres” of the functor $HBun$ from the 1–dimensional representations of the fundamental group of an elliptic curve to line over this curve. The next natural question is:

Is there a natural choice of representation in each fibre of $HBun$?

Amazingly, the positive answer is obvious from our previous work:

COROLLARY 4.23. *For every holomorphic line bundle L on the torus $T = \mathbb{C}/\langle 1, \tau \rangle$ of degree 0, there is one and only one unitary representation $\rho : \mathbb{Z}^2 \rightarrow S^1$ which induces L .*

Hence the functor $HBun$ induces a bijection between (isomorphism classes of) 1–dimensional unitary representations and isomorphism classes of holomorphic line bundles of degree 0.

PROOF. Existence follows by the theorem of Atiyah-Weil 4.18 as we can first pick any inducing representation and then modify it suitably using the above results. For uniqueness, assume ρ_1, ρ_2 are unitary representations mapping a to α_i and b to β_i . Write $\tau = u + iv$ with $v \neq 0$. By the above theorem, we can find a complex number $\theta = r + is$ such that

$$\frac{\alpha_1}{\alpha_2} = \exp(r) \exp(is)$$

$$\frac{\beta_1}{\beta_2} = \exp(ur - sv) \exp(i(us + rv))$$

Taking the modulus of both equations gives $r = 0 = ur - sv$, which implies that $\theta = 0$ and thus $\rho_1 = \rho_2$. \square

This should be very reassuring, since the Moduli space of one-dimensional unitary representations (with the complex subspace topology inherited from the Moduli space of one-dimensional representations) is homeomorphic to $S^1 \times S^1$, i.e. also to the moduli space of degree 0 line bundles.

4.5. From Narasimhan-Seshadri to Kobayashi-Hitchin

In the previous example, we proved that for elliptic curves, 1-dimensional unitary representations and line bundles of degree 0 are equivalent notions. The question of how this correspondence generalises to higher genus curves and higher rank representations is answered by the theorem of Narasimhan-Seshadri from 1965 in [26].

In order to state it, we need the following notions of slope and (semi-)stability.⁴

DEFINITION 4.24. The *slope* $\mu(\mathcal{F})$ of a torsion-free coherent sheaf \mathcal{F} on a compact Kähler manifold X with chosen Kähler form ω is defined to be

$$\mu(\mathcal{F}) = \frac{\deg_{\omega} \mathcal{F}}{\text{rk } \mathcal{F}}$$

DEFINITION 4.25. A holomorphic vector bundle E with sheaf of sections \mathcal{E} on a compact Kähler manifold is semistable if for every (automatically torsion-free) coherent subsheaf F with $0 < \text{rk}(F) < \text{rk}(\mathcal{E})$, we have:

$$\mu(\mathcal{F}) \leq \mu(\mathcal{E})$$

⁴This definition of stability actually fits into the bigger picture of Mumford’s geometric invariant theory, which we will not pursue further here.

The bundle is stable if, under the same assumptions, this inequality is strict.

Here $\deg = \deg_\omega$ denotes the degree with respect to the chosen Kähler form ω as motivated and defined in 4.2. Recall that for curves, the degree is independent of the chosen Kähler metric.

REMARK 4.26. One can show that in order to test stability of a bundle, it is enough to restrict attention to coherent subsheaves with torsion-free quotient sheaves (compare [35]).

Recall that a vector bundle E is *simple* if every holomorphic section of $\text{Hom}(E, E)$ is a scalar multiple of the identity endomorphism. We quote the following important property of stable bundles (see Corollary 7.14 in [23]):

LEMMA 4.27. *Every (ω) -stable bundle over a compact Kähler manifold (M, ω) is simple.*

It is easy to see that every simple bundle is indecomposable.

At this point, we should certainly mention the following important filtration (see [23]) relating general torsion-free coherent sheaves to semistable sheaves:

THEOREM 4.28. (*Harder-Narasimhan filtration*) *For every torsion-free coherent sheaf \mathcal{F} on a compact manifold X with chosen Kähler form ω , there is a unique filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_n = \mathcal{F}$$

by subsheaves such that for all $0 \leq i \leq n-2$, the quotient sheaf $\mathcal{F}_{i+1}/\mathcal{F}_i$ is the maximal (ω) -semistable subsheaf of $\mathcal{F}/\mathcal{F}_i$.

Finite-dimensional unitary representations are always semisimple, i.e. split as a direct sum of finitely many irreducible representations. Moreover, direct sums of unitary representations are unitary. Since monodromy respects direct sums of representations, we can therefore assume from now on that our unitary representations are irreducible.

For an elliptic curve, we can decompose any representation $\rho : \mathbb{Z}^2 \rightarrow \text{Gl}_n(\mathbb{C})$ into a sum of 1-dimensional components by simultaneously diagonalising the commuting images of the two generators. Hence we are reduced to the 1-dimensional case which we have solved in 4.4.2. We will therefore assume from now on that the genus g of our curve is bigger than 1.

In this situation, we have the following generalisation:

THEOREM 4.29. (*Narasimhan-Seshadri*) *Let X be a compact Riemann surface of genus $g > 1$. Then the functor $H\text{Bun}$ induces a bijection between conjugacy classes of irreducible unitary representations of $\pi_1(X)$ and isomorphism classes of stable holomorphic vector bundles on X of degree 0.*

REMARK 4.30. Since coherent subsheaves of bundles on curves are automatically locally free, corollary 4.4 shows that a bundle on a curve is stable if and only if the relevant inequality holds on all proper subbundles.

REMARK 4.31. One can both sides of the bijection a geometric structure, and hence turn them into Moduli spaces. Once this is done, the bijection is actually a homeomorphism.

About 18 years after the original proof presented in [26], Donaldson achieved a new proof of this theorem with differential-geometric means in [14], inspired by work of Atiyah and Bott in [4]. We decided to only present Donaldson's approach, since it is the one which will generalise later.

By Lemma 3.13 and our work on flat connections, a holomorphic bundle $(E, \bar{\partial}_E)$ comes from a unitary representation if and only if we can choose a hermitian metric on E such that the Chern connection ∇ is flat.

To answer the question when we can choose such a metric, it is helpful to think of these metrics as special cases of the following more general notion:

DEFINITION 4.32. Let E be a holomorphic bundle on some compact Kähler manifold (X, ω) and let Λ be the formal adjoint of $\omega \wedge (\cdot)$. A Hermitian metric h on E with Chern connection ∇ and curvature R_∇ is a *Hermite-Einstein metric* if there is a real constant $\lambda \in \mathbb{R}$ such that

$$i\Lambda R_\nabla = \lambda \text{id}_E$$

Using a Kähler identity for Λ (see [21]), this can be seen to be equivalent to the following equation of bundle homomorphisms $E \rightarrow \Omega^n(E)$:

$$iR_\nabla \wedge \omega^{n-1} = \frac{\lambda}{n} \omega^n \text{id}_E$$

According to [35], these metrics have been introduced to “give a differential-geometric interpretation of stability”. We should think of them as “nice” hermitian metrics whose Chern connection is not necessarily flat, but whose central curvature has a very controlled behaviour.

REMARK 4.33. Notice that this definition depends on both the Kähler metric ω and the holomorphic structure of our bundle E .

We firstly note that the constant λ is actually unique and essentially given by the slope $\mu(E)$ of our vector bundle:

LEMMA 4.34. *Given a Hermite-Einstein metric h on a bundle E with constant λ as above, we have*

$$\lambda = \frac{2\pi n}{\int_X \omega^n} \cdot \frac{\text{deg}_\omega(E)}{\text{rk}(E)} = \frac{2\pi}{(n-1)!} \cdot \frac{\mu(E)}{\text{Vol}(X)}$$

PROOF. Take traces of the second formulation of the definition of a Hermite-Einstein metric, integrate over X and use the well-known identity $c_1 = \frac{i}{2\pi} \text{tr} R_\nabla$ (compare [21]). \square

We ask ourselves how many distinct Hermite-Einstein metrics can occur. If $\lambda \in \mathbb{R}_{>0}$ and h is a Hermite-Einstein metric, then λh is also a Hermite-Einstein metric. For simple bundles, this is all that can happen (see [35]):

LEMMA 4.35. *If h_1, h_2 are two Hermite-Einstein metrics on a simple bundle E , then there exists some $\lambda \in \mathbb{R}_{>0}$ such that $h_1 = \lambda h_2$.*

The following definition is *not* standard terminology:

DEFINITION 4.36. A holomorphic vector bundle on a compact manifold with chosen Kähler form is *Kobayashi-Hitchin* if it is indecomposable and admits a Hermite-Einstein metric.

We are now able to state the first reduction in Donaldson's proof:

LEMMA 4.37. *Let X be a compact Riemann surface of genus $g > 1$. In order to prove the theorem of Narasimhan-Seshadri, it is sufficient to prove that holomorphic bundles on compact Riemann surfaces are Kobayashi-Hitchin if and only if they are stable.*

PROOF. We firstly need to prove that $HBun$ maps irreducible unitary representations to stable holomorphic bundles of degree 0. The vanishing degree is immediate since the connection is flat. We will skip the proof that holomorphic bundles associated to irreducible unitary representations of $\pi_1(X)$ of compact complex manifolds are indecomposable, see remark before Proposition 4.3 in [27]. Since our representation is unitary, we get an associated hermitian metric h whose flat Chern connection has our original representation as monodromy. This metric is clearly Hermite-Einstein, and hence by indecomposability, our bundle is Kobayashi-Hitchin, so stable.

To show that every stable bundle of degree 0 arises from an irreducible unitary representation, we use that it is Kobayashi-Hitchin, so it is irreducible and we can pick a Hermite-Einstein metric h . Since the degree of the bundle is zero, the constant λ vanishes. By the second formulation of the definition of Hermite-Einstein metrics, this implies that for the Chern-connection ∇ , we have

$$iR_{\nabla} \wedge \omega^{1-1} = iR_{\nabla} = 0$$

and hence our Chern-connection is flat, so is induced by a representation ρ . This representation is isomorphic to a unitary one since our connection is compatible with the hermitian metric h . Irreducibility of ρ follows by indecomposability of our bundle.

Finally, we want to show that if two irreducible unitary representations ρ_1, ρ_2 induce the same stable holomorphic bundle E , they must be isomorphic. Let h_1, h_2 be the associated metrics whose Chern connections ∇_1, ∇_2 have ρ_1, ρ_2 as monodromy. As shown above, the bundle E is stable and therefore simple (see 4.27). This implies by 4.35 that $h_1 = \lambda h_2$ for some positive real λ . From here, one can compute that ∇_1 is a Chern connection for h_2 and hence $\nabla_1 = \nabla_2$. This implies that $\rho_1 \cong \rho_2$. \square

In order to finish the proof of Narasimhan-Seshadri, Donaldson used analytic techniques to prove that a holomorphic vector bundle over a curve is Kobayashi-Hitchin if and only if it is stable.

The natural aim for generalisation makes us ask the following question:

For which compact Kähler-manifolds X is a bundle Kobayashi-Hitchin if and only if it is stable?

The conjecture that this holds for all compact Kähler manifold is known as the Kobayashi-Hitchin conjecture (expressed in 1980). We briefly outline its history, following [35].

Donaldson proved this result first over surfaces in 1985 (see [15]). One year later, Uhlenbeck and Yau extended the theorem to arbitrary compact Kähler manifolds. The fact that Kobayashi-Hitchin bundles have to be stable is the easier direction and has been originally proved by Kobayashi and simplified by Luebcke

(see theorem V.8.3 in [23]). The proof of the other direction of this theorem involves advanced analytic results (e.g. Uhlenbeck compactness), and a description of this proof would certainly exceed the limits of this essay.

