The $p-$adic Hodge Theory of Semistable Galois Representations

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Abstract. In this expository article, we will describe the equivalence between weakly admissible filtered $(\phi, N)-$modules and semistable $p-$adic Galois representations. After motivating and constructing the required period rings, we will focus on Colmez-Fontaine's proof that "weak admissibility implies admissibility".

It is certainly no exaggeration to call the absolute Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ the holy grail of algebraic number theory: it encapsulates all the hidden symmetries of the solutions of rational polynomials.

One way of studying this topological group is via its continuous finite-dimensional representations. Since the topology is profinite, there is an insufficient supply of such representations over the archimedean fields $\mathbb{R}$ or $\mathbb{C}$. We therefore work over the non-archimedean local fields $\mathbb{Q}_\ell$, where we encounter a richer theory.

Rather than thinking of $\mathbb{Q}$ and its closure as subfields of $\mathbb{R}$ and $\mathbb{C}$, we could also embed $\mathbb{Q}$ into one of its non-archimedean completions $\mathbb{Q}_p$ and thereby obtain a restriction map

$$Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to Gal(\overline{\mathbb{Q}}/\mathbb{Q})$$

These two observations indicate that ℓ-adic representations of the absolute Galois group of $\mathbb{Q}_p$ have deep relations to the absolute Galois group of $\mathbb{Q}$, and similar remarks apply to other number fields.

Inside the absolute Galois group G_K of a $p-$adic field K such as $\mathbb{Q}_p$, we have distinguished subgroups, namely the inertia and the wild inertia subgroup. In turns out that for $\ell \neq p$, the wild inertia (which is a pro-p-group) is mapped to a finite subgroup of $GL_n(\mathbb{Q}_\ell)$ by any continuous representation. In order to gain access to this part of G_K, we need to examine $p-$adic representations.

Galois representations can be constructed by taking the étale cohomology of schemes over K, and one often wants to establish comparison theorems between the cohomology of special and generic fibres of integral models. While for $\ell \neq p$, the $\ell-$adic étale cohomology groups are very well-behaved, this no longer holds true at the characteristic prime p: for example, the axioms of a Weil cohomology theory fail.

It was the aim to relate $p-$adic étale cohomology to a well-behaved Weil cohomology theory in characteristic p, namely (log-)crystalline cohomology, which led Fontaine to define several *period rings* producing certain comparison isomorphisms. He thereby realised a program already envisaged by Grothendieck.

More generally, one can use these period rings to produce $D-$functors from the category of $p-$adic Galois representations to certain linear-algebraic categories. While these functors will not be well-behaved on all representations, each of them restricts to a fully faithful functor on a certain sub-Tannakian category of so-called admissible representations.

The period ring B_{st} is designed to relate the $p-$adic étale cohomology of the generic fibre of an $\mathcal{O}_K$ -scheme with semistable reduction to the log-crystalline cohomology of its special fibre. The Fontaine-Mazur conjecture indicates, in a sense which we shall not make precise here, that the B_{st}-admissible representations are exactly the ones coming from geometry, and therefore deserve special attention.

In the groundbreaking paper "*Construction des reprsentations p-adiques semi-stables*", Colmez and Fontaine gives the first proof of a concrete characterisation of the essential image of the functor D_{st} associated to B_{st} (for a clearer proof using integral $p-$adic Hodge theory, see [**14**]).
The resulting equivalence of categories is very powerful, since it allows us to understand semistable representations in terms of very tractable and concrete linear algebraic objects - the so-called weakly admissible filtered $(\phi, N)-$modules.

The principal aim of this essay is to introduce the relevant objects, questions and definitions leading up to this deep theorem in a motivated manner and then present its proof in detail.

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Contents

CHAPTER 1

Introduction

One of the main focal points of twentieth century mathematics in general and the French school around Grothendieck in particular was the quest for the proof of the Weil conjectures:

1. From Weil Conjectures to ℓ-adic Cohomology

These form a collection of computationally very powerful statements about the number of points on smooth projective varieties over finite fields.

More precisely, they provide a functional equation for the *zeta function* $\zeta(X,s)$, describe this function as a quotient of certain integral polynomials $P_0, ..., P_{2dimX}$ in q^{-s}, make statements about the zeros ("Riemann hypothesis") of these polynomials, and finally relate their degrees to topological Betti numbers[1] of lifts to number fields (compare [**1**]).

While it took many years until these conjectures were proven in full glory by Deligne (see [**3**]) , the rough path to a proof was already envisaged by Weil (see [**15**] for a nice survey): Given a smooth projective variety X over $\mathbb{F}_q$, points of $X(\mathbb{F}_{q^n})$ are exactly fixed points of the n^{th} power of the Frobenius σ_q acting on $X(\overline{\mathbb{F}}_q)$. In topology, the *Lefschetz fixed-point theorem* expresses the number of fixed points[2] of a continuous self-map g of a (triangulable) compact space in terms of the action of g on rational cohomology. The idea was that one should be able to express the number of fixed points of the Frobenius in a similar way once we have an algebraic version of this result for general algebraically closed fields, but this of course required a definition of a cohomology theory in characteristic p.

In order for the proofs to go through smoothly, we need this theory to satisfy various nice properties which hold for the ordinary cohomology of complex projective varieties (these are axiomatised by the notion of a *Weil cohomology theory*)). Moreover, we need it to satisfy nice comparison theorems under reduction in order to relate the degrees of the $P_i's$ to topological Betti numbers.

Such a theory was developed by Grothendieck with the help of Artin: One first sets up the general framework of *Étale cohomology groups* $H^n_{ét}(X,-)$, and then uses these groups to define *ℓ-adic cohomology* groups as

$$H^i(\overline{X},\mathbb{Z}_\ell) = \varprojlim H^i_{ét}(\overline{X},\mathbb{Z}/\ell^k\mathbb{Z})$$

Here ℓ is any prime not equal to the characteristic p over which we work, X is a smooth projective variety over $\mathbb{F}_q$ (or any other field K of characteristic $\neq \ell$, such as $\mathbb{Q}$ or $\mathbb{Q}_p$), and $\overline{X}$ the base change to $\overline{\mathbb{F}}_q$ (or the algebraic closure $\overline{K}$).

[1] The Betti numbers of a variety over a number field are defined by considering it as a complex manifold - this is independent of the chosen embedding by GAGA.

[2] Assuming this number is finite.

These groups indeed define a Weil cohomology theory (this is *not* true for $\ell = p$), and there is a nice comparison isomorphism

$$H^i_{\acute{e}t}(\overline{X}, \mathbb{Z}_l) \cong H^i_B(X(\mathbb{C})^{an}, \mathbb{Z}_l)$$

where the right hand side denotes Betti cohomology of any smooth lifting.

REMARK 1.1. Notice that by tensoring up to $\overline{K}$, we have killed Galois-cohomological contributions and hence made the theory very geometric (e.g. by forcing $Spec(\mathbb{F}_p)$ to have the cohomology of a point). The étale cohomology of the non-geometric version of an $\mathbb{F}_p$-scheme can be computed by the Hochschild-Serre spectral sequence

$$H^i(G_K, H^j_{\acute{e}t}(\overline{X}, \mathbb{Q}_p)) \Rightarrow H^{i+j}_{\acute{e}t}(X, \mathbb{Q}_p)$$

This cohomology theory allowed Grothendieck to prove large parts of the Weil conjectures (except for the Riemann hypothesis, which is *not* to be confused with the Millennium problem going under the same name).

It is important to notice that the above comparison lemma gives us torsion information about the Betti cohomology groups of $X(\mathbb{C})^{an}$ at all primes $\ell \neq p$, but it breaks down for $\ell = p$. For the sake of proving the Weil conjectures, this causes no problems, but once we move on to more refined computational questions such as the determination of the $p-$adic valuation of the roots of the $P_i's$ (compare [**16**]) or more modern topics such as $p-$adic Galois representations via étale cohomology groups, a good cohomology theory at the characteristic prime is indispensable.

The solution to this problem was outlined by Grothendieck, worked out by his student Berthelot, and goes under the name of *crystalline cohomology.*

2. The General Setting

Before we begin with the mathematics, we should fix the setting of this paper. Unless mentioned otherwise, K will always denote a fixed field of characteristic zero equipped with a complete discrete valuation. We write:

- $\mathcal{O}_K$ its ring of integers (which is a discrete valuation ring)
- k for its residue field k which we shall assume to be *perfect* and of characteristic $p > 0$
- $K_0 = \mathrm{Frac}(W(k))$ for the maximal unramified subextension of $K/\mathbb{Q}_p$ - note that K/K_0 is totally ramified.
- $P = \hat{K^{nr}}$ for the completion of the maximal unramified subextension of K in $\overline{K}$
- $P_0 = W(\overline{k})$ for the maximal unramified subextension of $P/\mathbb{Q}_p$.
- $\mathbb{C}$ for the completion of the algebraic closure of K. Note that $\mathbb{C}_K = \mathbb{C}_P$. We will also use this notation for the usual complex numbers, but this should not cause confusion.

We will call such fields K a *p-adic* fields .

3. Crystalline Cohomology

Roughly speaking, the definition of crystalline cohomology solves the problem of writing down a Weil cohomology theory for smooth and proper schemes X over fields k (perfect and of positive characteristic p) which relates well to the algebraic de Rham cohomology of smooth and proper lifts to $W(k)$. We shall only outline several crucial results here and refer the interested reader to [**13**].

We shall briefly sketch the definition: Using the canonical map $Spec(k) \to Spec(W_n(k))$[3]., we can consider k-schemes as $W_n(k)$-schemes.

Using this, we can define a site S_n over X by enriching the usual Zariski opens $U \subset X$ by closed immersions $U \to T$ (over $W_n(k)$ - this gives extra flexibility) with a divided power structure on the vanishing ideal which is compatible with the one on $pW_n(k) \subset W_n(k)$:

$$\begin{tikzcd} U \arrow[r] \arrow[d, hook] & T \\ X & \end{tikzcd}$$

We then let $H^*(X/W_n)$ be the sheaf cohomology of $(X\mathcal{O}_X^n)$ (where $\mathcal{O}_X^n$ is the crystalline structure sheaf).

DEFINITION 1.2. Let X be a scheme over a perfect field k of characteristic $p > 0$. Its *crystalline cohomology* is then given by the $W(k)-$module

$$H^i_{cris}(X/W(k)) = \varprojlim H^i(X/W_n(k))$$

REMARK 1.3. The functor $\mathrm{Frac}(W(k)) \otimes_{W(k)} H^*_{cris}(X/W)$ on smooth and proper schemes over k is a Weil cohomology theory.

The point of this rather peculiar definition is that we can use closed immersions $X \to Z$ into $W_n(k)$-schemes to produce certain de Rham complexes

$$\mathcal{O}_D \otimes \Omega^\bullet_{Z/W_n(k)}$$

whose cohomology then computes $H^*(X/W_n(k))$. Applying these individual comparison maps all at once allows us to compare crystalline cohomology of a smooth and proper scheme over k to the de Rham cohomology of that same scheme and to the de Rham cohomology of its smooth lifts to $W(k)$.

THEOREM 1.4. *(Comparison de Rham - Crystalline, unramified case) Let X be a smooth and proper scheme over k. We then have a short exact sequence*

$$0 \to H^*_{cris}(X/W(k)) \otimes k \to H^*_{dR}(X/k) \to Tor_1^W(H^{*+1}(X/W(k)), k) \to 0$$

Moreover if Z is a smooth and proper scheme over $W(k)$ with special fibre

$$X = Spec(k) \times_{W(k)} Z$$

then crystalline and algebraic de Rham cohomology agree:

$$H^i_{cris}(X/W) \cong H^i_{dR}(Z/W)$$

We will soon refine this theorem and relate the crystalline cohomology of the *special fibre* to the de Rham cohomology of the generic fibre in the case of good (and later even semistable) reduction.

Before doing so, we should however notice a much more basic property of crystalline cohomology: the presence of a Frobenius. The absolute Frobenius endofunctor on the category of smooth and proper schemes over k yields a natural transformation

$$\phi : H^*_{cris}(-/W) \to H^*_{cris}(-/W)$$

Writing $K_0 = \mathrm{Frac}(W(k))$, this gives $(K_0 \otimes_{W(k)} H^*_{cris}(-/W), \phi)$ the following structure:

[3]Here $W_n(k) = W(k)/p^nW(k)$ should be thought of as a generalisation of $\mathbb{Z}/p^n\mathbb{Z}$.

DEFINITION 1.5. A $\phi-$*module* is a K_0-vector space D together with an endomorphism ϕ such that, writing $\sigma : K_0 \to K_0$ for the Frobenius, the map ϕ is $\sigma-$semilinear. This means that

$$\phi(\lambda x) = \sigma(\lambda)\phi(x)$$

for all $\lambda \in K_0$, and all $x \in D$.

We say such a *phi*-module is *finite* if it is finite-dimensional and ϕ is injective.

de Rham vs. Crystalline Cohomology for Good Reduction. Let X be a smooth and proper scheme over the complete DVR $\mathcal{O}_K$. We should think of X as a family of schemes fibered over a disc whose origin is the maximal ideal with perfect residue field k and whose punctured part is the generic ideal with residue field K, a finite totally ramified extension of $K_0 = \mathrm{Frac}(W(k))$.
We then have the following comparison theorem, which is significantly harder than the one stated above due to the presence of ramification:

THEOREM 1.6. *(Berthelot-Ogus) In the situation described above, write Y for the special and X_K for the generic fibre. We then have:*

$$H^*_{cris}(Y/W(k)) \otimes_{W(k)} K \cong H^*_{dR}(X_K/K)$$

The algebraic de Rham cohomology $H^*_{dR}(X_K/K) = \mathbb{H}^*(X_K, \Omega^\bullet_{X_K/K})$ (here we take hypercohomology) comes to us with the usual Hodge filtration whose i^{th} piece is $H^*(X_K, \Omega^{\geq i}_{X_K/K})$. We therefore see that in the case of good reduction, the triple $(H^*_{cris}(Y/W(k)) \otimes_{W(k)} K_0, \phi, Fil)$ has the following structure:

DEFINITION 1.7. A *filtered ϕ-module* as a triple (D, ϕ, Fil) where (D, ϕ) is a $\phi-$module as defined above and Fil is a filtration on $D \otimes_{K_0} K$.

It is called *finite* if its underlying $\phi-$module is finite.

4. Log-Crystalline Cohomology

The above nice comparison between de Rham cohomology on characteristic zero and crystalline cohomology in characteristic p relies on the assumption of good reduction. However, several naturally occurring varieties in number theory do not possess such a well-behaved reduction, a famous example being the Tate curve. We therefore introduce a less restrictive version of reduction:

DEFINITION 1.8. We say a proper flat scheme X over the complete DVR $\mathcal{O}_K$ has *semistable reduction* if X is regular with smooth generic fibre and such that the special fibre is a reduced divisor with normal crossings on X.

We run into difficulties when we want to prove comparison isomorphisms for crystalline cohomology in the case of semistable reduction as the crystalline side can be very badly behaved.
In order to fix this issue, we need to introduce an enrichment of crystalline cohomology, namely *log-crystalline cohomology.*
Let us start with a scheme X over $\mathcal{O}_K$ with semistable reduction as above and think about which extra structure we could take into account on its special fibre.

