Generalised Lie algebras in Algebra and Topology

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We discuss different generalisations of rational differential graded Lie algebras, and outline some recent applications to unstable homotopy theory [5][15], formal deformation theory [10], and the (generalised) homology of configuration spaces [7][8] away from characteristic zero.

1. Three applications of rational differential graded Lie algebras

Given a field k of characteristic zero, we recall the following notion:

Definition 1. A *(shifted) differential graded Lie algebra* $\mathfrak{g}$ over k consists of a chain complex $\ldots \to \mathfrak{g}_1 \xrightarrow{d} \mathfrak{g}_0 \xrightarrow{d} \mathfrak{g}_{-1} \to \ldots$ of k-vector spaces together with bilinear maps $[-,-] : \mathfrak{g}_i \times \mathfrak{g}_j \to \mathfrak{g}_{i+j-1}$ such that for all $x \in \mathfrak{g}_a$, $y \in \mathfrak{g}_b$, $z \in \mathfrak{g}_c$, we have:

(1) Antisymmetry: $[x,y] = (-1)^{ab}[y,x]$;
(2) Jacobi identity: $(-1)^{ac}[[x,y],z] + (-1)^{cb}[[z,x],y] + (-1)^{ba}[[y,z],x] = 0$;
(3) Leibnitz rule: $d([x,y]) = -[dx,y] - (-1)^a[x,dy]$.

Remark 2. The shifted grading convention arises naturally from Koszul duality; all Lie algebras appearing in this document are assumed to be shifted.

Rational differential graded Lie algebras have several classical applications:

Rational Lie models. Quillen [27] established an equivalence $(\mathcal{S}_*)_{\mathbb{Q},\geq 2} \simeq \mathrm{Lie}_{\mathbb{Q},\geq 2}$ between rational simply connected pointed spaces and differential graded Lie algebras $\mathfrak{g}$ with $\pi_i(\mathfrak{g}) = 0$ for $i < 2$. Under this correspondence, the rational n-sphere $S^n_{\mathbb{Q}}$ corresponds to the free Lie algebra on a class in degree n.

Rational homology of configuration spaces. Given a framed n-manifold M and an integer m, there is a (weight-preserving) isomorphism

$$\bigoplus_k H_*\left(\mathrm{Conf}_k(M) \otimes_{\Sigma_k} S^m; \mathbb{Q}\right) \cong H_*^{\mathrm{Lie}}\left(H_c^{-*}(M;\mathbb{Q}) \otimes \mathrm{Free}_{\mathrm{Lie}^{\mathbb{Q}}}(x_{n+m})\right). \tag{1}$$

Here $H_*^{\mathrm{Lie}}(-)$ denotes Lie algebra homology, and $H_c^*(-;\mathbb{Q})$ is compactly supported cohomology. The isomosphism (1) is due to Knudsen [18], and generalises work of Bödigheimer–Cohen–Taylor [4], Félix–Thomas [14], Totaro [34], and others. In practice, it is very useful; for example, we can read off that $H_*(\Omega^2 S^3; \mathbb{Q}) \cong \mathbb{Q}[1]$.

Rational deformation theory. Deformations of algebro-geometric objects over $\mathbb{Q}$ are controlled by rational differential graded Lie algebras. This general paradigm was first observed by Deligne [12], Drinfel'd [13], and Feigin, explored further by Hinich [16], Kontsevich–Soibelman [20], and Manetti [25], and finally formulated as an equivalence of ∞-categories by Lurie [22] and Pridham [26].

The Lurie-Pridham theorem identifies formal moduli problems over $\mathbb{Q}$, which encode deformation functors of algebro-geometric objects, with rational differential graded Lie algebras.

Remark 3. These applications extend to general fields of characteristic zero.

2. Settings away from characteristic zero

The three classical applications presented above use rational chain complexes; these model the ∞-category $\mathrm{Mod}_{\mathbb{Q}}$ of module spectra over $\mathbb{Q}$, i.e. $\mathbb{Q}$-local spectra.

It is possible to extend some of these results to other settings:

Modular settings. We could also work in chain complexes over a field k of characteristic p (e.g. $\mathbb{F}_p$), or over a complete local Noetherian base (such as $\mathbb{Z}_p$).