Hence we have the following crucially important so-called Kobayashi-Hitchin correspondence :

THEOREM 4.38. *(Uhlenbeck, Yau) Let E be a holomorphic bundle on a compact Kähler manifold (X, ω) . Then E is Kobayashi-Hitchin if and only if it is stable.*

To state this result for general holomorphic bundles, we introduce new notation:

DEFINITION 4.39. A holomorphic vector bundle on a compact Kähler manifold is called *polystable* if it is the direct sum of stable holomorphic bundles of same slope.

The more general Kobayashi-Hitchin correspondence then reads:

THEOREM 4.40. *Let E be a holomorphic bundle on a compact Kähler manifold (X, ω) . Then E admits a Hermite-Einstein metric if and only if E is polystable.*

4.6. The Unitary Nonabelian Hodge Theorem

We now want to generalise the theorem of Narasimhan-Seshadri to higher-dimensional manifolds by applying Donaldson's strategy to the Kobayashi-Hitchin correspondence. However, if we simply try to imitate the above proof, we run into difficulties since the fact that the constant λ of a Hermite-Einstein metric vanishes does *not* imply that its Chern connection ∇ is flat (but only that $R_\nabla \wedge \omega^{n-1} = 0$, where n is the dimension of our manifold X).

After studying the literature for a bit, we find the following rescuing classical theorem (see for example theorem 4.11 in [23]), which tells us that flatness of Chern connections of Hermite-Einstein metrics is governed by the intersection behaviour of the first two Chern classes with the Kähler form:

THEOREM 4.41. *Let E be a holomorphic bundle with a Hermite-Einstein metric h on some compact Kähler manifold (X, ω) of dimension n . Write ∇ for the associated Chern connection.*

Then ∇ is flat if and only if $\deg(E) = 0$ and $\int_X ch_2(E) \wedge \omega^{n-2} = 0$. Here and in the whole following text, we use the convention that $\omega^{-1} = 0$.

Here $ch_2(E) = \frac{1}{2}(c_1(E) - 2c_2(E))$ is the second Chern character.

Combining this result with the Kobayashi-Hitchin correspondence, we can now prove the higher-dimensional Narasimhan-Seshadri theorem, which will be the unitary version of the nonabelian Hodge theorem :

THEOREM 4.42. *Let X be a compact Kähler manifold. Then functor $HBun$ induces a bijection between*

- conjugacy classes of irreducible unitary representations of $\pi_1(X)$ and
- isomorphism classes of stable holomorphic vector bundles E on X with

$$\begin{aligned} \deg(E) &= 0 \\ \int_X ch_2(E) \wedge \omega^{n-2} &= 0 \end{aligned}$$

PROOF. Parallel to the proof of lemma 4.37 using theorem 4.40 and 4.41. \square

Using standard properties of the monodromy functor and the fact that all unitary representations are semisimple, we can extend this result to obtain:

THEOREM 4.43. *Let X be a compact Kähler manifold. Then $HBun$ defines a bijective correspondence between*

- *conjugacy classes of unitary representations of $\pi_1(X)$ and*
- *isomorphism classes of polystable holomorphic vector bundles E on X with*

$$\begin{aligned} \deg(E) &= 0 \\ \int_X ch_2(E) \wedge \omega^{n-2} &= 0 \end{aligned}$$

The General Nonabelian Hodge Theorem

The aim of this chapter is to extend the beautiful correspondence between unitary representations and certain holomorphic bundles we established in the previous chapter to general semisimple representations. Significantly different ideas are needed in order to handle the nonunitary behaviour.

We will start this chapter by describing a natural, at this stage purely speculative, nonabelian analogue of the abelian Hodge decomposition. Following the principle of *analogy of proportion*, this will inspire us by telling us where the objects of our general correspondence should live, and by hinting at its proof.

Unless mentioned otherwise, bundles in this chapter are understood to be complex bundles.

5.1. Inspiration from the Abelian Case

Recall that for a compact Kähler manifold X , the abelian Hodge theorem for the trivial complex line bundle gives us a decomposition (compare the proof of 3.15)

$$\mathrm{Hom}(\pi_1(X), (\mathbb{C}, +)) = H^1(X, \mathbb{C}) = H^0(X, \Omega_X^1) \oplus H^1(X, \mathcal{H}_X)$$

We can therefore interpret the abelian Hodge theorem as a correspondence between *additive* one-dimensional representations and pairs of global holomorphic 1-forms and cohomology classes in $H^1(X, \mathcal{H}_X)$ (which can of course be thought of as harmonic forms of type $(1, 0)$).

5.1.1. Nonabelian Sheaf Cohomology. The usual definition of sheaf cohomology relies on the fact that our sheaves are sheaves of abelian groups.

However, one can extend the definition at least in degrees 0 and 1 straightforwardly by imitating the Čech-cohomological approach. Let \mathcal{G} be a sheaf of nonabelian groups. We set $H^0(X, \mathcal{G}) = \mathcal{G}(X)$, the interesting part happens in degree 1:

If $\mathcal{U} = \{U_i\}_i$ is an open cover of X , a 1-cocycle for \mathcal{U} is defined to be a collection of sections $\{g_{ij} \in \mathcal{G}(U_i \cap U_j)\}$ satisfying the cocycle condition

$$g_{ik} = g_{ij}g_{jk}$$

We call two such cycles g and g' are cohomologous if there is a collection of sections $h_i \in \mathcal{G}(U_i)$ such that

$$g'_{ij} = h_i^{-1}g_{ij}h_j$$

We define $H^1(\mathcal{G}, \mathcal{U})$ to be the quotient *set* (this is *not* a group in general) obtained by identifying cohomologous cocycles. As usual, the cohomology of X is then defined to be the limit $H^1(X, \mathcal{G}) = \varinjlim_{\mathcal{U}} H^1(\mathcal{G}, \mathcal{U})$.

It is almost tautological from the definitions that for the sheaf $\mathcal{G} = \mathrm{Gl}_n(\mathcal{H}_X)$, the set

$$H^1(X, \mathrm{Gl}_n(\mathcal{H}_X))$$

is exactly the set of isomorphism classes of holomorphic vector bundles of rank n (compare 2.1.2. in [30]).

5.1.2. What the Nonabelian Hodge Theorem should say. Instead of studying homomorphisms from our fundamental group into the additive group of complex numbers, we want to examine homomorphisms into the general linear group $Gl_n(\mathbb{C})$, also known as complex representations. We now find the analogous elements of all terms involved in the abelian decomposition by replacing the sheaf \mathcal{H}_X by $Gl_n(\mathcal{H}_X)$:

- We have seen that $H^1(X, \mathbb{C}) = \text{Hom}(\pi_1(X), (\mathbb{C}, +))$, and therefore this should clearly be replaced by $\text{Hom}(\pi_1(X), Gl_n(\mathbb{C}))$.
- The space $H^0(X, \Omega_X^1)$ will be replaced by $H^0(X, Gl_n(\mathcal{H}_X) \otimes_{\mathcal{H}_X} \Omega_X^1)$, the space of endomorphism-valued global holomorphic one forms.
- We replace $H^1(X, \mathcal{H}_X)$ by $H^1(X, Gl_n(\mathcal{H}_X))$, which equals the set of isomorphism classes of holomorphic vector bundles.

It transpires that it is too optimistic to hope that the analogue correspondence, taken literally, holds true in the nonabelian case. However, the analogy reveals the vague nature of the objects appearing in the correspondence we are trying to establish:

Vague aim: Representations of $\pi_1(X)$ up to conjugation should correspond to pairs made up out of isomorphism classes of holomorphic vector bundles and endomorphism-valued holomorphic one-forms.

5.2. Metricity Defect and Absolute Metrification

We have seen how the unitary nonabelian Hodge theorem (generalising the theorem of Narasimhan-Seshadri) gives a correspondence between unitary representations of the fundamental group and certain holomorphic bundles. We want to extend the correspondence to semisimple representations by keeping track of an additional datum, namely an endomorphism-valued one-form, and we want that this generalised correspondence reduces to the unitary one in cases where this datum vanishes.

We therefore try to find a way to use such forms to measure how much a representation deviates from being unitary. Equivalently, we try to measure how far away the corresponding flat bundle is from admitting a compatible metric. Assume we are given a flat bundle (E, ∇) on some compact Kähler manifold (X, ω) (equivalently a representation of $\pi_1(X)$) and we want to encode it as a holomorphic bundle with an additional datum. There is of course the trivial way to do this: Just take the holomorphic bundle determined by $\nabla^{0,1}$ and keep track of the $\nabla^{1,0}$ -part as well, which is even a holomorphic connection. For unitary representations, we have seen that just the holomorphic structure is enough information to determine the representation uniquely.

This approach seems to be the wrong one for three reasons: It is trivial, the additional information does not vanish in the unitary case, and the additional information is not an endomorphism-valued one form.¹

The correspondence we will use for the general nonabelian Hodge theorem is a significantly more sophisticated extension of the unitary case, in particular, it should

¹Recall that differences of connections are endomorphism valued one-forms, but that individual connections are not.

be noted that in the nonunitary case, the holomorphic structure we associate will *not* be the one directly determined by the flat connection.

5.2.1. Compatibility Defect. Given a complex bundle E with a fixed metric h , we want to measure how far a given connection is from being compatible with h . To do this, we want to find a natural way to deform an arbitrary connection ∇ on E into a h -metric connection ∇_M .

There is certainly the naive way of just picking the Chern connection associated to h and $\nabla^{0,1}$. This technique is very asymmetric since it fixes the $(0,1)$ part and adapts the $(1,0)$ part - it corresponds to the philosophy that every smooth bundle with a flat connection is secretly already a holomorphic bundle.

The more useful symmetric way of approximating our connection by a metric one modifies both parts of ∇ in a coupled way. More precisely, we can consider connections of the form

$$\nabla + \Theta$$

where $\Theta^{1,0}$ and $\Theta^{0,1}$ are formal adjoints with respect to h .

Notice that we can deform every connection in this way to precisely one metric connection (see [5]):

LEMMA 5.1. *Let X , E and h be as above. For every flat connection ∇ , there is a unique metric connection ∇_h such that $\Theta_{\nabla,h}^{1,0} = \nabla^{1,0} - \nabla_h^{1,0}$ and $\Theta_{\nabla,h}^{0,1} = \nabla^{0,1} - \nabla_h^{0,1}$ are formal adjoints. Write $\Theta_{\nabla,h} = \Theta_{\nabla,h}^{1,0} + \Theta_{\nabla,h}^{0,1}$.*

The following operator therefore measures how far away ∇ is from being compatible with h :

DEFINITION 5.2. Given a connection ∇ on a bundle E with metric h as above, we define the *h -compatibility defect* to be the endomorphism-valued $(1,0)$ -form $\Theta_{\nabla,h}^{1,0} = \nabla^{1,0} - \nabla_h^{1,0} \in \mathcal{A}^{1,0}(End(E))$.

We also want to keep track of the best approximating connection:

DEFINITION 5.3. Given ∇ , E and h as above, the *h -metrisation* of ∇ is defined to be the connection ∇_h in the above notation.

This definition of the compatibility defect will turn out to be the right one since, expressed in terms that will be filled with meaning soon, the length of $\Theta_{\nabla,h}$ is exactly the “energy” of the metric h with respect to ∇ . Notice that our representation (E, ∇) is unitary if and only if $\Theta_{\nabla,h}^{1,0}$ vanishes for some h .

When we measure the distance from a point to a plane, we take the distance to the closest point. Analogously, in order to measure how far a fixed flat bundle (E, ∇) is from being compatible with some metric, we want to measure the distance to the “most compatible metric”. Hence we ask:

Given a flat bundle (E, ∇) , what is the most compatible metric?