The main insight is that the reduced normal crossing divisor $D = \bigcup D_i$ allows us to define the étale sheaf of functions which are *nonvanishing outside of* D by

$$\mathcal{M} = j_* \mathcal{O}^{\times}_{X \setminus D}$$

Write $i : Y \hookrightarrow X$ for the map from the special fibre

$$Y = Spec(k) \otimes_{Spec(\mathcal{O}_K)} X$$

into X.

This turns (X, M) and $(Y, i^{-1}M)$ into *schemes with a fine log-structure*, by which we *roughly* mean:

Definition 1.9. A *scheme with a fine log structure* is a triple $(X, \mathcal{M}, \alpha)$ where X is a scheme, $\mathcal{M}$ is an étale sheaf of monoids, and $\alpha : \mathcal{M} \to \mathcal{O}_X$ is a multiplicative map inducing an isomorphism

$$\alpha^{-1}(\mathcal{O}_X^{\times}) \to \mathcal{O}_X^{\times}$$

Such a log-structure is called *fine* if it is étale-locally finitely generated in a sense which we shall not spell out here in detail (see [**12**] for a careful exposition).

Observe that the complete DVR $\mathcal{O}_K$ has a canonical fine log-structure

$$M \to A$$

determined by its closed point. Fix any fine log-structure L on $Spec(k)$ once and for all - this gives us the log-structure

$$W_n(L) := L \oplus \ker(W_n(k)^{\times} \to k^{\times})$$

on $W_n(k)$, and this ring supports a divided power structure on the ideal $pW_n(k)$ as before.

From here, we proceed as before: Given a morphism $(Y, M) \to (Spec(k), L)$ of schemes with fine log-structures and a positive integer n, we obtain

$$(Y, M) \to (Spec(W_n(k)), W(L))$$

Again, we define a site S_n^{log} over (Y, M) by enriching the étale maps $U \to X$ slightly more structure than before: Rather than just adding a closed immersion with a divided power structure, we shall add an exact closed immersion

$$(U, M|_U) \to (T, M_T)$$

into a another log-scheme (T, M_T) over $(Spec(W_n(k)), W(L))$ and a divided power structure on the vanishing ideal of U in T which is compatible with the one on $W_n(k)$ as before.
We write $H^*_{log}(Y/W_n(k))$ for the sheaf cohomology of Y with respect to its structure sheaf on this site.

Definition 1.10. We define the *log-crystalline cohomology* of Y to be

$$H^i_{log-cris}(X/W(k)) = \varprojlim H^i_{log}(Y/W_n(k))$$

Again, this is a $W(k)$-module. Furthermore, the absolute Frobenius turns it again into a ϕ-module.

There is a crucial new structure appeaing: the *monodromy operator*

$$N : H^*_{log-cris}(X/W(k)) \to H^*_{log-cris}(X/W(k))$$

We will not attempt to define this operator precisely here, but just remark that it is a $p-$adic analogue of looking at the residue at the origin of the Gauß-Manin connection of a family of complex manifolds over $\mathbb{C}^*$ with semistable reduction at the origin. Another way to think about it is as a measure of "lack of smoothness" at the origin.

LEMMA 1.11. *The operator satisfies $N\phi = p\phi N$.*

We are now in a position to quote the extension of the theorem of Berthelot-Ogus to the semi-stable case:

THEOREM 1.12. *(Kato-Hyodo-(Fontaine-Illusie), see* [**13**]*) Let $X \to Spec(A)$ be a flat and proper scheme with semistable reduction. We obtain a morphism $(X, M) \to (Spec(\mathcal{O}_K), N)$ of schemes with log-structures.*
Then for each prime $\mathfrak{p}$ of A, we have a $K-$isomorphism between the log-crystalline cohomology of the special fibre Y and the de Rham cohomology of the generic fibre X_K:

$$\rho_{\mathfrak{p}} : H^*_{log-cris}(Y/W(k)) \otimes_{K_0} K \cong H^*_{dR}(X_K/K)$$

REMARK 1.13. There is in fact a very nice relationship between the isomorphisms for various choices of $\mathfrak{p}$, see [**12**].

We therefore see that in this case, the quadruple $(H^*_{logcrys}(Y/W(k)) \otimes_{W(k)} K_0, \phi, Fil, N)$ has the following structure:

DEFINITION 1.14. A *filtered (ϕ, N)-module* is a quadruple (D, ϕ, Fil, N) where (D, ϕ, Fil) is a filtered $\phi-$module as defined above, and $N : D \to D$ is an operator with $N\phi = p\phi N$.

Again, we call such a module *finite* if its underlying $\phi-$module is finite.
We write $MF_K^{\phi,N}$ for the category of finite filtered $(\phi, N)-$module over K.

The aim to compare these different cohomology theories was the core initial motivation for the study of period rings, and keeping them in mind will help us to increase our understanding of the more general settings in which we shall work later.

CHAPTER 2

Period Rings

In the last chapter, we have introduced the three main cohomology theories in our context:

$$H^*_{ét}, \quad H^*_{dR} \text{ and } H^*_{cris} \text{ (and } H^*_{log-cris})$$

We will now begin the difficult task of constructing the corresponding period rings B_{dR} and B_{cris} (and its variant B_{st}).

1. Gadgets for Comparison Isomorphisms

The theorems of the previous chapter give good comparison isomorphisms relating de Rham and (log-)crystalline cohomology - we could get from one to the other just by tensoring up to K (hence killing torsion information).

The compatibility of these two theories should not come as a surprise though, since the definition of crystalline cohomology imitates the de Rham case in positive characteristic. The next, much harder question is of course how to fit p-adic étale cohomology into this framework.

First, we recall the archimedean analogue - here we try to compare

$$H^*_{dR}(X/L) \text{ and } H^*_B(X(\mathbb{C})^{an}, \mathbb{Q})$$

for, say, a smooth projective variety X over a number field $L \subset \mathbb{C}$.

The usual integration pairing gives rise to an isomorphism

$$H^*_{dR}(X/L) \otimes_L \mathbb{C} \to H^*_B(X(\mathbb{C})^{an}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$

This isomorphism is usually *not* induced by a morphism on rational cohomology groups because of the presence of irrational *periods* such as $2\pi i = \int_{S^1} \frac{dz}{z}$. However, it is enough to add all rational combinations of numbers which occur by integrating integral cycles over algebraic differential forms on smooth projective varieties over number fields, and we don't need to go all the way up to $\mathbb{C}$ (see [**10**] for a detailed treatment). Writing B for the collection of such periods, we get:

$$H^*_{dR}(X, L) \otimes_L B \to H^*_B(X, \mathbb{Q}) \otimes_{\mathbb{Q}} B$$

We would like to examine the action of $Gal(\overline{L}/L)$ on this equation, but this fails since the induced automorphisms on $X(\mathbb{C})^{an}$ can be horribly discontinuous and will therefore not induce an isomorphism on Betti cohomology groups .
We could of course go straight up to the reals instead (notice that H^*_{dR} still denotes algebraic de Rham cohomology):

$$H^*_{dR}(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \to H^*_B(X(\mathbb{C})^{an}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$

Here, we indeed get a $Gal(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$ action on the left side by conjugation on $\mathbb{C}$ and on the right side by conjugation on both $H^*_B(X(\mathbb{C}^{an}, \mathbb{Q})$ and on $\mathbb{C}$. Note that we can recover real de Rham cohomology by taking fixed points on the right hand side.

Example. As an example to illustrate this, we can take

$$X = Spec(\mathbb{Q}[t, t^-1])$$

we check that indeed $H^*_{dR}(X(\mathbb{R}), \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ is spanned by $\frac{dx}{x}$ in degree 1, and maps to the class

$$\left[C \mapsto \frac{1}{2\pi i} \int_C \frac{dz}{z}\right] \in H^*_B(X(\mathbb{C})^{an}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$

Note that conjugation fixes both sides as it changes the orientation of C and sends i to $-i$.

We are now in a position to outline the content of the remainder of this chapter. Our aim is to find analogues of the above period isomorphism in the $p-$adic world - here Betti cohomology should be replaced by étale cohomology, and de Rham cohomology by either de Rham or crystalline or log-crystalline cohomology. Our isomorphisms should not just be isomorphisms of vector spaces, but of *structured objects*.

We will describe how the extra structures on these period rings arise and will quote the comparison isomorphisms - a construction will follow in the later sections of this chapter. An excellent reference for this material is [**23**].

Étale versus de Rham. Let us begin with the comparison between étale and algebraic de Rham cohomology for smooth and proper schemes X over our $p-$adic field K. Our aim is to find a $K-$algebra B_{dR} such that for all smooth and proper schemes X, we have

$$H^*_{dR}(X/K) \otimes_K B_{dR} \cong H^*_{\acute{e}t}\left(\overline{X}\right) \otimes_{\mathbb{Q}_p} B_{dR}$$

as structured objects. By this, we mean that both sides should come with the same additional data:

Since the right hand side has a G_K-action, this should hold for the left side should too, and we therefore want B_{dR} to be equipped with a $K-$linear G_K-action. We then require the isomorphism to intertwine the diagonal actions on both sides, where we let G_K act trivially on de Rham cohomology.

As the left hand side comes with a natural G_K-stable filtration, we also want that B_{dR} is filtered in a G_K-stable way. We have product filtrations on both sides, where we put the trivial filtration on $H^*_{\acute{e}t}$, and we want our isomorphism to preserve these filtrations.

If we also claim that $B^{G_K} = K$, we can even recover de Rham cohomology by taking Galois invariants on both sides:

$$H^*_{dR}(X/K) = H^*_{dR}(X/K) \otimes_K (B_{dR})^{G_K} \cong \left(H^*_{\acute{e}t}\left(\overline{X}\right) \otimes_{\mathbb{Q}_p} B_{dR}\right)^{G_K}$$

Étale versus Crystalline. Assume for a moment that we had found such a good such ring B_{dR}. Then given any smooth and proper scheme X over $\mathcal{O}_K$ with generic fibre X_K and special fibre Y, we can compare étale cohomology of X_K to de Rham cohomology of X_K, which can then compare to the crystalline cohomology of its good reduction Y using Berthelot-Ogus. We obtain the following comparison isomorphism:

$$(H^*_{cris}(Y/W(k)) \otimes_{W(k)} K) \otimes_K B_{dR} \cong H^*_{\acute{e}t}\left(\overline{X_K}\right) \otimes_{\mathbb{Q}_p} B_{dR}$$

A similar remark applies to log-crystalline cohomology, semistable reduction and the theorem of Kato-Hyodo-(Fontaine-Illusie).
However, these are *not* isomorphisms of structured objects yet as the left hand side is endowed with a Frobenius, while the right hand side is not.

The combined work of Tate and Grothendieck-Messing implies that given a smooth and proper abelian variety A over $\mathbb{Z}_p$, the following two objects determine each other:

- The $p-$adic Galois representation $V_p(A) = T_p(A_{\mathbb{Q}_p}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ associated to the *Tate-module*[1] of the generic fibre.
- The Hodge-filtered Dieudonné module[2] of the special fibre.

Grothendieck wanted to generalise this statement, and therefore asked if there was a way of passing directly from the crystalline cohomology of the special fibre Y to the étale cohomology of the generic fibre X_K - the *mysterious functor*.
The answer was given by Fontaine, who constructed a K_0-algebra B_{cris} with extra structure such that $B_{cris}^{G_K} = K_0$ and such that

$$H^*_{cris}(Y/W(k)) \otimes_{W(k)} B_{cris} \cong H^*_{\acute{e}t}(\overline{X_K}) \otimes_{\mathbb{Q}_p} B_{cris}$$

is an isomorphism taking all the present extra structure into account.

By this, we mean that B_{cris} is not just a filtered G_K-module and the isomorphism preserves the induced diagonal action and filtration, but that it also comes equipped with a *Frobenius* endomorphism

$$\phi : B_{cris} \to B_{cris}$$

such that the above comparison isomorphism preserves the two diagonal Frobenius morphisms (where we let it act trivially on $H^*_{\acute{e}t}(\overline{X_K})$). Observe that by taking G_K-invariants, we can unfortunately only recover $H^*_{cris}(Y/W(k)) \otimes_{W(k)} K_0$ and therefore again lose torsion information.

[1] The p-adic Tate module of an abelian variety A over a field K is the projective limit of the following system of $p-$torsion point of its K^s points $A(K^s)$:

$$Tp(A) = \varprojlim_n A(K^s)[p^n]$$

Here transition functions are given by taking p^{th} power. The $p-$adic Tate module is naturally a $\mathbb{Z}_p$-module, and as long as p does not divide $char(K)$, the $p-$adic Galois representation defined by this module in the obvious way is dual to the first étale cohomology group $H^1_{\acute{e}t}(A(K^s), \mathbb{Z}_p)$.

[2] The *Dieudonné*-module of a $p-$divisible formal group over a perfect field k of characteristic p is a certain $W(k)$-modules with a Frobenius-semilinear endomorphism. While there are quick ways do write down a definition in the modern literature (see for example [**22**]) one could rush through in a footnote, we instead refer the reader to [**6**] for a more detailed treatment. The Dieudonn'e module of the $p-$divisible group associated to an abelian variety over a finite field is just its crystalline cohomology.

Étale versus log-Crystalline. The *semistable period ring* B_{st} then plays an analogous role in the case of semistable reduction and log-crystalline cohomology: It is a $W(k)$-module B_{st}, endowed with additional structures, such that

$$H^*_{log-crys}(Y/W(k)) \otimes_{W(k)} B_{st} \cong H^*_{\acute{e}t}(\overline{X_K}) \otimes_{\mathbb{Q}_p} B_{st}$$

for all schemes X over $\mathcal{O})_K$ with semistable reduction.

Again claiming "same structures on both sides", we see that B_{st} should not just be filtered module with a linear G_K-action and a Frobenius ϕ, but should also come equipped with a monodromy operator N. In the above comparison isomorphism, the operators $N \otimes id + id \otimes N$ on the left and $1 \otimes N$ on the right will be compatible. We will again find $(B^{st})^{G_K} = K_0$, so that we can recover $H^*_{log-crys}(Y/W(k)) \otimes_{W(k)} K_0$ by taking invariants and again lose torsion.

2. Period-Rings in a Representation-Theoretic Setting

Before we give the detailed constructions of B_{dR}, B_{cris} and B_{st}, we shall reformulate our aims in a cleaner, more general representation-theoretic framework. For this, we want to think of the appearing étale cohomology groups just as specific examples of *p-adic Galois representations* , i.e. continuous homomorphisms [3]

$$G_K \to GL(V)$$

into the group of automorphisms of some finite-dimensional $\mathbb{Q}_p$-vector space V.

From this standpoint, each of the above rings $B = B_{dR}, B_{cris}, B_{st}$ assigns to a Galois representation V the $E = B^{G_K}$ vector space[4]

$$D_B(V) = (B \otimes_{\mathbb{Q}_p} V)^{G_K}$$

together with certain extra structures inherited from B and a comparison map

$$B \otimes_E D_B(V) \to B \otimes_{\mathbb{Q}_p} V$$

This fits into the general axiomatic framework of *admissible representations* (see [**5**]), whose basics we shall outline now:
Let F be a field (this will be $\mathbb{Q}_p$ in all our applications) and G any group (such as G_K) acting F-linearly on some $F-$algebra B (like all our period rings). Write $E = F^G$ for the ring of invariants (this is K in the de Rham and K_0 in the crystalline and semistable case). We will also assume that E *is a field.*

Then we can consider the following functor on finite-dimensional F-representations of G:

$$D_B : Rep_F(G) \to Vect_E$$
$$V \mapsto (B \otimes_F V)^G$$

We also have a *comparison morphism*

$$\alpha_{B,V} : B \otimes_E D_B(V) \to B \otimes_F V$$

This assignment is in general not very well-behaved, but the following conditions will ensure many nice properties:

[3] As always, the absolute Galois group G_K is equipped with its *profinite* topology.