Chromatic settings. For every prime p, chromatic homotopy theory constructs infinitely many ring spectra $K(0)=\mathbb{Q}, K(1), K(2), \ldots$ known as Morava K-theories. For $h > 0$, these satisfy $K(h)_* \cong \mathbb{F}_p[v_h^{\pm 1}]$ with $|v_h| = 2(p^h - 1)$; they may be thought of as "generalised fields" sitting in between $\mathbb{Q}$ and $\mathbb{F}_p$. Accordingly, the ∞-category $\mathrm{Sp}_{K(h)}$ of $K(h)$-local spectra interpolates between rational and p-local spectra. As is customary, we suppress p from our notation for Morava K-theories.

3. Generalised Lie algebras and their applications

Away from characteristic 0, differential graded Lie algebras are not homotopically well-behaved; for example, their "free functor" fails to preserve quasi-isomorphisms.

In recent years, more adequate substitutes were introduced for different applications:

	Lie models	Configuration spaces	Deformation theory
rational	Differential graded Lie algebras		
chromatic	Spectral Lie algebras $\downarrow E_*^\wedge(-)$ Hecke Lie algebras		
modular			Partition Lie algebras (derived algebraic geometry) Spectral partition Lie algebras (spectral algebraic geometry)

We will give a brisk outline of the definitions and recent applications of these generalised Lie algebras;

Spectral Lie algebras. Let $\mathcal{O}_{\mathrm{Comm}}$ be the commutative operad in spectra. Salvatore [31] and Ching [11] have defined the spectral Lie operad as the dualised bar construction $\mathbb{D}(\mathrm{Bar}(\mathcal{O}_{\mathrm{Comm}}))$; its algebras are called *spectral Lie algebras.* Over $\mathbb{Q}$, these are equivalent to the rational differential graded Lie algebras in Definition 1.

The free spectral Lie algebra on a spectrum $X \in \mathrm{Sp}$ is given by

$$\mathrm{Lie}_k^s(X) = \bigoplus_n \mathbb{D}(\Sigma\Pi_n^\diamond) \otimes_{h\Sigma_n} X^{\otimes n},$$

where $\Sigma\Pi_n^\diamond$ is the unreduced-reduced suspension of the n^{th} partition poset. This makes spectral Lie algebras susceptible to methods from combinatorial topology [1].

Unstable chromatic homotopy theory. Spectral Lie algebras were first linked to unstable chromatic homotopy theory by Behrens and Rezk [5] who, for each pointed space X, constructed a comparison map $c_X : \Phi(X) \to \mathrm{TAQ}_{S_{K(h)}}(S^X_{K(h)})$ from the Bousfield–Kuhn functor on X to the topological André–Quillen homology of the $\mathbb{E}_\infty$-ring $S^X_{K(h)}$ – the latter is always a spectral Lie algebra.

The map c_X is an equivalence for X a sphere [3], and also for special unitary and symplectic groups [6]. In [9], we proved (with Heuts) that c_X fails to be an equivalence on wedges of spheres and Moore spaces.

Heuts [15] later equipped the Bousfield–Kuhn functor $\Phi(X)$ with the structure of a spectral Lie algebra, and used this to establish an equivalence between $K(h)$-local spectral Lie algebras and a certain ∞-category $\mathcal{M}_{K(h)}$ of periodic spaces.

Hecke Lie algebras. Lie algebras in $\mathrm{Sp}_{K(h)}$ are not amenable to explicit computations, as their homotopy groups involve the $K(h)$-local homotopy groups of spheres. However, $\mathrm{Sp}_{K(h)}$ is equivalent to $K(h)$-local module spectra over a height h Morava E-theory E with action by the stabiliser group $\mathbb{G}$.

In [7], we introduced *Hecke Lie algebras* to describe the operations acting on the homotopy groups of spectral Lie algebras in $\mathrm{Mod}^\wedge_E$, the ∞-category of $K(h)$-local E-modules. In particular, $E^\wedge_*(\Phi(X))$ is a Hecke Lie algebra for any pointed space X.

Very roughly, Hecke Lie algebras are Lie algebras in E_*-modules, equipped with an additional additive action by the cohomology of Rezk's ring Γ, which is closely related to the Hecke algebra of $\mathrm{GL}_n(\mathbb{Z}_p)$ [29] [30]. There is an additional congruence at $p = 2$, and special care must be taken when composing operations.

These concrete algebraic structures facilitated recent computational advances:

Chromatic homotopy theory of configuration spaces. Computing the Morava K- or E-theory of unordered configuration spaces of manifolds M is a hard problem.