5.2.2. Harmonic Metrics. Given a smooth map $f : (M, g) \rightarrow (N, h)$ of Riemannian manifolds, with N having non-positive sectional curvature, we can associate the *energy density function*

$$e(f) : M \rightarrow \mathbb{R}_{\geq 0}$$

given by $x \mapsto \|df_x\|_2^2 = \text{tr}(df_x^* df_x)$ (see [5]). This function measures the local dilation of f .

For $K \subset M$ compact, we define the *energy functional* E_K to be

$$f \mapsto E_K(f) = \int_K e(f) dV$$

where dV is the natural volume form associated to g .

DEFINITION 5.4. A smooth map $f : (M, g) \rightarrow (N, h)$ of compact manifolds is *harmonic* if it is an extremal point of the energy functional E_K for all compact subsets K of M .

Given a flat bundle (E, ∇) over some Kähler manifold X with associated Riemannian metric g . The pullback \tilde{E} of our bundle to the universal covering \tilde{X} of X can be trivialised by some global sections e_1, \dots, e_n , horizontal with respect to the pulled back flat connection. We fix such a global frame once and for all.

Given now any metric h on E (not necessarily compatible with ∇), we obtain a pullback metric \tilde{h} on \tilde{E} for which we can choose a local unitary frames. Expressing these frames in terms of our fixed horizontal frame gives local matrix-valued functions, unique up to multiplication by unitary matrices. Hence these matrices glue to give a global function

$$\phi_h : \tilde{X} \rightarrow Gl(n)/U(n)$$

called the *classifying map*. The manifolds $Gl(n)/U(n)$ and \tilde{X} inherit Riemannian metrics.

DEFINITION 5.5. We say that a metric h is *harmonic* if its classifying map is a harmonic map of Riemannian manifolds.

Notice that this notion is independent of the initially chosen frame.

DEFINITION 5.6. A triple (E, ∇, h) of a bundle E with a flat connection ∇ and a harmonic metric h is called a *harmonic bundle*.

We want take these harmonic metrics as “most compatible” ones. This is justified by the fact that the energy of the classifying map Φ_h at a metric h can be seen to be $\|\Theta_{\nabla, h}\|^2$, so our metric is harmonic if and only if its compatibility defect has extremal (in fact minimal) length.

Even without using this fact, we can easily make the following observation:

LEMMA 5.7. *If h is a metric on a flat bundle (E, ∇) such that ∇ is compatible with h , then h is harmonic.*

PROOF. We can pick a global unitary horizontal frame for our pulled back bundle \tilde{E} on \tilde{X} . Then the classifying map with respect to this frame is constant, so harmonic. Since this property is independent of the choice of horizontal frame for \tilde{E} , we see that our metric is indeed harmonic. \square

We will briefly digress to give a reformulation of the harmonicity condition. The following definition can be taken as a purely formal abbreviation to make the subsequent lemma nicer:

DEFINITION 5.8. Let h be a metric on a flat bundle (E, ∇) with compatibility defect $\Theta_{\nabla, h}^{1,0}$ and h -metrisation ∇_h . Then the *pseudocurvature* of h is defined to be

$$P_h = (\nabla_h^{0,1} + \Theta_{\nabla, h}^{1,0})^2$$

We can now quote a more down-to-earth condition for harmonicity (see [38] for an overview of the proof):

LEMMA 5.9. *Let h be a metric on a flat bundle (E, ∇) on some compact Kähler manifold (X, ω) . Write Λ for the h -adjoint of $\omega \wedge (\cdot)$ and P_h for the pseudocurvature of h .*

Then h is harmonic if and only if $\Lambda P_h = 0$.

This in fact happens if and only if $P_h = 0$.

The next obvious question is if such a “most compatible” harmonic metric always exists, and, if so, whether it is unique. The following stunning answer is due to Corlette (see [11], special cases were proved by Diederich-Ohsawa in [13] and Donaldson in [16]).

THEOREM 5.10. *A flat bundle (E, ∇) admits a harmonic metric if and only if it is semisimple. If our bundle is also irreducible, then this metric is essentially unique.*

Here, a semisimple flat bundle is a bundle splitting up into a sum of simple flat bundles, or equivalently coming from a semisimple representation. A flat bundle is irreducible if it has no proper subbundles fixed under the action of ∇ , or equivalently if it comes from an irreducible representation.

REMARK 5.11. This theorem is often called the nonabelian Hodge-isomorphism for de Rham cohomology, reflecting the fact it tells us that every flat bundle corresponding to an irreducible representation (i.e. a certain nonabelian cohomology class) has an essentially unique harmonic representative.

The uniqueness result in Corlette’s theorem 5.10 and Lemma 5.1 imply that for a semisimple flat bundle, the h -metrisation ∇_h and the h -compatibility defect $\Theta_{\nabla, h}^{1,0}$ are independent of the choice of harmonic metric h .

We can use this to finally define a measure for how much a semisimple representation fails to be unitary:

DEFINITION 5.12. Let (E, ∇) be a semisimple flat bundle on a compact Kähler manifold X . The *metricity defect* $\theta_{\nabla} \in \mathcal{A}^{1,0}(E)$ of (E, ∇) is the h -compatibility defect of ∇ for any harmonic metric h .

We can also use our result to find the closest metric connection:

DEFINITION 5.13. Let E, ∇ and X be as above. The *absolute metrisation* ∇_M is just the h -metrisation for any harmonic metric h .

5.3. The Nonabelian Hodge Correspondence

We are finally able to define the desired correspondence for a compact Kähler manifold (X, ω) .

5.3.1. Definition of the functor $HiggsBun$. Recall that in the unitary case, we simply mapped a flat bundle to the holomorphic structure it defines. If we are given a general flat semisimple bundle (E, ∇) , we want to cook up an analogue out of the absolute metrisation ∇_M and the metricity-defect θ_{∇} . At this stage, lemma 5.9 becomes crucial: By considering the individual types of the equation

$$0 = P_h = (\theta_{\nabla} + \nabla_M^{0,1})^2$$

we obtain that

$$\begin{aligned} \left(\nabla_M^{0,1}\right)^2 &= 0 \\ \theta_\nabla \wedge \theta_\nabla &= 0 \\ \nabla_M^{0,1}(\theta_\nabla) &= 0 \end{aligned}$$

(When we take the wedge, we think of θ_∇ as an endomorphism-valued 1–form) These identities allow us to realise the vague aim expressed in 5.1.2: We have a holomorphic bundle $E_\nabla := (E, \nabla_M^{0,1})$ together with a holomorphic endomorphism-valued one-form θ_∇ on E_M . Equivalently, we can think of θ_∇ as a holomorphic map $E_\nabla \rightarrow E_\nabla \otimes \Omega_X^1$.

But θ_∇ is not an arbitrary such form since it squares to zero. Since such pairs $(E_\nabla, \theta_\nabla)$ will form the basic objects our representations correspond to, they deserve a new name (introduced by Hitchin in [20]):

DEFINITION 5.14. A *Higgs bundle* on a compact Kähler manifold X is a holomorphic vector bundle E on X , together with an endomorphism-valued holomorphic one-form

$$\theta \in \text{End}(E) \otimes \Omega_X^1$$

satisfying $\theta \wedge \theta = 0$. The form θ is called the *Higgs field*.

A useful way to think about Higgs bundles is as generalised holomorphic bundles, or holomorphic bundles with a “correction factor”. Many definitions and results therefore generalise from holomorphic to Higgs bundles.

Choosing local holomorphic coordinates z_1, \dots, z_n , we can express an endomorphism-valued holomorphic one-form locally as $\theta = \sum_j \theta_j dz_j$ for some holomorphic matrices θ_j . The condition $\theta \wedge \theta = 0$ is then equivalent to the claim that for all local coordinates, the matrices θ_j pairwise commute.

REMARK 5.15. Notice that by writing $D'' = \theta + \bar{\partial}_E$ (thinking of the Higgs field as a holomorphic map $E \rightarrow \Omega^1(E)$), we can see that specifying a Higgs bundle is equivalent to specifying a complex bundle E together with an operator D'' with $D''^2 = 0$ and an antiholomorphic Leibniz rule $D''(fe) = \bar{\partial}(f)e + fD''(e)$.

One can show that Higgs bundles form a category. And the above map on objects gives in fact rise to a functor. We obtain:

THEOREM 5.16. *There is a functor*

$$\text{HiggsBun} : \{\text{Semisimple Flat Bundles on } X\} \rightarrow \{\text{Higgs Bundles on } X\}$$

defined by mapping a flat bundle to the pair consisting of its metricity defect and the holomorphic structure determined by the absolute metrisation of ∇ :

$$(E, \nabla) \mapsto (E_\nabla, \theta_\nabla)$$

This functor restricts to the functor $HBun$ (see 4.21) on unitary representations in the obvious manner.

5.3.2. Injectivity of the functor *Higgsbun* on objects. From our previously stated results, we can easily see:

LEMMA 5.17. *The functor *Higgsbun* is injective on objects.*

PROOF. If $((E, \bar{\partial}_E), \theta) = \text{Higgsbun}(E, \nabla)$ is the Higgs bundle associated to some semisimple flat bundle, then we can pick a harmonic metric h and obtain a decomposition $\nabla = \nabla_M + \Theta_{\nabla, h}$ with

$$\begin{aligned} (\nabla_M)^{0,1} &= \bar{\partial}_E \\ \Theta_{\nabla, h}^{1,0} &= \theta \end{aligned}$$

This implies that ∇_M is the Chern connection of h and $\bar{\partial}_E$ and hence uniquely determined by these objects. By one of our previous remarks, ∇_M is independent of the harmonic metric we choose, and so depends purely on $\bar{\partial}_E$. A similar argument shows that the h -symmetric operator $\Theta_{\nabla, h}$ is already entirely determined by θ . \square

In fact, it can even be shown that this functor is fully faithful. We therefore obtain an embedding from the category of semisimple representations of $\pi_1(X)$ into the category of Higgs bundles, the image being a full subcategory. One last question needs to be answered in order to obtain the desired equivalence of categories: *What is the image of the functor *HiggsBun*?*

5.3.3. The Image of the functor *HiggsBun*. The determination of the image of this functor can be seen as a more elaborate version of theorem 4.43, which told us that the holomorphic bundles arising from irreducible unitary representations are exactly stable bundles of degree 0 such that the intersection number $ch_2(E) \cdot \omega^{n-2}$ vanishes.

Assume we start with a Higgs bundle $((E, \bar{\partial}_E), \theta)$ on some compact Kähler manifold (X, ω) . By remark 5.15, have an operator $D'' = \theta + \bar{\partial}_E$.

If h is *any* metric on E , write $\nabla^{h, D''} = \partial_E + \bar{\partial}_E$ for the Chern-connection with respect to h and $(E, \bar{\partial}_E)$, and write $\Theta^{h, D''} = \theta + \bar{\theta}$ for the sum of the Higgs field and its h -adjoint.

We know that if h is the harmonic metric for some flat bundle corresponding to the Higgs bundle we started with, then this bundle must have the flat connection $\nabla^{h, D''} + \Theta^{h, D''}$. Hence the problem of finding a preimage for our Higgs bundle is equivalent to finding a metric h such that the connection $\nabla := \nabla^{h, D''} + \Theta^{h, D''}$ is flat, and such that h is a harmonic metric for ∇ .

Again, theorem 5.9 is helpful at this stage: It tells us that if we have a metric h such that ∇ is a flat connection, then since the pseudocurvature of ∇^h is just $(D'')^2 = 0$, our metric h is automatically harmonic for ∇ .