[4] This space is a tensored version of the corresponding cohomology theory H^*_{dR}, H^*_{cris},, or H^*_{st} if our representation V is given by étale cohomology of a nice enough scheme.

DEFINITION 2.1. Given (F,G) as above, we call an $F-$algebra (F,G)-*regular* if

- The invariants B^G of B agree with the invariants $(\mathrm{Frac}(B))^G$ of the fraction field of B.
- If the line Fb spanned by $b \in B$ is $G-$stable, then b is a unit in B.
- B is a domain.

Note that these conditions are trivially satisfied whenever B is a field. In this setting, we have the following crucial lemma (see [**9**]):

LEMMA 2.2. *Assume B is $(F,G)-$regular. Then the comparison map is an injection and the dimension of $D_B(V)$ is bounded by the dimension of V.*
This inequality is an equality $\dim_B D_B(V) = \dim_F(V)$ *if and only if the comparison map $\alpha_{B,V}$ is an isomorphism, which in turn happens precisely if $B \otimes_F V$ is equivalent to a trivial $B-$representation of G*

We can therefore define the following class of well-behaved $G-$representations:

DEFINITION 2.3. Given an (F,G)-regular $F-$algebra B, a $G-$representation is called $B-$*admissible* if

$$\dim_B D_B(V) = \dim_F(V)$$

By the above lemma, this happens precisely if the comparison map is an isomorphism.

We write $Rep_F^B(G)$ for the full subcategory of $Rep_F(G)$ spanned by all $B-$admissible representations (as always on finite-dimensional vector spaces).

The category $Rep_F^B(G)$ and the functor $D_B(V)$ have all the nice properties we could ask for:

THEOREM 2.4. *(see [**9**]) Let B be (F,G)-regular. Then:*

- *$Rep_F^B(G)$ is a sub-Tannakian category of the neutral Tannakian category $Rep_F(G)$, which just means that it contains the unit, is closed under isomorphisms, subobjects, quotients, duals, direct sums, and tensor products.*
- *The functor*
$$D_B(V) : Rep_F^B(G) \to Vect_E$$
is an exact faithful tensor functor. The latter just means that it preserves the tensor product, duals and the unit.

REMARK 2.5. We will mostly consider *enhanced* versions of these functors, i.e. have objects with more structure on the linear algebra side.

3. An Analogy in Topology

The theory of admissible representations can be illustrated very nicely by its topological counterpart.

Let (X,x) be a sufficiently nice pointed space, and assume we want to study representations $G = \pi_1(X,x) \to GL(V)$ on some fixed vector space V.

Given a $G-$space Y (which corresponds to $Spec(B)$) and a real vector space V, the group $Aut(Y \times V) = Map(Y, GL(V))$ is acted on by G. One can check that the cocycles in $Z^1(G, Map(Y, GL(V)))$ parametrise $G-$equivariant structures on the trivial bundle $Y \times V$, and that the cohomology groups $H^1(G, Map(Y, GL(V)))$ classify such structures up to isomorphisms.

We want to attach linear algebra data to representations $\pi_1(X) \to V$, and we can do this by producing $G-$spaces Y with a $G-$invariant map down to X - one important class of examples of such spaces comes from coverings, but we shall not work in such a restricted setting.
We then have a map

$$Rep(G, GL(V)) = Z^1(G, Aut(X \times V)) \to Z^1(G, Aut(Y \times V))$$

which maps a representation of $G = \pi_1(X)$ to the pullback to Y of the silly equivariant structure it defines on X.
One can give a name to those representations which become trivial when pulled up:

Definition 2.6. Given a $G-$space Y with a $G-$invariant map $p : Y \to X$, a representation $\rho : \pi_1(X, x) \to GL(V)$ is called $Y-$admissible if its associated equivariant structure on $Y \times V$ is trivial, i.e. $H * (p) : H^1(G, Aut(X \times V)) \to H^1(G, Aut(Y \times V))$ maps ρ to zero.

One might expect a trivialisable $G-$equivariant structure to have no interesting structure left, but this is not true since we have *a preferred $G-$equivariant structure* coming from the fact that our bundle $Y \times V$ was trivial.
The topological $D-$functor should then map a representation ρ to the space of global sections $Y \to Y \times V$ for which $s(gy) = gs(y)$.

In the topological setting, it seems possible to often design spaces Y which make specific representations admissible by "resolving loops". As an example, the representation $\pi_1(S^1) \to GL_1(\mathbb{C})$ sending the generator to ζ_n would naturally let us take Y to be the $n-$fold cover of S^1.

4. The de Rham Ring of Periods B_{dR}

Recall that our goal is to find a filtered ring B_{dR} with a continuous linear G_K-action and a compatible filtration which gives the étale-de Rham comparison isomorphism for smooth and proper schemes - this means that all Galois representations induced by such schemes are admissible.

The most obvious ring with a G_K-action is clearly $\overline{K}$, but Hilbert 90 shows that the only admissible representations are those which are obtained from a *finite* quotient of G_K - this is by far not good enough for our purposes.

By analogy with the archimedean case, the field $\mathbb{C} = \mathbb{C}_K$ (i.e. the completion of the closure of K) seems like a better guess Even though it is not filtered yet and hence no good candidate for B_{dR}, Falting's theorem tells us that there is a nice G_K- equivariant isomorphism:

$$\mathbb{C} \otimes_{\mathbb{Q}_p} H^n_{\acute{e}t}(\overline{X_K}, \mathbb{Q}_p) = \bigoplus_q \mathbb{C}(-q) \otimes_K H^{n-q}(X, \Omega^q_{X/K})$$

for smooth and proper schemes X over K.
The following conjecture of Serre was proven by Sen and characterises $\mathbb{C}$-admissible representations precisely:

THEOREM 2.7. *(Sen) A p-adic Galois representation is admissible for $\mathbb{C}$ if and only if the homomorphism*

$$I_K \to G_K \to GL(V)$$

corresponding to the restriction to the inertia group [5] factors through a finite quotient.

At this stage, we want to force one of the simplest possible Galois representations to be admissible: The cyclotomic character[6] χ, which is just given by the *Tate module* of the group scheme $\mathbb{G}_m$, i.e. the dual of the first étale cohomology group of $\mathbb{G}_m$.

In the spirit of having equal structures on both sides, we require a similar twisting action to be present on B_{dR}.

Naïvely forcing it to be there leads us to consider B_{HT}, the *Hodge-Tate ring*, which is given by

$$B_{HT} = \bigoplus_n \mathbb{C}(n)$$

This is the associated graded of what will turn out to be the filtered ring B_{dR}. But what could the filtered ring itself be?
One obvious possibility would be to just take $\mathrm{Frac}(\mathbb{C}[[t]])$, but it turns out that this will not give us more than B_{HT} (see Example 4.1.3. in [**5**] for more details). We are therefore in need of a "less trivial" way to cook up a complete DVR out of a ring.

4.1. A Reminder on Witt Vectors and the Frobenius. For the sake of logical completeness, we briefly recall several basic facts about Witt vectors (see [**18**]).
Assume we start with a perfect[7] *ring* k of characteristic p. Our aim is to cook up some other complete local ring R where p does not divide zero and whose automorphisms are closely linked to the automorphisms of k.

Assuming k is a field, we can think schematically: We can picture being given a point with specified residue field k. Our task is then to find a small neighbourhood inside some scheme containing this point such that the germs of functions on this little neighbourhood are closely related to their values at this given point.

We first define what we want this ring of germs to look like:

DEFINITION 2.8. Let p be a fixed prime number. A ring W is called a *strict p-ring* if

- The element p is *not* a zero-divisor in W.
- The residue ring $R/(p)$ is perfect.

[5]The Inertia group consists of those Galois automorphisms which act trivially on the residue field.

[6]The cyclotomic character is the unique group homomorphism $\rho : G_K \to \mathbb{Z}_p^\times$ sending g to $\rho(g) = \varprojlim(a_{g,n}) \in \mathbb{Z}_p^\times$, where the numbers $a_{g,n} \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ describe the action of g on roots of unities, namely $g(\zeta_{p^n}) = \zeta_{p^n}^{a_{g,n}}$ for any primitive $(p^n)^{th}$ root of unity ζ_{p^n}.

[7]A ring is said to be *perfect* if exponentiating by its characteristic is an automorphism. For reduced rings, this is already implied by surjectivity of the p^{th} power map.

- The Krull topology on R defined by the filtration[8]

$$0 \subset ... \subset p^2W \subset pW \subset W$$

  is *complete* and *separated.*

We then have the following result containing all properties on Witt vectors we shall need later:

LEMMA 2.9. *(see* [**18**] *and* [**17**]*)*

- *Every perfect ring k of characteristic p is the residue ring of a strict p-ring $W(k)$, which is determined uniquely up to canonical isomorphism.*
- *There is a unique multiplicative section $T : k \to W(k)$ of the projection. Even though the punishment of damnatio memoriae might well be appropriate, the images of this map are usually called the Teichmüller representatives.*
  Every $x \in R$ has a unique expansion as

$$x = \sum_n T(a_n)p^n$$

  for $a_n \in k$.
- *W is a functor and T a natural transformation from the identity to W.*
- *There is a lift of the Frobenius $\sigma : k \to k$ to the ring homomorphism*

$$\phi : W(k) \to W(k)$$

$$\phi\left(\sum_n T(a_n)p^n\right) = \sum_n (T(a_n))^p p^n$$

4.2. The Blank Period Ring $W(R(\mathcal{O}_{\mathbb{C}}))$. The problem is that the machinery of Witt-vectors applies to perfect rings of characteristic p, and $\mathbb{C}$ has none of those properties. However, we can simply enforce them by brute force. One can do this in two different orders - we have decided to first make the p^{th}-power map bijective and then enforce characteristic p as this seems more natural to us.

First, we take $\mathcal{O}_{\mathbb{C}}$ instead of $\mathbb{C}$ so that p becomes non-invertible.

Secondly, we force the p^{th} power map to not just be surjective, but also injective by specifying what all the roots of a given element are. Set

$$R(O_{C_K}) = \lim_{x \mapsto x^p} (\mathcal{O}_{\mathbb{C}}) = \left\{(x_0, x_1, ...) \in \mathcal{O}_{\mathbb{C}} | x_{i+1}^p = x_i\right\}$$

Here we take the limit in the category of sets. Note that this is *NOT* a ring but just a monoid since exponentiating by p and addition do not commute (exactly because $p \neq 0$ in $\mathcal{O}_{\mathbb{C}}$).

Thirdly, we obtain a ring structure on this set by stabilising the ring operations in the obvious fashion:

$$(x + y)_n = \lim(x_{n+k} + y_{n+k})^{p^k}$$

$$(xy)_n = x_n y_n$$

[8]The sets $(r + p^nW)$ form a basis of opens for the Krull topology

We also note that the $\overline{k}$-algebra structure on $\mathcal{O}_{\mathbb{C}}$ given by inclusion of the Witt-vectors $W(\overline{k}) \subset \mathcal{O}_{\mathbb{C}}$ can be stabilised by defining

$$(\lambda x)_n = (\lambda^{p^{-n}} x_n)$$

Here, we used that $\overline{k}$ is perfect and hence $\lambda^{p^{-n}}$ is well-defined.

We call this ring $R(\mathcal{O}_{\mathbb{C}})$ and notice that 1 is of additive order p. It is the same ring as the one obtained from firstly passing to $\mathcal{O}_{\mathbb{C}}/p\mathcal{O}_{\mathbb{C}}$ and then enforcing perfection in the above way.
Note that G_K acts on $R(\mathcal{O}_{\mathbb{C}})$, and that this ring inherits a valuation by defining

$$v_R(x_0, x_1, ...) = v_p(x_0)$$

We quote the following theorem (see Lemma 4.3.3 in [**5**]):

LEMMA 2.10. *The ring $R(\mathcal{O}_{\mathbb{C}})$ with the topology inherited from the valuation v_R is both separated and complete.*

We can now define the ring which will be the basic ingredient in the construction of all other period rings:

DEFINITION 2.11. The *Blank period ring* of K is the ring of Witt-vectors of the perfectification of $\mathcal{O}_{\mathbb{C}}$, i.e. $W(R(\mathcal{O}_{\mathbb{C}}))$.

This lets us prove the following lemma, which will be very useful later:

LEMMA 2.12. *The (additive) group $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ acts on the units of $R(\mathcal{O}_{\mathbb{C}})$ congruent to $1 \mod m_{R(\mathcal{O}_{\mathbb{C}})}$ by exponentiation.*

PROOF. Indeed, if $1 + x \in 1 + m_{R(\mathcal{O}_{\mathbb{C}})}$ is a unit for x in and a_n and b_n are two sequences of integers converging to $x \in \mathbb{Z}_p$, then we know that

$$|(1+x)^{a_n} - (1+x)^{b_n}|_R = |(a_n - b_n)x + \frac{1}{2}(a_n^2 - b_n^2)x^2 + ...|_R \leq |a_n - b_n|_R |x|$$

tends to zero too. Since R is complete, this shows that exponentiating units in $1 + m_{R(\mathcal{O}_{\mathbb{C}})}$ by $p-$adic integers is well-defined. $\square$

We want to produce a G_K-equivariant morphism of $W(\overline{k})$-algebras from the blank period ring back to our original $\mathcal{O}_{C_{\mathbb{K}}}$, i.e. solve the following lifting problem:

$$\begin{array}{ccc} W(R(\mathcal{O}_{\mathbb{C}})) & \overset{\theta}{\dashrightarrow} & \mathcal{O}_{\mathbb{C}} \\ \downarrow & & \downarrow \\ R(\mathcal{O}_{\mathbb{C}}) & \xrightarrow{\theta_0} & \mathcal{O}_{\mathbb{C}}/p\mathcal{O}_{\mathbb{C}} \end{array}$$

We can do this by a general element $x = \sum_k T(a_k)p^k$ in $W(R(\mathcal{O}_{\mathbb{C}}))$ to

$$\theta(x) = \sum_n ((a_n)_n)p^n = \sum_n (a_n)^{\frac{1}{p^n}} \cdot p^n$$

Here $(a_n)^{\frac{1}{p^n}} = (a_n)_n$ is the chosen $(p^n)^{th}$ root of a_n.