For $M = \mathbb{R}^n$, the problem is of partcular interest, as the relevant groups parametrise Dyer-Lashof operations on $\mathbb{E}_n$-algebras. At *chromatic height* $h = 1$, the problem was solved by Langsetmo [21]. In *dimensions* $n = 2, 3, 4$, it was solved by Yamaguchi [35] and Tamaki [32] [33] with increasingly laborious methods. For general chromatic heights h and dimensions n, Ravenel stated a conjecture in [28].

With Hahn and Knudsen [8], we apply the theory of Hecke Lie algebras to Knudsen's spectral generalisation of (1) (cf. [19, Section 3.4.]) to compute the Morava K- and E-homology groups (at a prime p) of the configuration space of p points in $\mathbb{R}^n$, for all heights h and all dimensions n.

We carry out similar computations for configuration spaces of punctured surfaces. Letting h tend to infinity, we can read off previously unknown $\mathbb{F}_p$-homology groups.

One might hope to perform this computation without reference to E-theory by using spectral Lie algebras over $\mathbb{F}_p$. Their operations have been computed at $p = 2$ by Antolín-Camarena [2]. For p odd, partial progress has been made by Kjaer [17], but the Adem relations remain unknown. Our method from [7] does not immediately apply, as it uses $K(h)$-local Tate vanishing to identify orbits with fixed points.

However, algebraic geometry leads to other generalised Lie algebras over $\mathbb{F}_p$:

Formal moduli. The infinitesimal deformations of a given algebro-geometric object over a field k are described by a corresponding *formal moduli problem*, which is a functor of points defined on suitable Artin k-algebras satisfying a gluing axiom.

Away from characteristic 0, there are two variants of formal moduli problems, as algebraic geometry can either be based on simplicial commutative rings ("*derived algebraic geometry*") or on connective $\mathbb{E}_\infty$-rings ("*spectral algebraic geometry*").

Partition Lie algebras & spectral partition Lie algebras. Together with Mathew [10], we introduce two new generalisations of Lie algebras, called *partition Lie algebras* and *spectral partition Lie algebras*, over any base field k.

In characteristic 0, both recover the differential graded Lie algebras from Definition 1. In characteristic p, they are distinct from previously known generalisations (e.g. spectral Lie algebras or simplicial/cosimplicial restricted Lie algebras).

We then prove that our Lie algebras control fomal moduli problems in derived and spectral algebraic geometry, respectively. This generalises the Lurie–Pridham theorem from characteristic 0 to base fields of arbitrary characteristic (e.g. $\mathbb{F}_p$); we also offer a version over mixed characteristic bases (like $\mathbb{Z}_p$).

Our new Lie algebras are no longer governed by operads; instead, they are algebras over monads. Given a field k, we construct monads $\mathrm{Lie}^\pi_{k,\Delta}$ and $\mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}$ on the ∞-category Mod_k of k-module spectra. These preserve filtered colimits, geometric realisations, and are given on *coconnective* objects $X \in \mathrm{Mod}_{k,\leq 0}$ by

$$\mathrm{Lie}^\pi_{k,\Delta}(X) = \bigoplus_n \widetilde{C}^*(\Sigma\Pi_n^\diamond, k) \otimes^{\Sigma_n} X^{\otimes n} \;\; ; \;\; \mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}(X) = \bigoplus_n \widetilde{C}^*(\Sigma\Pi_n^\diamond, k) \otimes^{h\Sigma_n} X^{\otimes n}.$$

Here $\widetilde{C}^*(-,k)$ denotes the reduced k-valued singular cochains of a space, whereas $(-)^{\Sigma_n}$ and $(-)^{h\Sigma_n}$ denote strict invariants and homotopy invariants, respectively. The precise definition of strict fixed points uses the genuine equivariant topology of partition complexes and requires some care.

Future directions. As partition Lie algebras involve fixed points rather than orbits, one can adapt the arguments in [7] to compute their operations and relations. Following the strategy in [18] and [8] then leads to a new approach to the $\mathbb{F}_p$-homology of configuration spaces – a subject where many computations are yet to be done.

Two other tasks for the future are to give a Lie algebraic description of deformation theory in chromatic contexts, and to construct Lie models for the modular homotopy type of spaces (Koszul dual to Mandell's commutative models in [24]).

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