We see that a Higgs bundle (E, D'') lies in the image of *HiggsBun* if and only if we can choose a metric h such that the connection $\nabla := \nabla^{h, D''} + \Theta^{h, D''}$ is flat. This motivates the following definition of the curvature of metrics on Higgs bundles:

DEFINITION 5.18. Let h be a metric on a Higgs bundle (E, D'') . The *Higgs-curvature* $R_{h, D''}$ of h is defined to be the curvature of the associated connection $\nabla^{h, D''} + \Theta^{h, D''}$.

If the Higgs-curvature vanishes, we call h *Higgs-flat*

We see that a Higgs bundle lies in the image of $HiggsBun$ if and only if it admits a Higgs-flat metric.

This is a generalisation of the fact that a holomorphic bundle comes from a unitary representation if and only if there is a *flat metric* (i.e. a metric whose Chern connection is flat, see 4.5). In the unitary case, we saw that such a metric can be found if and only if the bundle is polystable, and satisfies some conditions on first and second Chern class. The key idea was to view flat metrics as special Hermite-Einstein metrics, and to then use the Kobayashi-Hitchin correspondence to tell us when such metrics exist.

Thinking of Higgs bundles as generalised holomorphic bundles, it is natural to suspect that this strategy will extend.

First, we generalise the notion of Hermite-Einstein metrics to Higgs bundles:

DEFINITION 5.19. Let h be a metric on a Higgs bundle $((E, \bar{\partial}_E), \theta)$ with Higgs-curvature R_h . Then h is called a *Higgs-Hermite-Einstein metric* if there is a real constant λ such that

$$\Lambda R_h = \lambda \text{id}_E$$

As always, Λ denotes the formal adjoint of the Kähler form.

REMARK 5.20. Simpson calls such metrics Hermitian-Yang-Mills metrics. We decided to not follow this convention to stress the similarity to the unitary case. Notice that for $\theta \neq 0$, Higgs-Hermite-Einstein metrics are *not* in general Hermite-Einstein metrics for our holomorphic bundle. For $\theta = 0$ however, the two notions agree.

We see that every Higgs-flat metric on a Higgs bundle is Higgs-Hermite-Einstein. As before, we are therefore lead to the question:

When does a Higgs bundle admit a Higgs-Hermite-Einstein metric?

In order to formulate the answer to this question, we need to introduce the natural generalisations of of the previous stability notions:

DEFINITION 5.21. A Higgs bundle $((E, \bar{\partial}_E), \theta)$ on a compact Kähler manifold is *semi-stable* if for every coherent subsheaf of \mathcal{E} with $0 < \text{rk}(\mathcal{F}) < \text{rk}(\mathcal{E})$ which is preserved ² by the Higgs field θ , we have an inequality of slopes:

$$\mu(\mathcal{F}) \leq \mu(\mathcal{E})$$

The Higgs bundle is *stable* if this inequality on slopes is strict.

The Higgs bundle is *polystable* if it is a direct sum of stable Higgs bundles of same slope.

The most naive guess would be that the obvious generalisation of the Kobayashi-Hitchin correspondence in these terms holds true. We are extremely lucky since this indeed turns out to be true, according to a theorem by C. Simpson (see [37], [38]):

THEOREM 5.22. *A Higgs bundle E has a Higgs-Hermite-Einstein metric if and only if it is polystable.*

²The Higgs field can be interpreted as a bundle morphism $\theta : E \rightarrow \Omega_X^1(E)$, so induces a map θ_* on sheaves of sections. We say θ preserves a coherent sheaf \mathcal{F} if $\theta_*(\mathcal{F}) \subset \mathcal{F} \otimes \Omega_X^1$.

Hence we know that all Higgs bundles in the image must be polystable. But when is the associated metric Higgs-flat? Recall that the flatness of Hermite-Einstein metrics on holomorphic bundles was determined by the intersection behaviour of the first two Chern classes with the Kähler form (see 4.41). Again, the most optimistic generalisation to Higgs bundles holds true (see [38]):

THEOREM 5.23. *Let E be a Higgs bundle with a Higgs-Hermite-Einstein metric h on some compact Kähler manifold (X, ω) of dimension n .*

Then h is Higgs-flat if and only if $\deg(E) = 0$ and $\int_X ch_2(E) \wedge \omega^{n-2} = 0$.

Again, we use the convention that $\omega^{-1} = 0$. Write

$$ch_2(E).w^{n-2} = \int_X ch_2(E) \wedge \omega^{n-2}$$

Combining the two above generalisations 5.22 and 5.23, we obtain:

THEOREM 5.24. *Let E be a Higgs bundle on a compact Kähler manifold (X, ω) . Then E comes from a semisimple flat bundle if and only if it admits a Higgs-flat metric, which happens if and only if it is polystable and satisfies*

$$\deg(E) = ch_2(E).w^{n-2} = 0$$

5.3.4. The Nonabelian Hodge Theorem. We are finally in a position to put all the pieces of the puzzle together and deduce the peak of the many generalisations we went through in this essay:

THEOREM 5.25. *(Nonabelian Hodge Theorem) Let (X, ω) be a compact Kähler manifold of dimension n . Then there is an equivalence of categories*

$$\left\{ \begin{array}{l} \text{Semisimple representations} \\ \text{of } \pi_1(X) \end{array} \right\} \xrightarrow{\text{HiggsBun}} \left\{ \begin{array}{l} \text{Polystable Higgs bundles } E \\ \text{with } \deg(E) = ch_2(E).w^{n-2} = 0 \end{array} \right\}$$

PROOF. We have seen that *HiggsBun* is injective on objects, and quoted that it is in fact a fully faithful functor. Theorem 5.24 determines the image of this functor. \square

This functor maps irreducible representations to stable Higgs bundles and preserves rank.

REMARK 5.26. We deviated in our presentation from most of the literature by treating representations as the preferred objects, from which we are naturally lead to consider Higgs bundles.

However, the correspondence can also be seen in a more symmetric light. We can define *harmonic bundles* to be bundles with a flat connection and a Higgs field which relate to each other via some harmonic metric, which is not part of the data. These can then be thought of as analogues of harmonic differential forms.

As indicated before, Corlette's proof can then be seen as an analogue of the Hodge-isomorphism for de Rham cohomology, whereas our determination of the image of *HiggsBun* can be rephrased as a an analogue of the Hodge isomorphism for Dolbeault cohomology (giving harmonic representatives for pairs of holomorphic bundles and endomorphism-valued one forms).

EXAMPLE 5.27. We give a simple example (see [20]) of how the existence of nontrivial Higgs-fields relates to the fundamental group.

Let X be a compact Riemann surface of genus $g > 1$ with canonical bundle $\mathcal{K} = \Omega_X^1$. Consider the bundle

$$E = \mathcal{K}^* \oplus \mathcal{H} \oplus \mathcal{K}$$

We have a Higgs field

$$\theta : E \rightarrow E \otimes \Omega_X^1 = E \otimes \mathcal{K} = \mathcal{H} \oplus \mathcal{K} \oplus \mathcal{K}^2$$

defined as

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(in the obvious notation) This is a Higgs field since $\theta \wedge \theta = 0$ for dimension reasons. Using that dual bundles have inverse Chern classes, we can see that $\deg(E) = 0$.

Coherent subsheaves are just subbundles as explained remark 4.30, and we immediately see that the only subbundle fixed by θ is \mathcal{K}^* . As the curve has genus bigger than one, this bundle has negative degree. This proves that the Higgs bundle (E, θ) is stable with $\deg(E) = 0$, and as $\theta \neq 0$, the Higgs bundle is induced by an irreducible nonunitary 3-dimensional representation of the fundamental group. This of course would not have been possible over an elliptic curve.

REMARK 5.28. One can in fact construct a quasiprojective variety whose points represent isomorphism classes of sums of stable Higgs bundles with vanishing Chern classes. Moreover, there is a moduli space for conjugacy classes of semisimple representations of the fundamental group. One can then prove ([38]) that the nonabelian Hodge correspondence is not just a bijection between those two sets, but in fact a *homeomorphism*.

EXAMPLE 5.29. Recall our discussion of moduli-spaces in 4.4: We realised that the moduli space of 1-dimensional representations of \mathbb{Z}^2 is $\mathbb{C}^* \times \mathbb{C}^*$, whereas the moduli space of holomorphic line bundles of rank one and degree zero is $S^1 \times S^1$.

Since endomorphisms of 1-dimensional complex vector spaces are just numbers, Higgs fields are just 1-forms, and so Higgs bundles of degree zero and rank one are just pairs of line bundles of degree zero and one-forms on the torus.

But the space of global holomorphic one-forms on a torus is one-dimensional. We therefore suspect that the moduli space of Higgs bundles of rank one and degree zero is $S^1 \times S^1 \times \mathbb{C}$, which is indeed homeomorphic to the moduli space $\mathbb{C}^* \times \mathbb{C}^* \cong S^1 \times S^1 \times \mathbb{R} \times \mathbb{R}$ of one-dimensional representations.

5.4. Further Discussion

The final step of our many generalisations is in fact just the beginning of the very fruitful theory of Higgs bundles. Many classical theorems (Riemann-Roch, Serre duality, Lefschetz decomposition) were generalised by Simpson (see [38]) to the nonabelian context.

A very striking application of the above correspondence comes from the fact that the obvious action of \mathbb{C}^* on Higgs bundles via

$$((E, \bar{\partial}_E), \theta) \mapsto ((E, \bar{\partial}_E), t\theta)$$

gives a completely nonobvious action of \mathbb{C}^* on the semisimple representation side.

One can then examine the fixed points of the induced action on the moduli space of representations, and obtain that if a representation is *rigid*³, then its isomorphism class is fixed by some complex non-root of unity.

This can be used to show that if a representation of the fundamental group is *properly rigid*⁴, then it must be induced by some *complex variation of Hodge structure*⁵. One can then prove that real Zariski-closures of images of representations arising in this way are very special groups: they are of *Hodge type*⁶.

This implies by contrapositive that if $\Gamma \subset W$ is a rigid dense lattice⁷ in a real algebraic group $W \subset Gl_n(\mathbb{C})$ which is *not* of Hodge type, then Γ cannot be a Kähler group.

The group $Sl_n(\mathbb{Z})$ can be shown to be rigid inside its real Zariski closure $Sl_n(\mathbb{R})$, and this group is *not* of Hodge type for $n > 2$. We showed in 3.19 with abelian methods that $Sl_2(\mathbb{Z})$ is not Kähler. The nonabelian techniques we just sketched allow us to obtain the same negative result for all $n > 2$, and we have therefore solved the Kähler problem for integral special linear groups:

THEOREM 5.30. *The groups $Sl_n(\mathbb{Z})$ are not Kähler for any $n > 1$.*

³A representation is rigid if it is isomorphic to all nearby representations

⁴A representation is properly rigid if it is rigid when viewed as a representation into the Zariski closure of its image

⁵A complex variation of Hodge structure is a flat bundle with a certain decomposition and a certain Hermitian form satisfying several axioms. These are motivated by the Hodge decomposition we get for each fibre-cohomology $H^k(\chi_b, \mathbb{C})$ in a *family* $\phi : \chi \rightarrow B$ of complex compact manifolds.

⁶A real algebraic group is of *Hodge type* if there is an action of \mathbb{C}^* on its complexification such that S^1 preserves W and such that the action of (-1) restricts to a Cartan involution on W .

⁷A rigid lattice is a discrete subgroup such that the quotient has finite volume and the identity representation is rigid.

Acknowledgement

First and foremost, I wish to thank my supervisor Prof. Ian Grojnowski for his advice, his deep insights and his stimulating encouragement. I also thank Prof. Danny Calegari for sharing his inspiring ideas. Finally, I am very grateful for the many interesting discussions with Julian Holstein, Stefan Schreieder, and Jan Weidner, which helped me to deepen my understanding of the material.