In order to use these maps to later make nontrivial statements about Galois representations, we need the following fact:

LEMMA 2.13. *The map θ is surjective and its kernel is the principal ideals generated by any element $x \in \ker(\theta)$ whose reduction to $R(\mathcal{O}_{\mathbb{C}})$ has valuation 1.*

4.3. The Construction of B_{dR}. We can use this map θ to produce a map from $W(R(\mathcal{O}_{\mathbb{C}}))[\frac{1}{p}]$ to $\mathbb{C}$:

$$\begin{tikzcd} W(R(\mathcal{O}_{\mathbb{C}})) \arrow[r, "\theta"] \arrow[d] & \mathcal{O}_{\mathbb{C}} \arrow[d] \\ W(R(\mathcal{O}_{\mathbb{C}}))\left[\frac{1}{p}\right] \arrow[r, "\theta"] & \mathbb{C} \end{tikzcd}$$

As mentioned in the very beginning, we want our period ring to contain the cyclotomic character. We will see in a moment that for the most obvious solution to this requirement to go through, we need to define a *logarithm.* At this point, we run into another problem: $W(R(\mathcal{O}_{\mathbb{C}}))[\frac{1}{p}]$ is *not* complete.

We therefore just complete it with respect to the $\ker(\theta)-$adic topology and obtain the ring

$$B_{dR}^{+} = \varprojlim_{n} W(R(\mathcal{O}_{\mathbb{C}}))\left[\frac{1}{p}\right]/(\ker(\theta)^{n})$$

The following lemma documents some of the nice properties enjoyed by B_{dR}^{+}:

LEMMA 2.14. *The ring B_{dR}^{+} is a complete DVR with*

- *maximal ideal* $\ker(\theta^{+} : B_{dR}$
- *residue field* $\mathbb{C}$
- *a natural G_K-action*
- *a natural G_K-stable decreasing filtration given by the preimages of the powers of the maximal ideal of B_{dR}^{+}*
- *a canonical injection $W(R(\mathcal{O}_{\mathbb{C}}))[\frac{1}{p}] \to B_{dR}^{+}$*

The cyclotomic uniformiser of B_{dR}^{+}. We can now find the cyclotomic character inside B_{dR}^{+}. Choose a nontrivial compatible system of $(p^n)^{th}$ roots of unity $\epsilon = (1, \zeta_p, ...) \in R(\mathcal{O}_{\mathbb{C}})$- this gives rise to an element $[\epsilon] \in W(R(\mathcal{O}_{\mathbb{C}}))$ by taking the Teichmüller lift.

Notice that

$$g([\epsilon]) = [(1, \epsilon_1^{a_{g,1}}, \epsilon_2^{a_{g,2}}, ...)] = [\epsilon]^{\chi(g)}$$

for the cyclotomic character χ (where exponentiation is well-defined by 2.12). We therefore want to conclude:

$$g(\log([\epsilon])) = \left(-\sum \frac{(1-g[\epsilon])^n}{n}\right) = \log([\epsilon]^{\chi(g)}) = \chi(g) \cdot \log([\epsilon])$$

However, there are subtleties involved: The logarithm converges with respect to the $\ker(\theta)-$adic topology on B_{dR}^{+}, whereas the $\mathbb{Z}_p$-adic exponentiation is defined for the $p-$adic topology on R. This issue can be fixed by refining the topology on B_{dR}^{+} - we shall call this refined topology *the mixed topology* and refer the interested reader to exercise 4.5.3. in [**5**].

Observe that for another choice ϵ' of compatible $(p^n)^{th}$ roots of unity, we have

$$\log([\epsilon]) = \lambda \log([\epsilon'])$$

for some $\lambda \in \mathbb{Z}_p^{\times}$.

We conclude that there is a canonical $1-$dimensional subspace of B_{dR} on which G_K acts via the cyclotomic character.
The following fact underlines the special role $\log([\epsilon])$ plays for B_{dR}^+:

LEMMA 2.15. *The element* $\log([\epsilon])$ *is a uniformiser for the complete discrete valuation ring* B_{dR}^+.

We now face one final problem: B_{dR}^+ is *not* $(\mathbb{Q}_p, G_K)-$regular as $\mathbb{Q}_p \cdot t$ is fixed under the action of G_K, but t is not a unit in B_{dR}^+. We resolve this problem by using brute force once more and finally arrive at the right definition of the crucial period ring B_{dR}:

DEFINITION 2.16. The *de Rham period ring* B_{dR} is the field of fractions

$$B_{dR} := \mathrm{Frac}(B_{dR}^+) = B_{dR}^+\left[\frac{1}{t}\right]$$

with filtration and G_K-action inherited from B_{dR}^+. The fixed field of B_{dR} is K.
We say a $p-$adic Galois representation is *de Rham* if it is admissible for B_{dR}.

We then have the following comparison isomorphism, which was our initial motivation for B_{dR}:

THEOREM 2.17. *Let X be a smooth and proper scheme over K. Then $H^*_{\acute{e}t}(X, \mathbb{Q}_p)$ is de Rham,*

$$D_{dR}(H^*_{\acute{e}t}(X, \mathbb{Q}_p)) = H^*_{dR}(X, K)$$

and the comparison isomorphism

$$B_{dR} \otimes_K H^*_{dR}(X, K) \cong B_{dR} \otimes_{\mathbb{Q}_p} H^*_{\acute{e}t}(X, \mathbb{Q}_p)$$

gives an isomorphism of filtered $K-algebras$ with filtration-preserving linear G_K-action.

5. The Crystalline Period Ring B_{cris}

Our next aim is to find a period ring realising a similar comparison isomorphism between crystalline and étale cohomology in the case of schemes with a smooth and proper model. The main new structure we need to take care of is the *Frobenius* coming from crystalline cohomology.
Motivating the various steps of the construction of B_{cris} in detail is an extremely difficult task even for experts and exceeds the aims of this exposition.
Since $R(\mathcal{O}_{\mathbb{C}})$ has characteristic p, it has a Frobenius which lifts to $W(R(\mathcal{O}_{\mathbb{C}}))[\frac{1}{p}]$.
The problem arises when we try to complete this ring at $\ker(\theta)$ since ϕ can turn $mod \ker(\theta)$-units into nonunits:

EXAMPLE. The following example appears in [**4**]: Choose a compatible system $\tilde{p} = (p, p^{\frac{1}{p}}, p^{\frac{1}{p^2}}, ...)$ extending p to $R(\mathcal{O}_{\mathbb{C}})$, and write

$$q : W(R(\mathcal{O}_{\mathbb{C}}))\left[\frac{1}{p}\right] \to B_{dR}^+$$

for the canonical map. Consider the element

$$\left[\tilde{p}^{\frac{1}{p}}\right] - p \in W(R(\mathcal{O}_{\mathbb{C}}))$$

An easy computation shows that $\theta([\tilde{p}^{\frac{1}{p}}] - p) \neq 0$, hence $q(x)$ does not lie in the maximal ideal of the local ring B_{dR}^+ and is therefore a unit.

However, $\zeta = \phi(x) = [\overline{p}] - p$ lies in $\ker(\theta)$ and thus $q(\phi(x))$ is a non-unit. Any natural Frobenius on B_{dR}^+ would therefore need to send the unit $q(x)$ to a nonunit, which is nonsense.

We see that we need a more sophisticated method for "fixing" the blank period ring $W(R(\mathcal{O}_{\mathbb{C}}))$. Rather than just inverting p, we only invert it "where needed" in order to produce elements of the form $\frac{x^n}{n!}$ for $x \in \ker\theta$ and define

$$A_{cris}^0 = W(R(\mathcal{O}_{\mathbb{C}}))\left[\frac{x^n}{n!}_{\theta(x)=0}\right] \subset W(R(\mathcal{O}_{\mathbb{C}}))[\frac{1}{p}]$$

Expressed in a slightly fancier language, we take the *divided power envelope* for

$$(W(R(\mathcal{O}_{\mathbb{C}})), \ker(\theta))$$

In order to be able to apply analytic techniques later, we *complete at* p and obtain a complete and separated topological ring

$$A_{cris} = \varprojlim_n A_{cris}^0/p^n$$

We state the following direct link to B_{dR}^+ as follows:

THEOREM 2.18. *There is a unique G_K-equivariant injection $j : A_{cris} \to B_{dR}^+$ extending the composite*

$$A_{cris}^0 \to W(R(\mathcal{O}_{\mathbb{C}}))\left[\frac{1}{p}\right] \to B_{dR}^+$$

Once we endow the right hand side with the mixed topology, the map j is even continuous.

One can prove that the image of j takes a very concrete form (here infinite sums indicate convergence in the $\ker(\theta)-$adic topology):

LEMMA 2.19. *Writing $\xi = [\overline{p}] - p$, there is an identification*

$$j(A_{cris}) \cong \left\{\sum w_n \frac{\xi^n}{n!} \;\;|\;\; w_n \in W(R(\mathcal{O}_{\mathbb{C}})) \text{ such that } |w_n|_p \to 0\right\}$$

We will often abuse notation and confuse A_{cris} with its image in B_{dR}^+. We quote more desirable facts which we will need later (see [**5**] for proofs):

THEOREM 2.20. *The constructed ring A_{cris} satisfies:*

- *The image of $\theta : A_{cris} \to B_{dR}^+ \to \mathbb{C}$ is contained in $\mathcal{O}_{\mathbb{C}}$.*
- *The natural G_K-action is continuous.*

In the de Rham case, we constructed our favourite "cyclotomic element" as

$$log([\epsilon]) = \sum (-1)^{n+1}([\epsilon] - 1)^n/n$$

by checking the convergence of this sum in the $\ker(\theta)-$adic topology.
Our above criterion immediately tells us that this expression lies in A_{cris}, and we therefore know that the sum $\sum(-1)^{n+1}([\epsilon] - 1)^n/n$ converges even $p-$adically in there. Again G acts on t by the cyclotomic character.

The desire for a nice theory of admissible representations forces us to invert t in order to obtain a $(\mathbb{Q}_p, G_K)-$regular ring

$$B_{cris} = A_{cris}[\frac{1}{t}]$$

We will now extend the Frobenius to B_{cris}, which was our original motivation for introducing this ring:

- The Frobenius on $W(R(\mathcal{O}_{\mathbb{C}}))$ extends naturally to A^0_{cris}
- By compatibility of Frobenius and $p-$adic topology, it extends naturally to the ring A_{cris}
- By $p-$adic convergence of the relevant power series for t, we can compute

$$\phi(t) = pt$$

Hence ϕ extends uniquely from A_{cris} to B_{cris}

It can then be shown that the G_K-invariants of B_{cris} are precisely K_0.
The ring B_{cris} can be considered as a G_K-stable K_0-subalgebra of B_{dR}, and

$$B_{cris} \otimes_{K_0} K \to B_{dR}$$

is injective, which yields a filtration on $B_{cris} \otimes_{K_0} K$. Since the Frobenius doesn't preserve $\ker(\theta)$, it will also not preserve the filtration.

Definition 2.21. We have constructed a $(\mathbb{Q}_p, G_K)$-regular $\mathbb{Q}_p$-algebra B_{cris} with a linear G_K-action with invariants K_0, a G_K-stable filtration Fil on $B_{cris} \otimes_{K_0} K$, and a Frobenius-semilinear endomorphism $\phi : B_{cris} \to B_{cris}$. This ring will be called the *crystalline cohomology ring*.
We will call a $p-$adic Galois representation *crystalline* if it is $B_{cris}-$admissible.

We can now state the following theorem, which realises Grothendieck's dream of the mysterious functor:

Theorem 2.22. *(Faltings, Fontaine, Kato, Messing, Tsuji)*
Let X be a smooth and proper scheme over $\mathcal{O}_K$ with generic fibre X_K and special fibre Y over the residue field k. The Galois representation $V = H^*_{ét}(\overline{X_K}, \mathbb{Q}_p)$ is crystalline, we have

$$D_{cris}(V) = K_0 \otimes_{W(k)} H^*_{cris}(Y, W(k))$$

and the comparison isomorphism

$$B_{cris} \otimes_{\mathbb{Q}_p} H^*_{ét}(\overline{X_K}, \mathbb{Q}_p) \cong B_{cris} \otimes_{W(k)} H^*_{cris}(Y, W(k))$$

is an isomorphism of filtered $\phi-$modules.

Once we tensor B_{cris} up to B_{dR}, we recover the previous isomorphism of Faltings.

Remark 2.23. There is in fact a definition of the crystalline period ring in terms of crystalline cohomology (see [**?**]) via

$$A_{cris}\left[\frac{1}{p}\right] = \left(\varprojlim H^0_{cris}\left(Spec(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})/W_n(k)\right)\right) \otimes_{W(k)} K_0$$

6. The log-Crystalline Period Ring B_{st}

We will now define the most refined period ring appearing in this essay, namely the one for log-crystalline cohomology and semistable reduction. Remember that we constructed B_{dR} from C_K by forcing the cyclotomic character to be present in B_{dR}. The step from B_{dR} to B_{cris} seems to be a good proof of Fontaine's ingenuity and is therefore very hard to motivate, but it is again much easier to explain the final step from B_{cris} to B_{st}.

Recall that our aim is to manufacture a period ring giving rise to a comparison isomorphism between étale and log-crystalline cohomology in the semistable case. We fix just one individual semistable variety over K with a semistable model over $\mathcal{O}_K$, namely the Tate curve $E_p = \mathbb{G}_m/p^{\mathbb{Z}}$, and hope for the best. We want its étale cohomology to be included in our period ring, and we might as well instead ask for the dual representation, namely the Tate module, to be contained.
The upshot is that one can compute this module (we will only quote the result here). Fix nontrivial compatible families of $(p^n)^{th}$ roots ϵ for 1 and $\tilde{q}$ of q.
There is a $\mathbb{Z}_p$-basis v, w for the Tate-module, and a 1-cocycle $c : G_K \to \mathbb{Z}_p$ for the G_K-module $\mathbb{Z}_p$ such that

$$\frac{g(\tilde{q})}{\tilde{q}} = \epsilon^{c(g)}$$

and

$$g(nv + mw) = (n\chi(g) + mc(g))v + mw$$

Our aim is now to extend the ring $B_{cris} \subset B_{dR}$ so that it contains this Galois representation. It is natural to guess that the cyclotomic element t should play the role of v since G acts on it by cyclotomic character.
But what could the counterpart of w be? It follows right from the definition that an element obeying the obvious rules for $\log([\tilde{p}])$ would be enough.
Indeed, it is easy to build this logarithm inside B_{dR} by simply defining $log(p) = 0$ and hence

$$log([\tilde{p}]) = \sum (-1)^{n+1} \left(\frac{[\tilde{p}]}{p} - 1 \right)^n / n$$

which can be checked to be convergent in B_{dR}^+ (see [**5**]).
However, this element is transcendental over B_{cris}, so we might as well take it to be an indeterminate X and define $B_{st} := B_{cris}[X]$ with Galois action

$$g(X) = c(g)t + Y$$

We extend the Frobenius in the most natural way by

$$\phi(X) = X^p$$

and define the *monodromy operator* N to be

$$N = -\frac{d}{dX}$$

It is readily checked that $N\phi = p\phi N$.
Our monodromy operator indeed behaves like a measure of nonsmoothness, as

$$B_{cris} = B_{st}^{N=0}$$

Given the choice of $\tilde{p}$ we made above, we have $B_{st} \subset B_{dR}$. This embedding is *not* canonical, but we should fix it from now on. It can be shown that even the map $B_{st} \otimes_{K_0} K \to B_{dR}$ is an injection, and therefore $B_{st} \otimes_{K_0} K$ inherits a filtration. This filtration is again *not* canonical, but depended on the choices of $\tilde{p}$.
We summarise the properties of B_{st} in the following definition:

DEFINITION 2.24. We have constructed a $(\mathbb{Q}_p, G_K)$-regular $\mathbb{Q}_p$-algebra B_{st} with a linear G_K-action with invariants K_0, a G_K-stable filtration Fil on $B_{st} \otimes_{K_0} K$, a Frobenius-semilinear endomorphism $\phi : B_{st} \to B_{st}$ and a monodromy operator N. This ring will be called the *semistable period ring*.
We will call a $p-$adic Galois representation *semistable* if it is $B_{st}-$admissible.