Background in Complex Geometry

In this section, we will briefly sketch the basics of complex geometry. It should serve as a reminder standardising notation rather than a precise and complete exposition, which can be found for example in [21].

A.1. The three perspectives on complex manifolds

The complex plane \mathbb{C} can be thought of as a two-dimensional \mathbb{R} -algebra satisfying the field axioms ¹. We can hence either choose to use the multiplicative structure in our considerations, or to ignore it and use only linear-algebraic techniques. Both of these viewpoints turn out to be very powerful in various areas, for example in the theory of number fields.

Applied to complex manifolds, this subtlety gives rise to three different worlds in which the objects of interest live, and it is the rich interplay between these worlds which gives complex geometry its attractive flavour. There are objects of smooth \mathbb{R} -valued, smooth \mathbb{C} -valued, and holomorphic nature.

Since we will need to treat all these perspectives at once, it is important to introduce effective and precise notation, which we shall now do. So assume X is a complex manifold of (complex) dimension n .

As a general rule for this essay, sheaves will always be denoted by curly letters, whereas the collection of their global sections will be denoted with normal letters.

A.1.1. Smooth real-valued objects. We may consider X as an ordinary $2n$ -dimensional real smooth manifold and forget the complex structure. For such smooth manifolds, basic differential geometry then provides the definitions of the following basic objects:

- The sheaf $\mathcal{C}_{X,\mathbb{R}}^\infty$ of *smooth real-valued functions*.
- *Smooth real vector bundles* E having (finite-dimensional) real spaces as fibres and smooth transition functions. We write $\mathcal{A}_{X,\mathbb{R}}^0(E)$ for the sheaf of smooth sections of E .
- The *real tangent bundle* $T_{X,\mathbb{R}}$ whose fibre at a point P is given by the space of derivations of the stalk $(\mathcal{C}_{X,\mathbb{R}}^\infty)_P$.
- The p^{th} exterior power $\Lambda^p T_{X,\mathbb{R}}^*$ of the (real-valued smooth) cotangent bundle. We write $\mathcal{A}_{X,\mathbb{R}}^p$ for the sheaf of sections of this bundle, whose sections are *real-valued p -forms*. To each smooth bundle E on X as above, we can then associate the sheaves $\mathcal{A}_{X,\mathbb{R}}^p(E)$ of E -valued p -forms, which are just smooth sections of $E \otimes_{\mathbb{R}} \Lambda^p T_{X,\mathbb{R}}^*$. Using the fact that there are

¹One can in fact show that it is the only finite-dimensional \mathbb{R} -algebra except for \mathbb{R} which defines an honest field, see [41]

smooth partitions of unity, one can show that the global section functor on smooth bundles preserves the monoidal structure given by tensor products. Therefore, we have $\mathcal{A}_{X,\mathbb{R}}^p(E) = \mathcal{A}_{X,\mathbb{R}}^p \otimes_{C_{X,\mathbb{R}}^\infty} \mathcal{A}_{X,\mathbb{R}}^0(E)$

A.1.2. Smooth complex-valued objects. We now want to present the complex-valued analogues of the above real valued concepts. We will hereby treat \mathbb{C} as a smooth real 2–manifold. As before, we forget the holomorphic structure on our manifold and view it as a smooth real manifold. For such smooth real manifolds, we can then define:

- The sheaf $\mathcal{C}_{X,\mathbb{C}}^\infty$ of *smooth complex-valued functions* consisting of all smooth maps from the real manifold X to the 2–dimensional real manifold \mathbb{C} . Notice that $\mathcal{C}_{X,\mathbb{C}}^\infty = \mathcal{C}_{X,\mathbb{R}}^\infty \otimes_{\mathbb{R}} \mathbb{C}$.
- *Smooth complex vector bundles* whose fibres are (finite-dimensional) complex spaces and whose transition functions are smooth. Equivalently, these are smooth real bundles together with a bundle endomorphism I with $I^2 = -id$.
- The *complex tangent bundle* $T_{X,\mathbb{C}}$ having fibre at P given by the space of complex derivations of the \mathbb{C} -algebra $(\mathcal{C}_{X,\mathbb{C}}^\infty)_P$. As in the previous case, an easy exercise shows that $T_{X,\mathbb{C}} = T_{X,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.
- The bundle $\Lambda^p T_{X,\mathbb{C}}^*$ whose sheaf of sections $\mathcal{A}_{X,\mathbb{C}}^p$ consists of *complex-valued p -forms*. Once more, we obtain a nice tensorial identity

$$\mathcal{A}_{X,\mathbb{C}}^p = \mathcal{A}_{X,\mathbb{R}}^p \otimes_{\mathbb{R}} \mathbb{C}$$

For smooth complex bundles E , the sheaf $\mathcal{A}_{X,\mathbb{C}}^p(E)$ is defined as sheaf of smooth sections of the bundle

$$\Lambda^p T_{X,\mathbb{C}}^* \otimes_{\mathbb{C}} E = \Lambda^p T_{X,\mathbb{R}}^* \otimes_{\mathbb{R}} E$$

(here E is viewed as a real bundle on the right hand side).

A.1.3. Holomorphic objects. We finally make use of the field-structure of \mathbb{C} by defining holomorphic functions from \mathbb{C} to itself to be (smooth) functions whose differential respects complex multiplication. A complex-valued function on \mathbb{C}^n is said to be holomorphic if it is so in every coordinate separately. We can then define holomorphic analogues of the above concepts:

- *Holomorphic functions*, which are complex-valued function on X such that every pullback along a chart has this property. We write \mathcal{H}_X for the sheaf of holomorphic functions.
- Holomorphic vector bundles E , which we define to be complex vector bundles $(\pi : E \rightarrow X)$ for which E is a complex manifold, π a holomorphic map, and we can trivialise the bundle by biholomorphisms.
- The *holomorphic tangent bundle* $T_X^{1,0}$, which has fibre at P given by the space of complex derivations of the \mathbb{C} -algebra $(\mathcal{H}_X)_P$ of germs of holomorphic functions at P .
- The sheaf Ω_X^p of *holomorphic p -forms*, which is the sheaf of holomorphic sections of the bundle $\Lambda^p \left(T_X^{1,0} \right)^*$. By tensoring this bundle with a holomorphic bundle E , we also get the sheaf $\Omega_X^p(E)$ of E -valued holomorphic p -forms.

A.1.4. Interplay between real, complex and holomorphic tangent bundle: The (p, q) -Decomposition. Notice that the complex manifold-structure we have on X gives rise to a special endomorphism I on the real tangent bundle $T_{X, \mathbb{R}}$ with $I^2 = -\text{id}$ as follows. The charts of our complex manifold give local trivialisations

$$\psi_* : T_{X, \mathbb{R}}|_U \rightarrow U \times \mathbb{C}^n$$

We can then glue the local operators $\text{id} \times i$ together to obtain I . Therefore the real tangent bundle actually is naturally a complex bundle. We warn the reader of arising subtleties: this does of course **not** mean that the real tangent bundle is equal to the complex tangent bundle (i.e. to its complexification). One should also remark that we have two distinct actions of \mathbb{C} on $T_{X, \mathbb{C}} = T_{X, \mathbb{R}} \otimes \mathbb{C}$ from the left by I and from the right by i . We of course use the complex structure obtained by the action from the right. We will see in our treatment of hermitian manifolds that for our purposes, it is not helpful to think of the real tangent bundle as a complex object. We should rather consider it as a real bundle with “a special operator I ”, and think of the holomorphic and the complex tangent bundle as the “complex objects”

Observe that we can extend every derivation on a holomorphic stalk to one on the complex smooth stalk which vanishes on antiholomorphic functions (by taking local coordinates). This in fact gives rise to an inclusion of bundles $T_X^{1,0} \hookrightarrow T_{X, \mathbb{C}}$. One can check that the image consists precisely of the set of vectors for which the two actions of \mathbb{C} described above agree.

Antiholomorphic functions are functions with \mathbb{C} -antilinear differential, and the analogously defined antiholomorphic tangent bundle $T_X^{0,1}$ also includes naturally into $T_{X, \mathbb{C}}$. This space consists of the $(-i)$ -eigenbundle for I .

We obtain a decomposition of the complex tangent bundle which will turn out to be crucial for the Hodge decomposition:

$$T_{X, \mathbb{C}} = T_X^{1,0} \oplus T_X^{0,1}$$

One can then check (using local holomorphic coordinates) that the composition

$$\phi = (T_{X, \mathbb{R}} \hookrightarrow T_{X, \mathbb{C}} \twoheadrightarrow T_X^{1,0})$$

is an isomorphism of smooth bundles, and gives the real tangent bundle $T_{X, \mathbb{R}}$ the structure of a complex vector bundle we already know from I .

If $\{z_i = x_i + iy_i\}_i$ are local holomorphic coordinates, then $\{dz_i = dx_i + idy_i\}_i$ is a coframe for the holomorphic cotangent bundle with dual frame $\{\frac{\partial}{\partial z_i} = \frac{1}{2}(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i})\}_i$. In this situation, we have $\phi(\frac{\partial}{\partial x_i}) = \phi(\frac{\partial}{\partial y_i}) = \frac{\partial}{\partial z_i}$ (where ϕ is the induced sheaf homomorphism).

Taking duals, we get a corresponding decomposition $T_{X, \mathbb{C}}^* = (T_X^{1,0})^* \oplus (T_X^{0,1})^*$. When we finally take exterior power of this bundle, our decomposition induces

$$\Lambda^k T_{X, \mathbb{C}}^* = \bigoplus_{p+q=k} (\Lambda^p T_X^{1,0})^* \wedge (\Lambda^q T_{X, \mathbb{C}}^{0,1})^*$$

On the level of sheaves of sections, this yields a decomposition

$$\mathcal{A}_{X, \mathbb{C}}(E) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(E)$$

of smooth differential forms (we drop the subscripts on the right hand side). for each complex bundle E . Note that forms of type $(p, 0)$ are *not* necessarily holomorphic, but rather locally of the form $f\alpha$ for some smooth section f and some holomorphic form α .

One can check using local coordinates that $d(\mathcal{A}^{p,q})(E) \subset \mathcal{A}^{p+1,q}(E) \oplus \mathcal{A}^{p,q+1}(E)$. Composing d with the two possible projections, we can therefore write $d = \partial + \bar{\partial}$ for $\partial : \mathcal{A}^{p,q}(E) \rightarrow \mathcal{A}^{p+1,q}(E)$ and $\bar{\partial} : \mathcal{A}^{p,q}(E) \rightarrow \mathcal{A}^{p,q+1}(E)$.

A.2. A Reformulation of Holomorphic Bundles

A posteriori, there is an equivalent formulation of holomorphic bundles in terms of a certain type of operators:

THEOREM A.1. *Let $(\pi : E \rightarrow X)$ be a smooth complex bundle on a complex manifold X . Then giving $(\pi : E \rightarrow X)$ the structure of a holomorphic vector bundle is equivalent to defining an operator $\bar{\partial}_E : \mathcal{A}^{0,0}(E) \rightarrow \mathcal{A}^{0,1}(E)$ such that the tweaked Leibniz rule*

$$\bar{\partial}_E(f \cdot e) = \bar{\partial}(f) \otimes e + f\bar{\partial}_E(e)$$

is satisfied for all smooth functions f and all smooth sections e of E and such that

$$\bar{\partial}_E^2 = 0$$

PROOF. We will only give a sketch of this proof since it uses the (comparatively hard, and rather unrelated) Frobenius theorem.

Given a holomorphic bundle E , and a local holomorphic frame e_1, \dots, e_n . We locally define $\bar{\partial}_E(\sum_i \alpha_i \otimes e_i) = \sum_i \alpha_i \bar{\partial} \alpha \otimes e_i$. Since transition functions are holomorphic, this defines an operator of the required form.

Conversely given such an operator, we can choose local frames e_1, \dots, e_n around any point such that $\bar{\partial}_E(e_i) = 0$ (this is the hard bit). Once this is done, we can consider the corresponding trivialisations, check that the transition functions are biholomorphic, and then put the natural complex structure on our smooth bundle. Holomorphic sections of this bundle are just smooth sections f for which $\bar{\partial}_E(f) = 0$. \square

A.3. A Reminder on Kähler manifolds

We fix one of several possible complex generalisations of Riemannian manifolds:

DEFINITION A.2. A *Hermitian manifold* is a complex manifold together with a Hermitian metric H on its holomorphic tangent bundle $T_X^{1,0}$.

Pulling back along ϕ (defined in A.1.4) and taking the real part, we get a Riemannian metric g on the real tangent bundle $T_{X,\mathbb{R}}$. We call g the *induced Riemannian metric*. Its complexification h defines a hermitian metric on $T_{X,\mathbb{C}}$.

Unfortunately, the restriction of h to the holomorphic tangent bundle **is not** equal to the metric H we started with - but we catch a (stress-)factor:

$$h = 2H$$

Notice that it is important to keep track of the details at this point since several subtleties arise. For example, the hermitian metric which is directly induced by H on the real tangent bundle considered as a complex bundle **is not** equal to the restriction of h to this subbundle. Essentially, h is only sesquilinear for the right

action on our complexified tangent bundle but not the left action given by I .

From now on, we will mainly use g or h instead of H .

The real $(1, 1)$ -form ω corresponding to the bilinear form $g(I(\cdot), (\cdot))$ is called the fundamental form. Using local coordinates, one can prove that the natural volume form dV associated to our Riemannian metric g is equal to $\frac{\omega^n}{n!}$.

DEFINITION A.3. A *Kähler metric* is a Hermitian metric whose fundamental form is closed. A *Kähler manifold* is a complex manifold for which we can choose a Kähler metric. When we say things like “*Assume (X, ω) is a Kähler manifold...*”, we of course make the Kähler form part of our data.

The property of being Kähler puts various strong constraints on the topological invariants of a manifold. For example, it is fairly immediate by Stokes’ theorem that if X is a compact Kähler manifold of dimension n , then the powers $\omega, \omega^2, \dots, \omega^n$ are not exact. This implies that the cohomology groups

$$H^0(X, \mathbb{C}), H^2(X, \mathbb{C}), \dots, H^{2n}(X, \mathbb{C})$$

do not vanish.

In this essay, we will find various much deeper topological consequences of the property of being Kähler.

The reason why we care so much about Kähler manifolds is that all projective algebraic varieties have this property. This can be seen by first showing that submanifolds of Kähler manifolds are Kähler, and then constructing the so-called Fubini study metric on $\mathbb{P}^n\mathbb{C}$ (see [44] for details).

A.4. The Hodge-Star-Operator

There are multiple ways in which this operator on an oriented inner product space over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ can be introduced. Whereas all definitions agree on Euclidean spaces, there are two distinct generalisations to Hermitian spaces, namely a \mathbb{C} -linear and a \mathbb{C} -antilinear operator. We will follow the latter approach, which is dominant in the literature. The Hodge- \star -operator is the composition of three maps: The Riesz map, the Wedge map and the dictionary between the two: the Orientation map. Our approach will show how some of the apparent arbitrariness arises and that the antilinearity dates back to the fact that Hermitian inner products are sesquilinear rather than bilinear. We have:

- The Riesz map: Given any finite-dimensional inner product space $(W, \langle \cdot, \cdot \rangle)$, we have a natural map $R : W \rightarrow W^*$ given by $v \mapsto (w \mapsto \langle w, v \rangle)$. Whereas this map is \mathbb{R} -linear in the real case, it is *not* \mathbb{C} -linear but rather \mathbb{C} -antilinear in the complex case - and this can not be fixed by swapping the order in the definition. The map R is an antiisomorphism and shall be called the *Riesz map* (in the case $\mathbb{K} = \mathbb{R}$, antiisomorphisms and isomorphisms are of course the same).
- The Wedge map: Fix some vector space V of dimension n and let k, r be numbers between 0 and n . Since the wedge-product is bilinear, we obtain an associated \mathbb{K} -linear map by currying:

$$\begin{aligned} W_r : \Lambda^k V &\rightarrow \text{Hom}(\Lambda^r, \Lambda^{k+r}) \\ \alpha &\mapsto (\beta \mapsto \beta \wedge \alpha) \end{aligned}$$

For $r = n - k$, this map is injective and thus by dimension-considerations an isomorphism.

- The Orientation map: Given a \mathbb{K} -vector space $(V, \langle \cdot, \cdot \rangle)$ over \mathbb{K} of dimension n together with a chosen form $dV \in \Lambda^n V$ of top degree (i.e. an *orientation*). The map $O : \mathbb{K} \rightarrow \Lambda^n V$ given by $\lambda \mapsto \lambda dV$ defines an isomorphism of vector spaces.

If we now fix an oriented inner product space $(V, \langle \cdot, \cdot \rangle, dV)$ and some integer k between 0 and n , we get a canonical inner product $\langle \cdot, \cdot \rangle_k$ on the space $\Lambda^k V$ with the following property: Whenever e_1, \dots, e_n is a $\langle \cdot, \cdot \rangle$ -orthonormal basis for V , then $\{e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k}\}_{i_1 < \dots < i_k}$ is a $\langle \cdot, \cdot \rangle_k$ -orthonormal basis for $\Lambda^k V$. We will drop the lower index from now on. We can now compose the Riesz map R and the wedge map W_k on the exterior power through a translation via the orientation map O . The composite \star is an antiisomorphism:

$$\star = \left(\Lambda^k V \xrightarrow{R} (\Lambda^k V)^* = \text{Hom}(\Lambda^k V, \mathbb{K}) \xrightarrow{O_*} \text{Hom}(\Lambda^k V, \Lambda^n V) \xrightarrow{W_k^{-1}} \Lambda^{n-k} V \right)$$

Hence the Hodge- \star -operator is the unique operator with $s \wedge \star t = \langle s, t \rangle dV$

A.4.1. Tensorial Extension. It is crucial for our applications that this correspondence is stable “under tensorial extension by hermitian spaces” in the following sense: Let V and k be as before and $(E, \langle \cdot, \cdot \rangle_E)$ be any other inner product space (without orientation, over the same field). Then $\Lambda^k V \otimes E$ inherits an inner product by extending $\langle \alpha \otimes e, \beta \otimes f \rangle = \langle \alpha, \beta \rangle_k \cdot \langle e, f \rangle_E$ and we therefore get a Riesz map $R : \Lambda^k V \otimes E \rightarrow (\Lambda^k V \otimes E)^* = (\Lambda^k V)^* \otimes E^*$.

We run into slight difficulties when we want to extend our \wedge -product to elements of $\Lambda^k V \otimes E$. For our previous method to work, we want elements of rank k and elements of rank $n-k$ to have products lying in a 1-dimensional space which we can again identify with \mathbb{K} . Therefore, the aim is to find a product which “kills off” the E -component. A first *naive* try would be to define $(\alpha \otimes e) \wedge (\beta \otimes f) = \alpha \wedge \beta \cdot \langle e, f \rangle_E$. A second thought reveals that this expression is *not* well-defined on tensors as our original operator \wedge is \mathbb{C} -bilinear whereas our inner product $\langle \cdot, \cdot \rangle$ is \mathbb{C} -antilinear in the second entry.

We therefore use the tautological pairing $E^* \times E \rightarrow \mathbb{K}$ to define a pairing

$$\wedge : (\Lambda^k V \otimes E) \times (\Lambda^r V \otimes E^*) \rightarrow \Lambda^{k+r} V$$

Let $W_r : \Lambda^k V \otimes E^* \rightarrow \text{Hom}(\Lambda^r V \otimes E, \Lambda^{k+r} V)$ be the map we obtain by currying. By a similar argument as before, this is an isomorphism for $r = n - k$.

We can now repeat our earlier definition to define the \star_E -operator as:

$$\Lambda^k V \otimes E \xrightarrow{R} \text{Hom}(\Lambda^k V \otimes E, \mathbb{K}) \xrightarrow{O_*} \text{Hom}(\Lambda^k V \otimes E, \Lambda^n V) \xrightarrow{W_k^{-1}} \Lambda^{n-k} V \otimes E^*$$

Again $s \wedge \star t = \langle s, t \rangle dV$ by construction. For all $\alpha \in \Lambda^k V, e \in E$, we also have:

$$\star_E(\alpha \otimes e) = (\star \alpha) \otimes R(e)$$

Moreover, one can check that $\star_{E^*} \circ \star_E = (-1)^k \text{id}$ (on k -forms) if we give E^* the dual metric.

APPENDIX B

GAGA

Serre’s famous paper “Géométrie Algébrique et Géométrie Analytique” establishes a tight correspondence between algebraic and analytic varieties. In this chapter, we will give a precise formulation of the statement of GAGA, and sketch applications illustrating the strength of this theorem and its ramifications with abelian Hodge theory.

GAGA says that the theory of coherent algebraic sheaves over a projective variety is essentially equivalent to the theory of coherent analytic sheaves over the same variety considered as an analytic space. We will now fill these words with meaning. In accordance with Serre’s original paper [32], we adopt the convention that varieties are not necessarily irreducible. We will restrict ourselves to the case of analytic and algebraic varieties, but it should be mentioned that the theory can be extended to certain schemes over \mathbb{C} .

B.1. Analytic Varieties

B.1.1. Affine Analytic Varieties. Recall the following basic definition:

DEFINITION B.1. Let $U \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$ be an open set. A continuous function $f : U \rightarrow \mathbb{C}$ is holomorphic if it is holomorphic in each variable separately.

REMARK B.2. One can show that the continuity assumption is in fact redundant, and that every holomorphic function has a power series expansion around each point in its domain (see [42] for details).

DEFINITION B.3. A subset $X \subset \mathbb{C}^n$ is an affine analytic variety if for all $x \in X$, there are holomorphic functions f_1, \dots, f_k , defined on an open neighbourhood U of x in \mathbb{C}^n , such that

$$X \cap U = \{z \in U \mid \forall i : f_i(z) = 0\}$$

We equip X with the subspace topology inherited from \mathbb{C}^n .

The following is an example of a rather wild affine analytic variety:

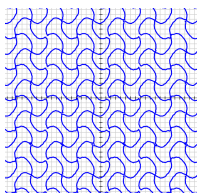


Figure 1: Real part of the analytic variety $\sin(x + 2 \sin(y)) = \cos(y + 3 \cos(x))$

The above notion of a holomorphic function descends to analytic affine varieties:

DEFINITION B.4. Let $U \subset X$ be an open subset of an affine analytic variety $X \subset \mathbb{C}^n$. A function $f : U \rightarrow \mathbb{C}$ is called *holomorphic* if for all $P \in U$, we can find an open neighbourhood W of P in \mathbb{C}^n and a holomorphic function $g : W \rightarrow \mathbb{C}$, such that:

$$f|_{W \cap U} = g|_{W \cap U}$$

This allows us to define the analytic analogue of the sheaf \mathcal{O}_X of regular functions on an affine algebraic variety:

DEFINITION B.5. Given an affine analytic variety X , we define the sheaf \mathcal{H}_X of (germs of) holomorphic functions by:

$$U \subset X \text{ open} \mapsto \{f : U \rightarrow \mathbb{C} \mid f \text{ holomorphic}\}$$

From now on, an affine analytic variety is understood to be the pair (X, \mathcal{H}_X) .