THEOREM 2.25. *(Tsuji) Let X be a proper scheme over $\mathcal{O}_K$ with semistable reduction, with generic fibre X_K and special fibre Y over the residue field k. We endow the special fibre with its canonical log-structure.*

The Galois representation $V = H^*_{ét}(\overline{X_K}, \mathbb{Q}_p)$ is semistable, we have

$$D_{st}(V) = K_0 \otimes_{W(k)} H^*_{log-cris}(Y, W(k))$$

and the comparison isomorphism

$$B_{st} \otimes_{\mathbb{Q}_p} H^*_{ét}(\overline{X_K}, \mathbb{Q}_p) \cong B_{st} \otimes_{W(k)} H^*_{cris}(Y/W(k))$$

is an isomorphism of filtered $(\phi, N)-$modules.
Once we tensor B_{st} up to B_{dR}, we recover the previous isomorphism of Faltings.

We have now constructed all necessary period rings, and have the following chain of implications:

Crystalline implies Semistable implies de Rham

REMARK 2.26. In all our constructions above, we had a fixed p-adic field K in mind. However, it is obvious that once we forget the Galois action, *any* other such field K' with $K \subset K' \subset \overline{K} \subset \mathbb{C}_K = \mathbb{C}'_K = \mathbb{C}$ leads to the same result for B_{dR}, B_{cris} and B_{st}: All constructions started with $\mathbb{C}$.

The two Galois actions are compatible in the sense that an element in G_K preserving K' acts on the period rings in the same way as the corresponding element in $G_{K'}$.

CHAPTER 3

Semistability and Weak Admissibility

In the previous chapters, we have seen that the semistable period ring naturally arises as gadget comparing the cohomology of special and generic fibres of integral models, and that we can more generally use it to study certain Galois representations. More precisely:

DEFINITION 3.1. To each continuous $p-$adic Galois representation V, the additive functor D_{st} assigns a finite filtered $(\phi, N)-$module given by:

$$D_{st}(V) = (B_{st} \otimes_{\mathbb{Q}_p} V)^{G_K}$$

Two simple-minded questions occur immediately:

QUESTION. *When is a representation crystalline?*

QUESTION. *What is the essential image of D_{st}?*

QUESTION. *Is the functor D_{st} fully faithful?*

We introduce the following terminology:

DEFINITION 3.2. A finite filtered $(\phi, N)-$module is called *admissible* if it comes from a semistable Galois representation.

We will not touch the third question at all, but it does have an affirmative answer. The main aim of the rest of this essay is to first find a good candidate for the essential image (the so-called weakly admissible modules) and then prove that this is indeed what we need.

We will first show that the answer is completely dictated by the behaviour on the inertia group in a certain precise sense. To analyse this reduced case, we will then need to introduce certain invariants of filtered $(\phi, N)-$modules and study how they relate for admissible ones. This will then give us necessary conditions for admissibility and thus naturally lead us to the definition of weakly admissinle modules.

1. Reduction to Closed Residue Field

Recall the basic short exact sequence of groups

$$1 \to I_K \to G_K \to G_k \to 1$$

Since our understanding of the absolute Galois groups of perfect fields of characteristic p is rather good, it is very natural to wonder if we can reduce questions about semistability of representations and admissibility of (ϕ, N)-modules to the inertia group. Even though this question makes sense, it is not quite well-defined yet.

For the representation side, we want to first express I_K as a Galois group G_P of some new $p-$adic field P and then relate the representations of G_p and G_K. While it is straightforward to define P abstractly, we want to carry out its construction *inside* the period rings in order to facilitate comparison theorems.

We fix the period rings $B_{cris} \subset B_{st} \subset B_{dR}$ of the extension $\overline{K}/K$ and think of K_0 and K as sitting inside B_{cris}.
We need the following easy auxiliary lemma (see [**9**] for a proof):

LEMMA 3.3. *The map*

$$R(\mathcal{O}_{\mathbb{C}}) = \varprojlim_n (\mathcal{O}_{\mathbb{C}}/p\mathcal{O}_{\mathbb{C}}) \xrightarrow{x \mapsto x_0} \mathcal{O}_{\mathbb{C}}/p\mathcal{O}_{\mathbb{C}} \to \overline{k}$$

has a unique section, and this section is given by mapping an element $a \in \overline{k}$ to sequence $(b_0, b_n, ...) \in R(\mathcal{O}_{\mathbb{C}})$ whose n^{th} term b_n is given by the Teichmüller representative of $(a^{p^{-n}}, 0, 0, ...)$ in $W(\overline{k}) \subset \mathbb{C}$ (taking $(p^n)^{th}$ roots makes sense since $\overline{k}$ is perfect).

This map is injective, and we will use it to identify $\overline{k}$ with a subfield of $R(\mathcal{O}_{\mathbb{C}})$ from now on.

We therefore have a natural inclusion

$$W(\overline{k}) \subset W(R(\mathcal{O}_{\mathbb{C}})) \subset B^+_{cris} = A_{cris}\left[\frac{1}{p}\right]$$

Define $P_0 \subset B_{cris}$ to be the field of fractions of $W(\overline{k})$, and set

$$P = P_0 \otimes_{K_0} K \subset B_{dR}$$

This is a field as P_0 and K are linearly disjoint over K_0. The field P_0 (and P) are in fact the completions of the maximal unramified extensions of K_0 (or K) inside $\overline{K}$ (which sits in B^+_{dR} by Hensel's Lemma).

Define $\overline{P}$ to be the algebraic closure of P *inside* B_{dR}, one checks that $\overline{P} = P_0\overline{K}$. This closure is again contained in B^+_{dR} (by Hensel's lemma), and indeed algebraically closed.
A basic exercise in $p-$adic analysis reveals that the natural restriction map is well-defined and gives rise to an isomorphism of profinite groups

$$I_K = Gal(\overline{K^{nr}}, K^{nr}) \cong Gal(\overline{P}/P) = Gal(\hat{\overline{K^{nr}}}, \hat{K^{nr}})$$

We should therefore think of I_K as the Galois group of the p-adic field P from now on.

Once we forget the actions, the period rings of K and P are identical by 2.26, and we therefore obtain the following commutative diagram (given a profinite group G, write $Rep_{\mathbb{Q}_p}(G)$ for the category of continuous finite-dimensional $p-$adic representations):

$$\begin{tikzcd} Rep_{\mathbb{Q}_p}(G_K) \arrow[r, "D^K_{st} = (B^K_{st} \otimes_{\mathbb{Q}_p} -)^{G_K}"] \arrow[d, "(-)|_{\overline{P}}"'] & MF^{\phi,N}_K \arrow[d, "P_0 \otimes_{K_0} (-)"'] \\ Rep_{\mathbb{Q}_p}(G_P) \arrow[r, "D^P_{st} = (B^P_{st} \otimes_{\mathbb{Q}_p} -)^{G_P = I_K}"] & MF^{\phi,N}_P \end{tikzcd}$$

We can now formulate the vague questions raised in the beginning of this section precisely:

QUESTION. *How does semistability of* $V \in Rep_{\mathbb{Q}_p}(G_K)$ *relate to semistability of* $V|_{\overline{P}} \in Rep_{\mathbb{Q}_p}(G_P)$*?*

QUESTION. *How does admissibility of a filtered* (ϕ, N) *module* $D \in MF_K^{\phi,N}$ *relate to the admissibility of* $P_0 \otimes_{K_0} D \in MF_P^{\phi,N}$*?*

The answer to both of these questions is as nice as it could possibly be: both conditions are equivalent in both cases. In the rest of this section, we shall prove this.

1.1. Completed Unramified Galois Descent. In order to prove the first equivalence, we need to apply the method of *completed unramified Galois-descent* . As the name suggests, this is an extension of ordinary Galois descent, which we now recall in the setting of $G_F = Gal(\overline{F}/F)$

LEMMA 3.4. *(Galois descent) Let* V *be be an* $\overline{F}-$*vector space and assume* G_F *acts on* V *in a* G_K-*semilinear way which makes*

$$Fix(v) = \{\sigma \in G_F | \sigma v = v\}$$

a finite index subgroup of G_F *for all* $v \in V$*.*
Then there is a V *basis of* G_F*-invariant vectors for* V*, i.e.*

$$V \cong \overline{F} \otimes_F V^{G_F}$$

The proof of this basic fact can be found in [**20**], or carried out as an exercise in basic Galois theory by the reader.

We now want to use this statement to prove the following theorem:

THEOREM 3.5. *Let* D *be a finitely generated* $P = \hat{K^{nr}}$*-vector space together with a continuous* $P-$*semilinear* $K-$*linear* $G_k = G_K/I_K \cong Gal(K^{nr}/K)$*-action.*
Then D^{G_k} *is a finitely generated* $K-$*vector space and the natural comparison map*

$$\alpha_M : P \otimes_K D^{G_k} \to D$$

is an isomorphism P*-vector spaces with a semilinear* G_k*-action.*

PROOF. Write $\dim_P(D) = n$.
Step 1): (Reduction to Lattices): There is a G_k-stable $\mathcal{O}_P$-lattice Λ inside the $P-$vector space D.

Indeed, pick any $\mathcal{O}_P$-lattice Λ_0 inside D, and a basis $e_1, ..., e_n$ for Λ^0. The continuous $P-$semilinear action gives rise to a continuois map (in general *not* a homomorphism)

$$\rho : G_k \to GL_n(P)$$

describing the images of these basis elements.
The subset $GL_n(\mathcal{O}_P) \subset GL_n(P)$ is open, and hence so is its preimage

$$H = \rho^{-1}(GL_n(\mathcal{O}_P)) \subset G_k$$

This preimage is a subgroup containing precisely those $g \in G_k$ which map Λ_0 to itself.

By compactness of G_k, the space of right cosets of H in G is finite. Choose representatives $g_1, ..., g_r$, and let $\tau : G_k \to Map(V, V)$ be the action homomorphism before choosing bases. Now consider

$$\Lambda = \sum_i \tau(g_i)(\Lambda_0)$$

Note that elements in H also fix all of the above translates of Λ_0.
The $\mathcal{O}_P-$module Λ is G_k-stable: Indeed, choose general elements $g = h \cdot g_j$ (with $h \in H$) in G_K and $x = \sum_i \tau(g_1)a_1 + ... + \tau(g_n)a_n$ in Λ. We then have:

$$\tau(g)x = \sum_i \tau(h)\tau(g_j g_i)a_i \subset \sum_i \tau(g_j g_i)\Lambda_0 \subset \Lambda$$

A final check reveals that Λ is indeed a lattice.

We therefore have constructed a G_k-stable $\mathcal{O}_P$-lattice Λ. To prove out claim, it is enough to show that the natural map

$$\phi : \mathcal{O}_P \otimes_{\mathcal{O}_K} \Lambda^{G_k} \to \Lambda$$

is an isomorphism of $\mathcal{O}_P$-modules.

This is the point where we remember that we do have Galois descent for algebraic extensions, and when trying to relate P and K to such algebraic extensions, it is a natural idea to pick a uniformiser $\pi \in \mathcal{O}_K$, which is automatically one for $\mathcal{O}_P$ too (and fixed by G_k) and "divide it out" everywhere:

Indeed, we get a map of representations in $Rep_{\overline{k}}(G_k)$:

$$\phi \bmod \pi : \overline{k} \otimes_k (\Lambda^{G_k}/\pi\Lambda^{G_k}) \to \Lambda/\pi\Lambda$$

and as a first step to our main goal, we shall try to establish the following result:

Step 2): The reduction ϕ mod π is an isomorphism of $\overline{k}$-vector spaces with a semilinear action of G_k.

It would be very helpful if $(\Lambda^{G_k}/\pi\Lambda^{G_k})$ happened to be equal to $(\Lambda/\pi\Lambda)^{G_k}$ since we could then apply classical Galois-descent. The obvious short exact sequence of G_k-modules

$$0 \to \Lambda \xrightarrow{\pi} \Lambda \to \Lambda/\pi\Lambda \to 0$$

gives rise to a long exact sequence

$$0 \to \Lambda^{G_k} \xrightarrow{\pi} \Lambda^{G_k} \to (\Lambda/\pi\Lambda)^{G_k} \to H^1(G_k, \Lambda) \to ...$$

which tells us that $\Lambda^{G_k}/\pi\Lambda^{G_k} = (\Lambda/\pi\Lambda)^{G_K}$ if and only if $H^1(G_k, \Lambda) = 0$ (here and everywhere else, we mean *continuous* group cohomology).

It is unfortunate that this is the relevant cohomology group rather than $H^1(G_k, \Lambda/\pi\Lambda)$, as we can compute the latter as follows: By classical Galois descent, we have

$$\Lambda/\pi\Lambda = \overline{k} \otimes_k (\lambda/\pi\Lambda)^{G_k}$$

Writing $d = \dim_{\overline{k}} \Lambda/\pi\Lambda$, this implies that

$$\Lambda/\pi\Lambda \cong \overline{k}^d$$

as G_k−modules. Since $H^1(G_k, \overline{k}) = 0$, we deduce that $H^1(G_k, \Lambda/\pi\Lambda) = 0$. But how does this help us with computing $H^1(G_k, \Lambda)$?
The argument above can be easily generalised to show that for any finite index subgroup $H \subset G$ and any $k \geq 0$, the G/H−module $\pi^k\Lambda^H/\pi^{k+1}\Lambda^H$ has

$$H^1(G/H, \pi^k\Lambda^H/\pi^{k+1}\Lambda^H) = 0$$

Since $\Lambda^H = \varprojlim_k(\Lambda^H/\pi^k\Lambda^H)$, it follows that $H^1(H, \lambda^H) = 0$ (this is a standard fact of the cohomology of finite groups, see Lemma 3 in [**19**] for a proof.) Since $G_K = \varprojlim_{H \leq G \text{ finite index}}(G/H)$ and $\Lambda = \varinjlim_H \Lambda^H$ forms a compatible system of pairs of groups with modules, we have by theorem 7 in [**19**]:

$$H^1(G_K, \Lambda) = H^1\left(\varprojlim_{H\leq G \text{ finite index}} (G/H) \quad , \quad \varinjlim_H \Lambda^H\right) = \varinjlim_H H^1(G/H, \Lambda^H) = 0$$

Hence we conclude that

$$\Lambda^{G_k}/\pi\Lambda^{G_k} = (\Lambda/\pi\Lambda)^{G_K}$$

In order to complete step 2), we have to prove that

$$\phi \text{ mod } \pi : \overline{k} \otimes_k (\Lambda/\pi\Lambda)^{G_k} \to \Lambda/\pi\Lambda$$

is an isomorphism, which just follows by classical Galois descent.