B.1.2. Analytic Varieties. A manifold is locally euclidean, an algebraic variety locally affine, and a scheme locally a spectrum of a ring. Following this philosophy, we can interpret affine analytic varieties as local spaces of more general objects we will define now:

DEFINITION B.6. An analytic variety is a pair (X, \mathcal{H}_X) , where X is a Hausdorff space and \mathcal{H}_X is a sheaf of continuous complex-valued functions on X , satisfying the following local-triviality condition:

There exists an open cover $X = \bigcup X_i$ of X such that each $(X_i, (\mathcal{H}_X)|_{X_i})$ is isomorphic to some affine algebraic variety (Y_i, \mathcal{H}_{Y_i}) . This means that there is a homeomorphism $\phi_i : X_i \rightarrow Y_i$ whose pullback map gives isomorphisms

$$\phi_i^* : \mathcal{H}_{Y_i}(\phi_i(U)) \rightarrow \mathcal{H}_X|_{X_i}(U)$$

for all $U \subset X_i$ open. We call ϕ_i the (analytic) charts.

The sheaf \mathcal{H}_X is said to be the holomorphic structure sheaf of X , and its elements are holomorphic functions.

Notice that every complex manifold is an analytic variety.

Finally, we have a natural notion of morphisms in our arising category of analytic varieties:

DEFINITION B.7. Given two analytic varieties X, Y , a holomorphic map $f : X \rightarrow Y$ is a continuous map that pulls back holomorphic functions to holomorphic functions.

B.2. Coherent Analytic and Algebraic Sheaves

The general definition of coherent sheaves introduced by Serre in [31] is slightly cumbersome. However, we are only interested in sheaves of modules over the holomorphic / algebraic structure sheaf $\mathcal{H}_X / \mathcal{O}_X$. In both cases, one can prove that the following very nice definition is equivalent (we use Oka's theorem in the analytic case):

DEFINITION B.8. Let X be an analytic / algebraic variety with holomorphic / algebraic structure sheaf $\mathcal{R} = \mathcal{H}_X / \mathcal{O}_X$.

A sheaf \mathcal{M} of \mathcal{R} -modules is called a *coherent* analytic / algebraic sheaf if it is locally finitely presented. This means that for each $P \in X$, there is an open

neighbourhood U and morphisms of \mathcal{R} -modules making the following diagram exact for some integers n, m :

$$(\mathcal{R}|_U)^n \rightarrow (\mathcal{R}|_U)^m \rightarrow \mathcal{M}|_U \rightarrow 0$$

Coherent analytic / algebraic sheaves have many desirable properties, which turn out to be crucial in the proof of GAGA:

- They form an abelian category for a fixed space X .
- The support (i.e. the set of points with nonzero stalk) of an algebraic coherent sheaf is Zariski-closed.
- For $X \subset \mathbb{P}^n(\mathbb{C})$ a projective algebraic variety, and \mathcal{F} a coherent algebraic sheaf, the cohomology groups $H^i(X, \mathcal{F})$ are finite-dimensional vector spaces.
- For every coherent algebraic sheaf \mathcal{F} on $X = \mathbb{P}^n(\mathbb{C})$, there is some m such that \mathcal{F} is (globally) isomorphic to a quotient of $(\mathcal{O}_X(m))^p$.

B.3. The Analytification functor

Since polynomials are holomorphic functions, it is suggestive that analytic spaces and sheaves are generalisations of their algebraic counterparts. We will formulate this claim more precisely.

B.3.0.1. On Spaces. Recall that algebraic varieties are defined to be pairs (X, \mathcal{O}_X) of spaces and sheaves of \mathbb{C} -valued functions which are locally isomorphic to affine algebraic varieties with their algebraic structure sheaves, and which are separated. We call the local isomorphisms algebraic charts.

DEFINITION B.9. Let (X, \mathcal{O}_X) be an algebraic variety.

One can show that there is a unique analytic variety (X^{an}, \mathcal{H}_X) on the set X such that every algebraic chart $\phi : (U, \mathcal{O}_X|_U) \rightarrow (V, \mathcal{O}_V)$ to an affine algebraic variety V is also an analytic chart $\phi : (U, \mathcal{H}_X|_U) \rightarrow (V, \mathcal{H}_V)$.

One can also verify that every regular map $f : X \rightarrow Y$ gives rise to a corresponding holomorphic map $f^{an} : X^{an} \rightarrow Y^{an}$.

Thus $(-)^{an}$ is a functor from the category of algebraic varieties to the category of analytic varieties.

B.3.1. On Sheaves. We will now define the analytification functor.

Recall that for any space X , there is an equivalence of categories between local homeomorphism to X and sheaves over X . The equivalence maps a local homeomorphism $\pi : F \rightarrow X$ to its sheaf of sections \mathcal{F} , and we call $\pi : F \rightarrow X$ the étalé space of \mathcal{F} . We can use this to define the analytification of algebraic sheaves:

DEFINITION B.10. Let X be an algebraic variety and \mathcal{F} be a sheaf of \mathcal{O}_X -modules, with associated étalé space $\pi : F \rightarrow X$. Write $\phi : X^{an} \rightarrow X$ for the (continuous) identity map.

First, we pull $\pi : F \rightarrow X$ back along ϕ by giving

$$F' = \{(x, f) \in X^{an} \times F \mid \phi(x) = \pi(f)\}$$

the subspace topology and taking the first projection π_1 as local homeomorphism.

$$\begin{array}{ccc} F' & \xrightarrow{\pi_2} & F \\ \downarrow \pi_1 & & \downarrow \pi \\ X^{an} & \xrightarrow{\phi} & X \end{array}$$

We obtain an associated sheaf \mathcal{F}' , but this is not a sheaf of $\mathcal{H}_{X^{an}}$ -modules yet.

To rectify this issue, we extend our scalars and define the analytification \mathcal{F}^{an} of \mathcal{F} to be the sheaf

$$\mathcal{F}^{an} = \mathcal{F}' \otimes_{\mathcal{O}'_X} \mathcal{H}_{X^{an}}$$

One can check that every morphism $\phi : \mathcal{F} \rightarrow \mathcal{G}$ naturally induces a morphism of \mathcal{H}_X -modules $\phi^{an} : \mathcal{F}^{an} \rightarrow \mathcal{G}^{an}$.

Hence we obtain a functor

$$(-)^{an} : \{\text{Sheaves of } \mathcal{O}_X\text{-modules over } X\} \rightarrow \{\text{Sheaves of } \mathcal{H}_{X^{an}}\text{-modules over } X^{an}\}$$

We list several important properties of this functor (compare [34]):

- It is an exact functor.
- It maps coherent algebraic sheaves to coherent analytic sheaves.
- \mathcal{F} and \mathcal{F}^{an} have the same support, i.e. the same set of points with nonzero stalk.
- For all q , the map $\phi : X^{an} \rightarrow X$ induces a natural morphism

$$\epsilon : H^q(X, \mathcal{F}) \rightarrow H^q(X^{an}, \mathcal{F}^{an})$$

(easy to see via Čech cohomology)

B.4. The statement of GAGA

We are now finally ready to give the precise statement of GAGA:

THEOREM B.11. *Let X be a projective variety, i.e. a Zariski-closed algebraic variety inside $\mathbb{P}_n(\mathbb{C})$. Then, the functor*

$$(-)^{an} : \{\text{Coherent algebraic sheaves over } X\} \rightarrow \{\text{Coherent analytic sheaves over } X^{an}\}$$

is a cohomology-preserving equivalence of categories.

By this, we mean:

- *If \mathcal{F}, \mathcal{G} are two coherent algebraic sheaves over X , then every morphism $f : \mathcal{F}^{an} \rightarrow \mathcal{G}^{an}$ is induced by a unique morphism $g : \mathcal{F} \rightarrow \mathcal{G}$.*
- *For each coherent analytic sheaf \mathcal{F} on X^{an} , there is a coherent algebraic sheaf \mathcal{G} , determined uniquely up to isomorphism, with $\mathcal{G}^{an} = \mathcal{F}$.*
- *For every coherent algebraic sheaf \mathcal{F} , the homomorphism $\epsilon : H^q(X, \mathcal{F}) \rightarrow H^q(X, \mathcal{G})$ induced on cohomology is an isomorphism.*

Notice that since we need to pass to the category of *quasi-coherent* sheaves to obtain *enough injectives* and hence be able to compute sheaf-cohomology, the preservation of cohomology does not follow immediately from the equivalence of categories but requires extra work.

B.5. Applications

We can now sketch the proofs of two very strong theorems:

B.5.1. Chow's Theorem. This theorem tells us that if we restrict ourselves to closed projective analytic varieties, wild behaviour as seen in the above figure cannot occur.

THEOREM B.12. (*Chow*) *Let X be a closed analytic variety in $\mathbb{P}^n(\mathbb{C})$. Then X is a projective algebraic variety.*

PROOF. Let $\mathcal{A}(X)$ be the ideal sheaf of holomorphic functions on open subsets of $\mathbb{P}^n(\mathbb{C})$ whose restrictions to X vanish.

By a theorem of Cartan, $\mathcal{H}_X = \mathcal{H}_{\mathbb{P}^n(\mathbb{C})}/\mathcal{A}(X)$ is then a coherent analytic sheaf on $\mathbb{P}^n(\mathbb{C})$. It has support exactly X .

By GAGA, there is a coherent algebraic sheaf \mathcal{F} on $\mathbb{P}^n(\mathbb{C})$ with $\mathcal{F}^{an} = \mathcal{H}_X$, and as we remarked above, the two sheaves have the same support

Since \mathcal{F} is a coherent algebraic sheaf, its support X is Zariski-closed. \square

With little more effort, one can even show that every compact analytic variety has the structure of an algebraic variety.

B.5.2. Invariance of Betti numbers under field-automorphisms. To every field-automorphism $\sigma : \mathbb{C} \rightarrow \mathbb{C}$, we have an associated map $\sigma : \mathbb{P}^n(\mathbb{C}) \rightarrow \mathbb{P}^n(\mathbb{C})$ given by

$$[t_0 : \dots, t_n] \mapsto [\sigma(t_0), \dots, \sigma(t_n)]$$

If X is a non-singular projective algebraic variety in $\mathbb{P}^n(\mathbb{C})$, then so is $\sigma(X)$. The following result is surprising, since it says that certain algebraic manipulations cannot change certain topological invariants:

THEOREM B.13. *Let $X \subset \mathbb{P}^n(\mathbb{C})$ be a non-singular projective algebraic variety. Then X^{an} and $\sigma(X)^{an}$ have the same Betti-numbers b_n .*

PROOF. Let $\Omega^p(X)$ be the coherent sheaf of regular differential forms of degree p , then $(\Omega^p(X))^{an}$ is the sheaf of holomorphic p -forms.

By the abelian Hodge theorem 3.12, we have

$$b_n(X^{an}) = \sum_{p+q=n} \dim(H^q(X^{an}, \Omega^p(X)^{an}))$$

By GAGA, $H^q(X^{an}, \Omega^p(X)^{an}) = H^q(X, \Omega^p(X))$.

We can check with Čech cohomology that $H^q(X, \Omega^p(X)) \cong H^q(\sigma(X), \Omega^p(\sigma(X)))$. Going back the above chain of implications proves the claim. \square

We conclude with a neat application of this last result, conjectured by A. Weil, which is particularly interesting for number theorists:

COROLLARY B.14. *Let V be a non-singular projective variety over an (abstract) number field K . Every embedding $\sigma : K \rightarrow \mathbb{C}$ gives rise to a nonsingular projective variety V_σ over \mathbb{C} .*

Then the Betti numbers of this variety are independent of the chosen embedding.

B.5.3. Algebraicity of Holomorphic Bundles on Projective Varieties.

An algebraic vector bundle on a smooth complex algebraic variety X is a smooth vector bundle for which the transition functions are regular maps. By the usual yoga, these can be seen to be equivalent to locally free sheaves of (finite-dimensional) \mathcal{O}_X -modules. Analogously, there is an equivalence of categories between holomorphic bundles and locally free sheaves of (finite-dimensional) \mathcal{H}_X modules.

From now on let $X \subset \mathbb{P}^n(\mathbb{C})$ be a smooth projective variety (or equivalently by Chow a projective complex manifold). We can then apply GAGA to the two above sheaf-theoretic formulations to obtain:

THEOREM B.15. *Let $X \subset \mathbb{P}^n(\mathbb{C})$ be a smooth projective variety. The embedding of the category of algebraic vector bundles into the category of holomorphic vector bundles is an equivalence of categories.*

Bibliography

- [1] M.F. Atiyah, *On the Krull-Schmidt theorem with applications to sheaves*, Bulletin de la S.M.F., tome 84, pp.307-317, 1956
- [2] M.F. Atiyah, *Vector Bundles over an Elliptic Curve*, Proceedings of the London Mathematical Society, Third series, 7, pp. 414-452, 1957
- [3] M.F. Atiyah, *Complex Analytic Connections in Fibre Bundles*, Transactions of the American Mathematical Society 85, pp. 181-207, 1957
- [4] M.F. Atiyah, *The Yang-Mills equations over Riemann surfaces*, R. Bott, Philosophical Transactions of the Royal Society London, A308, pp 523-615, 1982
- [5] J. Amorós, M. Burger, K. Corlette, D. Kotschick, D. Toledo, *Fundamental Groups of Compact Kähler Manifolds*, American Mathematical Society, 1996
- [6] O. Baues, V. Cortés, *Aspherical Kähler Manifolds with Solvable Fundamental Group*, Geometriae Dedicata, 122:215-229, 2006
- [7] J. Bertin, J.-P. Demailly, L. Illusie, C. Peters, *Introduction to Hodge Theory*, American Mathematical Society, 2002
- [8] I. Biswas, J.N. Iyer, *Holomorphic Connections on some Complex Manifolds*, Vol. 334, Issue 9, 2007, pp. 577-580
- [9] C. Birkenhake, H. Lange, *Complex Abelian Varieties*, Springer Verlag, 2004
- [10] I. Biswas, M. Mj *One-relator Kähler Groups*, preprint from arXiv
- [11] K. Corlette, *Flat G-bundles with Canonical Metrics*, Journal of Differential Geometry, No. 28, pp. 361-382, 1988
- [12] A. Deitmar, *Topologie* (lecture notes), available from the author's webpage, 2007
- [13] K. Diederich, T. Ohsawa, *Harmonic Mappings and Disc Bundles over Compact Kähler Manifolds*, Publications of the Research Institute for Mathematical Sciences, 21, pp. 819-833, 1985
- [14] S. Donaldson, *A new Proof of a Theorem of Narasimhan and Seshadri*, Journal of Differential Geometry, Vol. 18, No. 2, pp 269-277, 1983
- [15] S. Donaldson, *Anti self-dual Yang-Mills Connections over Complex Algebraic Surfaces and Stable Vector Bundles*, Proceedings of the London Mathematical Society, Third series, 50, pp. 1-26, 1985
- [16] S. Donaldson, *Twisted Harmonic Maps and Self-Duality Equations*, Proceedings of the London Mathematical Society, 55, pp. 127-131, 1987
- [17] A. Grothendieck, *Sur la Classification des Fibres Holomorphes sur la Sphere de Riemann*, American Journal of Mathematics, Vol. 79, No. 1, 1957
- [18] P. de la Harpe, *Topics in Geometric Group Theory*, University of Chicago Press, 2000
- [19] F. Hirzebruch, *Prospects in Mathematics*, Annals of mathematics studies, no. 70, 1971
- [20] N. Hitchin, *The Self-Duality Equations on a Riemann Surface*, Proceedings of the London Mathematical Society, third series, 55, pp. 59-126, 1987
- [21] D. Huybrechts, *Complex Geometry*, Springer-Verlag, 2005
- [22] F. Johnson, E. Rees, *On the Fundamental Group of a Complex Algebraic Manifold*, Bulletin of the London Mathematical Society, 1987
- [23] S. Kobayashi, *Differential Geometry of Complex Vector Bundles*, Iwanami Shoten, Publishers, and Princeton University Press, 1987
- [24] R.C. Lyndon, *Cohomology Theory of Groups with a Single Defining Relation*, Annals of Mathematics, Vol. 53, No. 3, 1950
- [25] M. Nieper-Wißkirchen, *Chern Numbers and Rozansky-Witten Invariants of Compact Hyper-Kähler Manifolds*, World Scientific Publishing Co., 2004
- [26] M.S. Narasimhan, C.S. Seshadri, *Stable and Unitary Vector Bundles on a Compact Riemann Surface*, The Annals of Mathematics, Second Series, Vol. 82, No. 3, pp. 540-567, 1965

- [27] M.S.Narasimhan, C.S. Seshadri, *Holomorphic Vector Bundles on a Compact Riemann Surface*, Mathematische Annalen, Colume 155, No. 1, pp.69-80, 1964
- [28] C. Okonek, M. Schneider, H. Spindler, *Vector Bundles on Complex Projective Spaces*, Birkhäuser Verlag, 1988
- [29] N.R.O'Brian, D.Toledo, Y.L.L.Tong, *The Trace Map and Characteristic Classes for Coherent Sheaves*, American Journal of Mathematics, Vol. 103, No. 2, 1981
- [30] J. Rosenberg, *Topology, C^* -Algebras, and String Duality*, Regional Conference Series in Mathematics, No. 111, American Mathematical Society, 2009
- [31] J.P. Serre, *Faisceaux Algébriques Cohérents*, The Annals of Mathematics, 2nd Ser., Vol. 61. No. 2. (Mar.,1955).pp.197-278
- [32] J.-P. Serre, *Géométrie Algébrique et Géométrie Analytique*, Annales de l'Institut Fourier, tome 6, 1956.
- [33] J.-P. Serre, *Sur la topologie des variétés algébriques en caractéristique p* , Symposium International de Topologia Algebraica, Universidad Autonoma de Mexico, pp.24-53, 1958
- [34] I.R. Shafarevich, *Algebraic Geometry II*, Springer-Verlag Berlin Heidelberg, 1996
- [35] B. Siebert, *Die Existenz einer Hermite-Einstein-Metrik auf einem stabilen Vektorraumbündel*, Diplomarbeit, Universität Göttingen, 1989
- [36] J. Simonis, *A Class of Indecomposable Algebraic Vector Bundles*, Mathematische Annalen, Vol. 192, No. 4, pp. 262-278, 1971
- [37] C.T. Simpson, *Constructing Variations of Hodge Structure Using Yang-Mills Theory and Applications to Uniformization*, Journal of the American Mathematical Society, Vol 1, No. 4, pp. 867-918, 1988
- [38] C.T. Simpson, *Higgs bundles and local systems*, Publications Mathématiques de l'I.H.É.S, tome 75, p. 5-95, 1992
- [39] C.T. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety I*, Publications Mathématiques de l'I.H.É.S, tome 79, p. 47-129, 1994
- [40] C.T. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety II*, Publications Mathématiques de l'I.H.É.S, tome 80, p. 5-79, 1994
- [41] W. Soergel, *Algebra* (lecture notes), available from the author's webpage
- [42] J. L. Taylor, *Several complex variables with connections to algebraic geometry and Lie groups*, American Mathematical Society, 2002.
- [43] K.K. Uhlenbeck, S.-T. Yau, *On the existence of Hermitian-Yang-Mills connections in stable vector bundles*, Communications on Pure and Applied Mathematics, 39, pp. 257-293, 1986
- [44] C. Voisin, *Hodge Theory and Complex Algebraic Geometry, Volume 1*, Cambridge University Press, 2003
- [45] C. Voisin, *On the homotopy types of compact Kähler and complex projective manifolds*, Inventiones Mathematicae, Bd. 157, pp. 329-343, 2004

Index

- \mathcal{C} -covering space, 7
- Abelian Hodge theorem, ordinary, 22
- Abelian Hodge theorem,
 - representation-theoretic, 24
- Acyclic resolution, 15
- Algebraic variety, 66
- Analytic variety, 65
- Analytification of algebraic varieties, 66
- Analytification of sheaves, 66
- Atiyah-class, 36
- Atiyah-Weil, theorem of, 37

- Cartan-Hadamard, theorem of, 27
- Classification of abelian(ised) Kähler groups, 24
- Classifying map of a metric, 49
- Coherent sheaf, analytic and algebraic, 65
- Compatibility defect, 48
- Complex variation of Hodge structure, 56
- Connection, flat, 11
- Connection, holomorphic, 13
- Curvature of a connection, 11

- Degree of a coherent torsion-free sheaf, 42
- Degree of line bundles on Riemann surfaces, 31
- Determinantal Bundle, 32
- Differential operator, 16
- Differential operator, elliptic, 17
- Discrete vector bundle, 9

- Energy density function, 48
- Energy functional, 49

- Finite groups are projective, 3
- Finiteness theorem, 18
- Flat metric, 53
- Formal adjoint, 17
- Functor *HiggsBun*, 51
- Functor HBun, 39

- GAGA, 64
- Grothendieck group, 33
- Groups of Hodge type, 56

- Harder-Narasimhan filtration, 41
- Harmonic bundle, 54
- Harmonic differential forms, 20
- Harmonic map of Riemannian manifolds, 49
- Harmonic metric, 49
- Hermite-Einstein metrics on holomorphic bundles, 42
- Hermitian manifold, 61
- Hermitian metrics on unitary bundles, 23
- Hermitian-Yang-Mills metric on a Higgs bundle, 53
- Higgs bundle, 51
- Higgs curvature, 52
- Higgs field, 51
- Higgs-flat metric, 52
- Higgs-Hermite-Einstein metric on Higgs bundle, 53
- Hirzebruch-Riemann-Roch formula, 33
- Hodge-Star-Operator, 62
- Holomorphic bundle, ordinary formulation, 59
- Holomorphic structure sheaf \mathcal{H}_X , 65
- Holomorphic vector bundle,
 - operator-theoretic formulation, 61

- Indecomposable holomorphic bundle, 30
- Iwasawa manifold, 25

- Jet bundle, 35

- Kähler group, 3
- Kähler problem, 3
- Kobayashi-Hitchin bundle, 42
- Kobayashi-Hitchin correspondence, 44
- Kobayashi-Hitchin correspondence for Higgs bundles, 53

- Local system, 8
- Lyndon's theorem, 27

- Metricity defect, 50
- Metrisation, absolute, 50
- Metrisation, h-, 48
- Moser's theorem, 17

- Narasimhan-Seshadri, higher dimensional theorem of, 44
- Narasimhan-Seshadri, theorem of, 41
- Nonabelian Hodge Theorem, general, 54
- Nonabelian Hodge theorem, unitary, 44
- Nonabelian sheaf cohomology, 46

- One-relator groups, 27

- Picard group, 34
- Polystable Higgs bundle, 53
- Polystable holomorphic bundle, 44
- Projective group, 3
- Properly rigid representation, 56
- Pseudocurvature, 49

- Rank of a coherent sheaf, 31
- Representation-problem, 15
- Resolution-problem, 15
- Rigid representation, 56

- Semi-stable Higgs bundle, 53
- Semistable holomorphic bundle, 40
- Serre's twisting sheaf, 34
- Simple vector bundle, 41
- Singularity set of a coherent sheaf, 30
- Slope, 40
- Special algebra, 37
- Stable Higgs bundle, 53
- Stable holomorphic bundle, 40

- Todd class, 33
- Torsion-free coherent sheaf, 32