Since ϕ mod π is an isomorphism, it is now enough to prove the following claim:
Step 3): The $\mathcal{O}_P$-modules $\mathcal{O}_P \otimes_{\mathcal{O}_K} \Lambda^{G_k}$ and Λ are both free of same rank

$$d := \dim_{\overline{k}} \Lambda/\pi\Lambda$$

It is easy to check that both are free.
As $\Lambda/\pi\Lambda$ is a d−dimensional trivial G_K-space, we have

$$d = dim_k(\Lambda/\pi\Lambda) = dim_k((\Lambda/\pi\Lambda)^{G_k}) = dim_k(\Lambda^{G_k}/\pi\Lambda^{G_k})$$

To see that $\dim_{\mathcal{O}_\mathcal{K}}(\Lambda^{G_k})$ and $dim_k(\Lambda^{G_k}/\pi\Lambda^{G_k})$ agree, we take a lift $\{w_1, ..., w_d\}$ of a k−basis $\{v_1, ..., v_d\}$ of $\Lambda^{G_k}/\pi\Lambda^{G_k}$. This set generates: For every $w \in \Lambda^{G_k}$, we can first bring it to $\pi\Lambda^{G_k}$ by subtracting a linear combination of the w_i's with coefficients in $\mathcal{O}_\mathcal{K}$. We then divide the remainder by π and reiterate the same process. We obtain a power series expression for the coefficients, and it is clear that they converge in $\mathcal{O}_P$. If there were a smaller generating set of this torsion-free module, it would drop down to a smaller basis of $\Lambda^{G_k}/\pi\Lambda^{G_k}$ - a contradiction.
It is also clear that $\dim_{\mathcal{O}_P}(\Lambda) = dim_k(\Lambda/\pi\Lambda)$.

We have therefore established *Step 3)*, which completes the proof. □

1.2. Reduction to the Inertia. With the method of completed unramified Galois descent at our disposal, we can now straightforwardly prove the following comparison theorem:

COROLLARY 3.6. *Let V be a $p-$adic Galois representation in $Rep_{\mathbb{Q}_p}(G_K)$, and, as always, write $P = \hat{K^{nr}}$ for the completion of the maximal unramified extension of K in $\overline{K}$. As before, $P_0 = W(\overline{k})$ is the maximal unramified subextension of $P/\mathbb{Q}_p$.*

Then, the natural map

$$P \otimes_K D_{st}^K(V) \to D_{st}^P(V)$$

is an isomorphism of filtered (ϕ, N)-modules with G_k-action.

PROOF. (Sketch) Note that the map is a morphism of (ϕ, N)-modules with G_k-action, and that, by completed unramified Galois descent applied to K_0 , P_0, $D = D_{st}^P(V)$, and $D^{G_k} = D_{st}^K(V)$, it is in fact an isomorphism of $(\phi, N)-$modules with G_k-action. One last subtlety remains: We have to check that the morphism preserves filtrations. This can be done by first checking that the G_k-action on each filtered piece $Fil^i(D_{st}^P(V))$ is continuous and then applying descent to those pieces. We refer the reader to the proof of 6.3.8 in [**5**] for this final tedious detail. □

We can now show that semistability is indeed determined by what happens on the inertia:

COROLLARY 3.7. *A $p-$adic Galois representation V of G_K is semistable if and only if it is semistable as a representation of $I_K = Gal(\overline{P}/P) = G_P$.*

PROOF. The isomorphism $P \otimes_K D_{st}^K(V) \to D_{st}^P(V)$ shows that

$$\dim_{K_0}(D_{st}^K(V)) = \dim_{P_0}(D_{st}^K(P))$$

□

We therefore see:

PRINCIPLE. *n order to answer which $p-$adic Galois representations of G_K are semistable and which filtered $(\phi, N)-$modules are admissible, it is always enough to answer the corresponding questions over P.*

We will therefore from now on mostly work over P, and the fact that the residue field is algebraically closed will be very helpful soon.

2. Weak Admissibility - a Necessary Criterion

Our general aim is to find criteria for when a filtered (ϕ, N)-module comes from a semistable representation. As a first step, we shall try to find good necessary criteria - a second, much harder step will be to prove that they are in fact sufficient. As justified above, we work over P with residue field $\overline{k}$ algebraically closed.

2.1. One-Dimensional Semistable Representations. Let us start modestly and try to understand $1-$dimensional semistable representations of G_P and $1-$dimensional admissible $(\phi, N)-$modules.
We have at least an integer's worth of those, namely the cyclotomic character

$$\chi : G_P \to \mathbb{Z}_p^*$$

and all its powers.
We will now see that all $1-$dimensional crystalline / semi-stable representations

of G_P are of this form (an analogous result for G_K would allow us to twist by arbitrary unramified[1] characters):

LEMMA 3.8. *A* $1-$*dimensional representation* $\rho : G_P \to \mathbb{Q}_p^*$ *is semistable (crystalline) if and only if it is given by* $\rho = \chi^i$ *for some integer* i*.*

PROOF. We prove the semistable case, the crystalline one goes through in exactly the same way.
By admissibility, $(B_{st} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is trivial, so we can choose a nonzero invariant vector

$$b \otimes v \in B_{st} \otimes_{\mathbb{Q}_p} V$$

(this is where $dim(V) = 1$ is helpful) and conclude that the $\mathbb{Q}_p$-subspace

$$\langle b \rangle \subset B_{st}$$

spanned by $b \in B_{st}$ is G_K-invariant.
Assume $b \in Fil^i(B_{st}) \backslash Fil^{i+1}(B_{st})$, then we have

$$t^{-i} b \in Fil^0(B_{st}) \backslash Fil^1(B_{st})$$

Using the augmentation map $\theta : B_{st} \to \mathbb{C}$, this means that $\theta(t^{-i}b)$ generates a one-dimensional G_P-invariant P_0-vector space V in $\mathbb{C}$.
Considering V as a G_P-representation in P_0-vector spaces, the obvious generalisation of Sen's theorem 2.7 shows that

$$\rho : G_P \to GL_{P_0}(V)$$
$$(g, \theta(t^{-i}b)) \mapsto \theta(t^{-i}(g \cdot b))$$

maps the inertia of P to a finite subgroup, which means (since $I_P = G_P$) that the image of the whole Galois group in $GL_{P_0}(V)$ is finite.

By injectivity of θ on the span $\langle (t^{-i} \cdot b) \rangle$, this implies that the action of G_P on this space also factors through a finite quotient. This representation is abstractly given by $\tau = \chi^{-i}\rho$ and is also semistable.

Hence we are reduced to proving that if a semistable (in our case one-dimensional) representation τ of G_P has finite image, then it is trivial.

This follows easily as follows: Let $H = \ker(\tau) \subset G_P$, then $L = \overline{P}^H$ is a finite Galois extension of P with $K = Gal(L/P) = G_P/H$.
We already quoted above that $(B_{st})^H = (B_{st})^{G_L} = L_0$, the closure of the maximal unramified extension of $\mathbb{Q}_p$ inside L. But as the residue field of L is a finite extension of the residue field of P, and the latter is closed, we have

$$L_0 = \mathrm{Frac}(W(\mathcal{O}_L / \pi_L \mathcal{O}_L)) = P_0$$

We conclude that $(B_{st})^H = P_0$ and compute:

$$D_{st}(V) = (B_{st}^H \otimes_{\mathbb{Q}_p} V)^K = P_0 \otimes_{\mathbb{Q}_p} V^K$$

Since V is semistable, this is one-dimensional, and thus $V^K = V$ which means $H = 0$, hence τ is the trivial representation.
This concludes the proof that $\rho = \chi^i$. □

[1]A representation ρ is *unramified* if it acts trivially on the inertia subgroup $I_K \subset G_K$, i.e. if $g \in G_K$ acts trivially on the residue field, then $\rho(g) = id$.

2.2. Necessary Conditions: the One Dimensional Case. We have described one-dimensional semistable representations of G_P completely. The next step is to find a criterion which tells us when a one-dimensional $(\phi, N)-$module is admissible. We are in the fortunate position of only having to deal with powers of the cyclotomic character, whose (ϕ, N)-module we will describe now:

LEMMA 3.9. *The $(\phi, N)-$module $D_{st}(V)$ corresponding to the representation $V = \langle t^i \rangle \in B_{st}$ given by the i^{th} power of the cyclotomic character $\chi : G_P \to \mathbb{Z}_p^*$ is given as follows: A one-dimensional filtered P_0-vector space D generated by an element d in $Fil^{-i} \backslash Fil^{-i+1}$, together with the endomorphism $\phi(v) = p^{-i}v$ and the operator $N = 0$.*

PROOF. This is a chain of easy checks. We have:

$$D_{st}(V) = (B_{st} \otimes_{\mathbb{Q}_p} V)^{G_P} = \langle t^{-i} \otimes t^i \rangle \subset B_{st} \otimes B_{st}$$

For all of the following computations, only the left copy of B_{st} carries extra structure, the right copy should just be thought of as a convenient way of expressing χ^i.
Using once more that $\phi(t) = pt$, we see that $\phi(t^{-i} \otimes t^i) = p^{-i}(t^{-i} \otimes t^i)$ (as ϕ only acts on the left copy of B_{st} and not on the Galois representation).
Since $t^{-i} \in B_{cris}$, we have $N = 0$.
It is also clear that $t^{-i} \otimes t^i \in Fil^{-i} \backslash Fil^{-i+1}$. □

Our goal is to spot special properties of admissible filtered (ϕ, N)-modules, and staring for a moment at this lemma lets us notice a special relationship between two of the occurring structures:

The largest degree i for which Fil^i is nontrivial agrees with the p-adic valuation of the eigenvalue of the Frobenius ϕ.

We introduce some language to express this more concisely, and start by defining the invariant t_H of filtered vector spaces, which measures at which point the filtration collapses.

DEFINITION 3.10. Let (D, Fil) be a one-dimensional filtered $K-$vector space. Write $t_H(V)$ for the maximal i such that $Fil^i(V) = V$ and $Fil^{i+1}(V) = 0$.

The next invariant is a measure for the action of the Frobenius element ϕ. Due to semilinearity, this action is not quite diagonal, but we can fix this by measuring valuations:

DEFINITION 3.11. Let (D, ϕ) be a one-dimensional $\phi-$ module on K, and assume ϕ is bijective (or equivalently injective).
Pick any nonzero element $d \in D$, write $\phi(d) = \lambda d$ for some $\lambda \in K_0$, and define

$$t_N(D) := v_p(\lambda)$$

We note that by $\sigma-$semilinearity, this number is independent of the choice of d (whereas λ *is not*).

With these two definitions, we can rephrase our previous observation concisely:

COROLLARY 3.12. *A $1-$dimensional filtered (ϕ, N)-module over P is admissible if and only if $t_N = t_H$.*

For the sake of understanding the $1-$dimensional case, the introduction of t_N and t_H is of course just language, but extending the definition to higher dimensions will help us in a second.

2.3. Necessary Conditions Implied by the One-Dimensional Case. Our current aim is to find necessary conditions for admissibility, and we have solved this problem in the $1-$dimensional case.
Given now a semistable $p-$adic Galois representation V of G_P of dimension n, it is a natural idea to try to produce a new semistable $1-$dimensional representation, obtain the restrictions on its $(\phi, N)-$module described above, and then express these restrictions in terms of V.
A candidate for this new representation is readily found, namely the top exterior power $\Lambda^n V$. Before we go on, we should of course check the following fact:

LEMMA 3.13. *If V is a semistable Galois representation of G_P, then so is $\Lambda^r V$ for all r.*

PROOF. This is just one of the many nice consequences of the fact that for a $(\mathbb{Q}_p, G_P)$-regular B, the category of $B-$admissible representations is a sub-Tannakian category of $Rep_{\mathbb{Q}_p}(G_P)$. □

We can therefore examine the (ϕ, N) module $D_{st}(\Lambda^n V)$ attached to $\Lambda^n V$, and deduce by the $1-$dimensional case that

$$t_N(D_{st}(\Lambda^n V)) = t_H(D_{st}(\Lambda^n V))$$

But we also have $D_{st}(\Lambda^n V) = \Lambda^n D_{st}(V)$, and thus

$$t_N(\Lambda^n D_{st}(V)) = t_H(\Lambda^n D_{st}(V))$$

It is therefore natural to extend the definitions of t_H and t_N to higher dimensional $(\phi, N)-$modules as follows:

DEFINITION 3.14. Let (D, Fil) be an $n-$dimensional filtered $K-$vector space. Define $t_H(V) = t_H(\Lambda^n V)$.

DEFINITION 3.15. Let (D, ϕ) be a $\phi-$ module of dimension n on K, and assume ϕ is bijective. We define $t_N(D) = t_N(\Lambda^n D)$.

We have therefore proven that if D is an admissible filtered (ϕ, N)-module, then $t_H(D) = t_N(D)$.
This condition is not sufficient as the most basic higher-dimensional case shows:

EXAMPLE. If it were, then one could prove an analogous result for K itself by showing that t_N and t_H do not change by tensoring up to $P_0 \otimes_{K_0} -$.

We can therefore look at $2-$dimensional $p-$adic Galois representations of $G_{\mathbb{Q}_p}$ for which a very detailed analysis can be carried out (see theorem 8.3.6. in [**5**]).
Start with the 2-dimensional representation V given by the sum of powers of the cyclotomic character:

$$\chi^0 \oplus \chi^{-2}$$

This representation has Hodge-Tate weights $(0, 2)$.
We find that $D = D_{st}(V) = D_{cris}(V)$ is the $\mathbb{Q}_p$-vector space

$$V = \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2$$

with

$$\phi(e_1) = e_1, \phi(e_2) = p^2 e_2$$

and filtration given by:

$$Fil^i(D) = \left\{ \begin{array}{cc} \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2 & \text{if } i \leq 0 \\ \mathbb{Q}_p e_2 & \text{if } i = 1, 2 \\ \{0\} & \text{else} \end{array} \right\}$$

The monodromy operator N vanishes, andd we have $t_N(D) = t_H(D) = 2$.

We note that the characteristic polynomial of ϕ is $f = (T-1)(T-p^2)$.

We now define a new filtered $(\phi, N)-$module E by taking the same (ϕ, N)-module as for D, but endowing it with the filtration given by:

$$Fil^i(D) = \left\{ \begin{array}{cc} \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2 & \text{if } i \leq 1 \\ \{0\} & \text{else} \end{array} \right\}$$

Also for this filtration, we have $t_N(E) = t_H(E) = 2$.

However, this module is *not* admissible: Assume for the sake of contradiction that we had $E = D_{cris}(V)$.
If $V = U_1 \oplus U_2$ would split as a direct sum, then E would split as a direct sum and we could find two linearly independent subspaces on which $t_N = t_H$. This is clearly not the case.
If V does *not* split as such a direct sum, then $Fil^1(E)$ would sill need to be $1-$dimensional by theorem 8.3.6. in [**5**], but this is also not the case.

We deduce from this that $t_N(E) = t_H(E)$ is *not* a sufficient criterion for admissibility.

2.4. Necessary Conditions from Submodules. The above analysis shows that we need to find more necessary conditions - they key idea is to look at sub-filtered-(ϕ, N)-modules. We shall again start with the one-dimensional case and reduce the problem to a question about our period rings.

So assume $D = D_{st}(V)$ is an admissible filtered (ϕ, N)-module with a 1-dimensional sub-filtered-(ϕ, N)-module D_0 spanned over K_0 by the vector e_0. Assume

$$t_H(D) = h$$

so that $e_0 \in Fil^h(D) \backslash Fil^{h+1}(D)$.

One can prove, using the so-called isoclinic decomposition, that monodromy operators N are always nilpotent (we will not discuss the argument here), hence N must be the zero operator on the $1-$dimensional space D_0.
Hope: There will be a nontrivial statement relating t_N and t_H. In order to relate D_0 to the period ring, we make use of the admissibility of D - i.e. notice that

$$D_0 \subset (B_{st} \otimes V)^{G_P}$$

We can therefore write

$$e_0 = \sum_i b_i \otimes v_i \in B_{st} \otimes V$$

for some $\mathbb{Q}_p$-basis $\{v_i\}$ of V such that all b_i lie in $Fil^h(B_{st})$ (this just follows from the definition of the diagonal filtration).

Vague Claim: The period vectors appearing in the above decomposition behave like e_0.
Indeed, we can write

$$\sum_i \phi(b_i) \otimes v_i = \phi(e_0) = \lambda e_0 = \sum_i (\lambda b_i) \otimes v_i$$

for some $\lambda \in K_0$ - this shows that ϕ acts in a very simple way by $\phi(b_i) = \lambda b_i$ for all i. Similarly, we see that $N(b_i) = 0$ for all i (so that $b_i \in B_{cris} = B_{st}^{N=0}$).

We see that in order to understand how $t_H(D_0)$ and $t_N(D_0)$ relate, we have to answer the following question:

QUESTION. *If $b \in B_{cris}$ is any nonzero vector on which the Frobenius acts diagonally with eigenvalue λ, what is the largest filtered piece in which it can lie?*

Let us reduce this question a bit further: Assume λ has $p-$adic valuation r, then

$$\phi\left(\frac{b}{t^r}\right) = p^{-r}\lambda b$$

and so $c = \frac{b}{t^r}$ is scaled by the action of ϕ by some μ with $v_p(\mu) = 0$.
We therefore need to know the answer to the following

QUESTION. *Given an element $c \in B_{cris}$ on which ϕ acts by an element in $W(\overline{k})^\times$, what's the largest filtered piece in which it can lie?*

Let us reduce the problem even further by simplifying the scaling factor μ. We need the following basic lemma about Witt vectors over algebraically closed fields (see [**5**] for a proof):

LEMMA 3.16. *If k is any perfect field of characteristic $p > 0$, then the map*

$$W(\overline{k})^\times \to W(\overline{k})^\times$$

$$x \mapsto \frac{\sigma(x)}{x}$$

is surjective.

Writing $\mu = \frac{\sigma(\tau)}{\tau}$ and $d = \frac{c}{\tau}$, we have

$$\phi(d) = \frac{\phi(c)}{\sigma(\tau)} = \frac{\mu c}{\mu \tau} = d$$

We are therefore finally reduced to the following rhyme:

QUESTION. *Given an element $c \in B_{cris}$ fixed by ϕ,*
what's the largest filtered piece in which it can lie?

At this point, we need an early theorem by Fontaine. Its proof is very tedious, and we therefore refer the interested reader to [**3**], Proposition 5.3.6 for a complete treatment.

THEOREM 3.17. *The elements in $Fil^0 B_{cris}$ which are fixed by the Frobenius are precisely the $p-$adic numbers $\mathbb{Q}_p \subset \mathrm{Frac}(W(\overline{k})) \subset B_{cris}$.*

We immediately obtain:

COROLLARY 3.18. *In the above situation, we have*

$$Fil^i(B_{cris})^{\phi=1} = \{0\}$$

for all $i > 0$.

This allows us to give answers to the chain of questions we raised before: If $c \in B_{cris}$ is acted on by the identity, then the largest filtered piece which can contain c is $Fil^0 B_{cris}$. Therefore, d can also at most lie in $Fil^0 B_{cris}$. As $b = t^r c$, we obtain that b can lie at most in $Fil^r B_{cris}$, where $r = v_p(\lambda)$. Taking i such that $b = b_i \in Fil^h(B_{cris})$ is nonzero, we see that

$$t_H(D_0) = h \leq r = v_p(\Lambda) = t_N(D_0)$$

We have therefore established the following fact:

LEMMA 3.19. *If D is an admissible filtered $(\phi, N)-$module and D_0 is a one-dimensional sub-filtered-(ϕ, N)-module of D, then*

$$t_H(D_0) \leq t_N(D_0)$$

Notice that by the old determinant trick, we can immediately obtain more necessary criteria by applying the previous argument to more than just one-dimensional subobjects. Indeed, if D_0 is a sub-filtered-(ϕ, N)-module of the admissible module D of *any* dimension r, then

$$\Lambda^r D_0 \subset \Lambda^r D$$

is a one-dimensional submodule of an admissible module, and the above analysis and our definition of t_N and t_H imply:

THEOREM 3.20. *If D is an admissible filtered (ϕ, N)-module over the field P with closed residue field, then D is finite and*

$$t_N(D) = t_H(D)$$

and for any sub-filtered-(ϕ, N)-module $D_0 \subset D$, we have

$$t_H(D_0) \leq t_N(D_0)$$

This lets us introduce the following notation:

DEFINITION 3.21. A filtered $(\phi, N)-$module D is *weakly admissible* if it is finite and

$$t_H(D_0) \leq t_N(D_0)$$

for all subobjects with equality for $D_0 = D$.

We can easily extend the previous statement to general p-adic fields by checking:

LEMMA 3.22. *If D is a filtered (ϕ, N)-module over a field K_0, then*

$$t_H(D \otimes_{K_0} P_0) = t_H(D)$$
$$t_N(D \otimes_{K_0} P_0) = t_N(D)$$

We therefore obtain the following general necessary criterion for admissibility:

THEOREM 3.23. *Every admissible filtered (ϕ, N)-module over a p-adic field K is weakly admissible.*

It turns out that this time, we will not be able to find weakly admissible modules which are not admissible. This lead Fontaine to conjecture the following in [**8**]:

CONJECTURE. *(Fontaine) Every weakly admissible module is admissible.*

The aim of the final section is to present the first proof of this fact, which was given by Fontaine and Colmez in [**2**]. By doing this, we give a very concrete description of the essential image of the functor D_{st}.

CHAPTER 4

Weak Admissibility - a Sufficient Criterion

The main aim of this section is to present the original proof by Colmez and Fontaine of the fact that weak admissibility implies admissibility. This means that the tensor functor

$$Rep^{st}_{\mathbb{Q}_p} \to MF^{\phi,N}_{wa}$$

from $p-$adic Galois representations to weakly admissible filtered $(\phi, N)-$modules is not just full and faithful (as stated above), but also essentially surjective.
We will later again use that we can work with P instead of K, keeping in mind that our previous work shows that this is no loss of generality.

1. The functor V_{st}

When we try to establish that a functor $F : \mathcal{C} \to \mathcal{D}$ between two categories restricts to an essential surjection between two full subcategories $\mathcal{C}_0$ and $\mathcal{D}_0$, it is often a good idea to first produce some natural functor $G : \mathcal{D} \to \mathcal{C}$ going in the opposite direction such that $F(G(Y)) \cong Y$ for all $Y \in \mathcal{D}_0$.
In our case, we take

$$F = D_{st}$$

$$\mathcal{C}_0 = Rep^{st}_{\mathbb{Q}_p}(V) \subset \mathcal{C} = Rep_{\mathbb{Q}_p}(V)$$

$$\mathcal{D}_0 = MF^{\phi,N}_{w.a.} \subset \mathcal{D} = MF^{\phi,N}$$

In other words, given the filtered $(\phi, N)-$module

$$D = (B_{st} \otimes_{\mathbb{Q}_p} V)^{G_K}$$

we want to produce a $p-$adic Galois representation $V_{st}(D)$ that agrees with V if it was semistable.
If this is the case, then we have a comparison isomorphism of filtered $(\phi, N)-$modules with G_K-action:

$$B_{st} \otimes_{K_0} D \cong B_{st} \otimes_{\mathbb{Q}_p} V$$

Recall that G_K acts diagonally on the right and only on B_{st} on the left, ϕ does the opposite, the filtration is diagonal on the left and only on the period ring on the right, and N acts like a derivation on the left and only on B_{st} on the right.

So let us try to first spot some necessary criteria for a general element

$$x = \sum_i b_i \otimes v_i \in B_{st} \otimes_{\mathbb{Q}_p} V$$

to lie in $\mathbb{Q}_p \otimes V \cong V$ and then check if they are sufficient.

If $x \in V$, then $b_i \in \mathbb{Q}_p$ for all i. We compute action by Frobenius, monodromy operaror, and filtration for $x \in V$ to be:

$$\phi(x) = x$$

$$N(x) = 0$$

$$x \in Fil^0(B_{st} \otimes_{\mathbb{Q}_p} V) = Fil^0(B_{st} \otimes_{K_0} D)$$

Hence, it is very natural to define the following functor:

Definition 4.1. We define the functor $V_{st} : MF^{\phi,N} \to Rep_{\mathbb{Q}_p}(G_K)$ by

$$D \mapsto \{v \in B_{st} \otimes_{K_0} D | \phi(v) = v, Nv = 0, v \in Fil^0(B_{st} \otimes_{K_0} D)\}$$

Note that this is a topological $\mathbb{Q}_p$ vector space on which G_K acts linearly and continuously.

We will now verify that the necessary conditions we established above are indeed also sufficient - the crucial input here is once more that $Fil^0(B_{cris}^{\phi=1}) = \mathbb{Q}_p$:

Lemma 4.2. *If V is a semistable Galois representation, then*

$$V \cong V_{st}(D_{st}(V))$$

Proof. We just checked the inclusion "$\subseteq$".
In order to verify the reverse inclusion, let

$$x = \sum_i b_i \otimes v_i \in V_{st}(D_{st}(V)) \subset B_{st} \otimes_{\mathbb{Q}_p} V$$

for a given basis $\{v_i\}_i$ of V.
Since $N(x) = \sum_i N(b_i) \otimes v_i = 0$, we have $N(b_i) = 0$ and thus $b_i \in B_{cris}$ for all i.
Similarly, we conclude that $\phi(b_i) = b_i$ and $b_i \in Fil^0(B_{cris})$ for all i.
Hence all b_i lie in $Fil^0(B_{cris})^{\phi=1} = \mathbb{Q}_p$. □

This theorem is of great value since for a given filtered $(\phi, N)-$module D, we now know the representation which would be semistable and associated to it if D happened to be admissible. Indeed, if $D = D_{st}(W)$, then

$$D_{st}(V_{st}(D)) = D_{st}(V_{st}(D_{st}(W))) = D$$

2. Establishing Admissibility

We see that if indeed all weakly admissible filtered $(\phi, N)-$modules happened to be admissible, we would in particular know that $V_{st}(D)$ were finite-dimensional and semistable for all weakly admissible D.

We will now prove this weaker claim, which will be very helpful later:

Theorem 4.3. *Let D be a nonzero finite filtered $(\phi, N)-$module such that*

$$t_H(D') \leq t_N(D')$$

for all sub-filtered-(ϕ, N)-modules $D' \subset D$ - this in particular applies to D weakly admissible.
Then $V = V_{st}(D)$ is a finite-dimensional $\mathbb{Q}_p$-vector space and the G_K-action defined by it is semistable.

PROOF. Recall that in order to show that

$$V = Fil^0(B_{st} \otimes_{K_0} D)^{\phi=1,N=0} \subset B_{st} \otimes_{K_0} D$$

is semistable, we need to show that the comparison map

$$B_{st} \otimes_{K_0} D_{st}(V) \to B_{st} \otimes_{\mathbb{Q}_p} V$$

is an isomorphism. Phrased differently, we are given the inclusion

$$V \subset B_{st} \otimes_{K_0} D$$

and we need to check whether or not the $\mathbb{Q}_p$-vector space $D_{st}(V)$ is a solution to the following problem:

PROBLEM. *Given a G_K-stable subspace $V \subset B_{st} \otimes_{K_0} D$, find a $\mathbb{Q}_p$-vector space E such that $V = B_{st} \otimes_{\mathbb{Q}_p} E$ as G_K-spaces.*

Let us forget for a moment that we have other structures (like ϕ, N and Fil) and that we already have a candidate for the solution. Our aim for now is just to construct such an E.

This reminds us a lot of descent for vector-spaces, and indeed the following lemma tells us that if B_{st} happened to be a field V were closed under scaling by elements in B_{st}, things would be as nice as they possibly could be:

LEMMA 4.4. *Let F be a field and let $E = F^G$ be the fixed field of some subgroup $G \subset Aut(F)$ (here F/E will not necessarily be finite).*
Given any finite-dimensional E-vector space Δ, the group G naturally acts E-linearly on the $F-$vector space

$$W = F \otimes_E \Delta$$

Then an F-subspace $L \subset W$ comes from tensoring up some $E-$subspace $\Delta' \subset \Delta$ as

$$L = F \otimes_E \Delta'$$

if and only if L is $G-$stable

The proof is an easy exercise and we will therefore leave it as an exercise to the reader.

The idea now is simply "force" B_{st} to be a field and V to be closed under scaling by working over its fraction field $C_{st} \subset B_{dR}$ and considering the $C_{st}-$space generated by V. We can then obtain the necessary statements in this context and reduce back down again.

Observe that the G_K-action extends in the obvious way to C_{st} and that $C_{st}^{G_K} = K_0$. Hence we consider the G_K-stable C_{st}-subspace L of $C_{st} \otimes_{K_0} D$ *generated by* V. The above lemma then yields a K_0-subspace $D' \subset D$ such that

$$C_{st} \otimes_{K_0} D' = L \subset C_{st} \otimes_{K_0} D$$

Notice that also the monodromy operator N and the Frobenius ϕ extend to C_{st}, and since $V \subset (C_{st} \otimes_{K_0} D)^{\phi=1,N=0}$, we can conclude that D' is fixed by ϕ and N and hence not just a subspace, but a sub-(ϕ, N)-module of D. Notice that $V_{st}(D') = V$ as it both contains V and is contained in $V_{st}(D) = V$.

GOAL. *Prove that V is a finite-dimensional $\mathbb{Q}_p$ vector space, that $D' = D_{st}(V)$ and that*

$$B_{st} \otimes_{\mathbb{Q}_p} \cong B_{st} \otimes_{K_0} D'$$

Once tensored up to C_{st}, all these statements hold, so let us choose witnesses: We can take elements $\{v_1, ..., v_r\} \in V$ forming a C_{st}-basis of L, and we can pick a K_0-basis $\{d_1, .., d_r\}$ for D' which will automatically tensor up to give a C_{st}-basis for L.

We shall start by proving that V is a finite-dimensional $\mathbb{Q}_p$-vector space, and the obvious approach to doing this is to show that $\{v_1, ..., v_n\}$ is a $\mathbb{Q}_p$-basis.
So let $v \in V$ be arbitrary and pick elements $c_i \in C_{st}$ such that

$$v = \sum_i c_i v_i$$

Set $w = v_1 \wedge ... \wedge v_n$. We can now pick out the i^{th} coefficient by wedging with all other basis vectors in the following sense:

$$v_1 \wedge ... \wedge v_{i-1} \wedge v \wedge v_{i+1} \wedge ... \wedge v_r = c_i w$$

This does not seem to make things any easier, until we notice that we can also express any wedge of vectors in V as $cd_1 \wedge ... \wedge d_r$ for some $c \in B_{st}$ (as we can express each v_i as a $B_{st}-$linear combination of the $d_j's$). One easily checks that even more is true, namely that any wedge of r vectors in V lies inside

$$W = V_{st}(\Lambda^r D')$$

Writing $w = bd_1 \wedge ... \wedge d_r$, this tells us that $c_i w \in W$. In order to prove that $\dim_{\mathbb{Q}_p}(V) < \infty$, it is therefore sufficient to show that $W = \mathbb{Q}_p w$.
This is the point where the assumptions on D enter the stage:
The sub-(ϕ, N)-module D' satisfies

$$t_H(D') = t_H(\Lambda^r D') \leq t_N(\Lambda^r D') = t_N(D')$$

The $(\phi, N)-$module $E := \Lambda^r D' \subset B_{st} \otimes E$ is one-dimensional, and essentially the same arguments as in 2.4 allow us to show that E is admissible.
More precisely, choose a generator e for E such that $e \in Fil^{t_H(E)} \backslash Fil^{t_H(E)+1}$. As before $Ne = 0$ by nilpotence, and thus $e \in B_{cris}$. Pick $a_0 \in W(k)^\times$ such that $\phi(\frac{e}{t^{t_n(E)}}) = a_0 e$. By 3.16, we can again write $a_0 = \frac{\phi(x)}{x}$ for some $x \in W(\overline{k})$. This implies that $\frac{e}{xt^{t_n(E)}} \in Fil^{t_H(E)-t_N(E)}(B_{cris}^{\phi=1})$ is a *nonzero* vector. Recalling the fundamental fact that $Fil^0(B_{cris})^{\phi=1} = \mathbb{Q}_p$, we deduce that $t_H(E) - t_N(E) = 0$ as if it were negative, e would need to vanish. Therefore $E = \Lambda^r D'$ is admissible and therefore $dim_{\mathbb{Q}_p}(W) = 1$. This proves that $\dim_{\mathbb{Q}_p}(V) < \infty$.

The admissibility equation $B_{st} \otimes_{K_0} \Lambda^r D' = B_{st} \otimes_{\mathbb{Q}_p} W$ also tells us that we can express $d = d_1 \wedge ... \wedge d_r$ in terms of W, i.e. that b is invertible in B_{st}.

By Cramer's rule, this implies that we can write all d_i's as $B_{st}-$linear combinations of the $v_i's$, and hence that

$$B_{st} \otimes_{\mathbb{Q}_p} V = B_{st} \otimes_{K_0} D'$$

This finally proves that $V = V_{st}(D)$ is semistable with $D_{st}(V) = D'$. $\square$

We immediately obtain an easy dimension condition for admissibility:

COROLLARY 4.5. *Let D be a filtered weakly admissible $(\phi, N)-$module of dimension $d > 0$. Then D is admissible if and only if* $\dim(V_{st}(D)) \geq d$.

PROOF. The submodule $D_{st}(V_{st}(D)) \subset D$ is admissible of dimension $\dim(V_{st}(D)$. This implies the claim. □

For simple objects in $MF_{w.a.}^{\phi,N}$, we obtain an even easier criterion:

COROLLARY 4.6. *Let D be a (nonzero) simple object in the category of weakly admissible filtered $(\phi, N)-$modules. Then D is admissible if and only of $V_{st}(D) \neq 0$.*

PROOF. One direction is obvious. For the other, note that $D_{st}(V_{st}(D))$ is a nonzero subobject, hence it equals all of D. □

3. Existence of Admissible Filtrations on Simple Modules

In this section, we will complete the first big step towards a proof of Fontaine's conjecture: We want to show that for every weakly admissible module, we can change the filtration and make it admissible (i.e. find an *admissible filtration*). In the section thereafter, we will then establish that admissibility is preserved under "small" changes of filtration.
From now on, we shall work again with the field P - we have seen in 3.7 that admissibility results for this case immediately generalise to K.

Let us begin with the simple case where D is a simple object in the category of finite $(\phi, N)-$modules. First of all notice that this forces N to vanish. By our previous work, namely 4.5, we know that in order to produce an admissible filtration on D, it is enough to choose a filtration Fil turning (D, Fil) into a weakly admissible filtered (ϕ, N)-module such that $V_{st}(D, Fil) \neq 0$.

This is the point where an actual classification of simple finite $\phi-$modules is needed - and where the closure of the residue field of P is useful. The necessary result was known long before Fontaine's work, and we will therefore only quote it here:

THEOREM 4.7. *(Dieudonné-Manin) Given a p-adic field P with closed residue field $\overline{k}$, the category of finite $\phi-$modules is semisimple*[1] *and its simple objects are precisely given by the following pairwise nonisomorphic objects:*
For each rational α, choose unique coprime $r \in \mathbb{Z}$, $s \in \mathbb{N}_{>0}$ such that $\alpha = \frac{s}{r}$. Let $D_{[\alpha]}$ be the P_0-vector space generated by a basis

$$e_1, ..., e_r$$

and endowed with the $\phi-$operator given by

$$\phi(e_1) = e_2, \ \dots \ , \phi(e_{r-1}) = e_r, \phi(e_r) = p^s e_1$$

Assuming this result, we can now prove the following:

THEOREM 4.8. *Every simple finite $(\phi, N)-$module has an admissible filtration.*

[1] A category is *semisimple* if all objects are sums of simple ones, and all sums of simple objects exist.

PROOF. Fix an $\alpha = \frac{s}{r}$ as above such that $D = D_{[\alpha]}$. By 4.5, it is sufficient to find a filtration Fil and a nonzero "witness" element

$$\zeta \in V^{st}(D) = Fil^0(B_{st} \otimes V)^{\phi=1,N=0}$$

It is natural to first forget about the filtration and simply try to produce a nonzero element in $(B_{cris} \otimes V)^{\phi=1}$. We make the Ansatz

$$y = \sum_{i=1}^{r} b_i \otimes e_i$$

and see that $\phi(y) = y$ if and only if

$$\phi(b_1) = b_2 \ , \ \dots \ , \ \phi(b_{r-1}) = b_r \ , \ \phi(b_r) = p^{-s} b_1$$

We therefore want an element $b_{r,s} \in B_{cris}$ with $\phi^r(b_{r,s}) = p^{-s} b_{r,s}$. Note that it is sufficient to solve the problem for $s = 1$: As the Frobenius is a ring endomorphism, we then just set $b_r := b_{1,r}$.

Before we think more about these elements b_r, let us examine the possible filtrations on $D_P = P \otimes_{P_0} D$ which make $D = D_{[\alpha]}$ weakly admissible. Simplicity helps us: In order for (D, ϕ, Fil) to be admissible, we just have to check that $t_H(D) = t_N(D) = -s$. This happens precisely if $-r$ is the largest integer which we can decompose as $-s = i_1 + \dots + i_s$ with $e_j \in Fil^{i_j}(D)$ for all j.
The filtration which will make the construction of our witness easiest is obtained by defining

$$Fil^i D_P = \left\{ \begin{array}{cc} D_P & \text{if } i \leq -s \\ P \cdot e_2 \oplus \dots \oplus P \cdot e_n & \text{if } -s < i \leq 0 \\ \{0\} & \text{if } s > 0 \end{array} \right\}$$

Now recall that we want our "witness" element

$$y = \sum_{i=1}^{r} (\phi^i(b_r))^s \otimes e_i$$

to be of filtration zero - the only thing we have to claim for this to happen is

$$\phi^i(b_r) \in Fil^1 D_K$$

We have therefore reduced the problem of constructing an admissible filtration on all simple (ϕ, N)-modules to the following claim about B_{cris} which we will prove separately: $\square$

LEMMA 4.9. *For all $r \in \mathbb{N}_{>0}$, there is a $b_r \in Fil^1 B_{cris}$ such that*

$$\phi^r(b_r) = p b_r$$

PROOF. Our aim is to show that restricting the natural augmentation

$$\theta : B^+_{cris} \to \mathbb{C}$$

to the additive subgroup

$$Z_r = \{z \in B^+_{cris} \mid \phi^r(z) = pz\}$$

has nontrivial kernel.
Fontaine applied a very clever trick to prove this fact: he shows that $\theta|_{Z_h}$ is surjective, but *not* bijective, which of course implies the claim.

Seeing that the G_K-equivariant map is not bijective is easy by simply noticing that the sets of fixed points disagree:

$$Z_h^{G_P} = \{0\}$$
$$\mathbb{C}^{G_P} = P \neq \{0\}$$

Proving that the map is surjective is substantially harder. We shall assume that $p > 2$ - a similar but slightly different argument goes through for $p = 2$. Let us start off by making two reductions:
First of all, it is clearly sufficient to show that the restriction

$$\theta : Z_r^0 := Z_h \cap \theta^{-1}(p\mathcal{O}_{\mathbb{C}}) \to p\mathcal{O}_{\mathbb{C}}$$

is surjective, which is desirable since $Z_r^0 \subset A_{cris}$.
Secondly, it is enough that the composite

$$\theta : Z_r^0 \to p\mathcal{O}_{\mathbb{C}} \to p\mathcal{O}_{\mathbb{C}}/p^2\mathcal{O}_{\mathbb{C}}$$

is surjective: If this is the case, then we can approximate the preimage better and better, and these approximations will converge $p-$adically inside Z_r^0, which is a closed subset of a complete and separated space.

In order to prove the last reduction, we will use the inclusion $W(R(\mathcal{O}_{\mathbb{C}})) \subset A_{cris}$. Fix $a \in p\mathcal{O}_{\mathbb{C}}$.

First step: The goal $\phi^r(z) = pz$ naturally leads us to make the following Ansatz:

$$z = \sum_{i \in \mathbb{Z}} [u^{p^{r \cdot i}}] p^{-i}$$

for which $\phi^r(z) = pz$ is obvious, assuming convergence. Here u is assumed to be an element of $R(\mathcal{O}_{\mathbb{C}})$.
Let us check that this makes sense. The infinite sum

$$\sum_{i \leq 0} [u^{p^{r \cdot i}}] p^{-i}$$

converges by definition of the Witt-vectors inside $W(R(\mathcal{O}_{\mathbb{C}}))$.
For the positive half, we notice that once we *assume* that $u \in p\mathcal{O}_{\mathbb{C}}$, we can rewrite the sum as

$$\sum_{i > 0} \frac{(p^{ri})!}{p^i} \frac{[u^{p^{r \cdot i}}]}{(p^{ri})!}$$

Since $\frac{[u^{p^{r \cdot i}}]}{(p^{ri})!} \in A_{cris}$ (as $u \in \ker(\theta)$) and we p-adically completed, this sum converges. We therefore see that the definition of z makes sense and that $\phi^r(z) = pz$.

Our next aim is to make a good choice of u such that $\theta(z) \equiv a \mod p^2$. We compute $\theta(z)$:

$$\theta(z) = \sum_{i \geq 0} [u^{p^{1-r \cdot i}}] p^i + \sum_{i > 0} \frac{(p^{ri})!}{p^i} \theta\left(\frac{[u^{p^{r \cdot i}}]}{(p^{ri})!} \right) \equiv [u] + p[u^{p^{1-r}}]$$

Our aim is to make this equal to $a \mod p^2$, and we therefore set

$$x = u^{p^{1-r}} \in \mathcal{O}_{\mathbb{C}}$$

such that $u = x^{p^r}$. This implies:

$$x^{p^r} + px \equiv a \mod p^2$$

Picking a $(p^r)^{th}$ root α of p, we can simplify this further by setting $y = \frac{x}{\alpha}$:

$$y^{p^r} + \alpha y \equiv p^{-1}a \mod p^2$$

We can solve this equation in $\mathcal{O}_{\mathbb{C}}$, which finishes the proof of the lemma. $\square$

We have now finished the proof that simple finite $(\phi, N)-$modules have admissible filtrations. Our next aim is:

GOAL. *Show that all finite $(\phi, N)-$modules over P can be endowed with admissible filtrations.*

To measure how "non-simple" a given (ϕ, N)-module is and then apply induction, we introduce the following notion:

DEFINITION 4.10. The *length* $l(D)$ of a (ϕ, N)-module D is the supremum over all numbers r for which there is a strictly increasing chain of sub-(ϕ, N)-modules

$$D_0 \subsetneq D_1 \subsetneq ... \subsetneq D_r$$

Note that (nonzero) simple modules are precisely those of length 1. Given a (ϕ, N)-module D of length $l(D) > 1$, we can find a nontrivial short exact sequence

$$0 \to D' \to D \to D'' \to 0$$

with $l(D'), l(D'') < l(D)$. By induction, D'_P and D''_P then have admissible filtrations Fil' and Fil''.
We define a filtration on D_P by first choosing a $P-$linear section of the projection $D_K \to D''_K$ of vector spaces and then defining

$$Fil^i(D_P) = (Fil')^i(D'_P) + s(Fil'')^i(D''_K)$$

We have therefore turned the above sequence into a short exact sequence of *filtered* (ϕ, N)-modules - the following question arises:

QUESTION. *Given a short exact sequence of filtered (ϕ, N)-modules whose outer modules are admissible, does this force the middle one to be admissible too?*

It is easy to see that the corresponding claim for weak admissibility is true:

First, one can prove the following straightforward lemma about the behaviour of t_N and t_H under short exact sequences:

LEMMA 4.11. *The functions t_N and t_H are additive on short exact sequences.*

We can now easily deduce:

LEMMA 4.12. *If the outer two modules in a short exact sequence*

$$0 \to D' \xrightarrow{f} D \xrightarrow{g} D'' \to 0$$

of filtered (ϕ, N)-modules are weakly admissible, then so is the inner one.

Proof. By additivity, the equality $t_H(D) = t_N(D)$ is clear.
If $E \subset D$ is a sub-filtered-(ϕ, N)-module, we define

$$E' = E \cap D'$$

$$E'' = g(E)$$

and obtain a short exact sequence

$$0 \to E' \to E \to E'' \to 0$$

which implies

$$t_H(E) = t_H(E') + t_H(E'') \leq t_N(E') + t_N(E'') = t_N(E)$$

□

However, in order to give a full affirmative answer to the question raised above, we need to introduce the machinery of the fundamental complex (developed by Fontaine and Emerton-Kisin), which we shall do in the following section.

4. The Fundamental Complex

Our aim is to measure how admissibility changes under short exact sequences, and it is therefore natural to look for a *homological criterion for admissibility.* More precisely, we are looking for a sequence of functors

$$H^i : MF^{(\phi,N)} \to Vect_{\mathbb{Q}_p}$$

which map short exact sequences to long ones and such that we can read off admissibility from the vanishing of some H^i - this would then clearly imply our initial claim about admissibility of middle modules nested between two admissible ones. In this section, we work with a general p-adic field K again.

It is natural to look for such a collection of functors with $H^0(D) = V_{st}(D)$ and, being economical, ask for H^1 to measure admissibility.
One problem is that the category of *filtered* (ϕ, N)-vector spaces is not abelian, so we *cannot* simply apply the methods of homological algebra to produce a candidate. Even though it might well be possible to resolve this issue by the methods of homotopical algebra instead, an ad-hoc approach will be completely sufficient for our purposes here - so let us try to define the required complex

$$V_{st}^{\bullet}(D) = V_{st}^0(D) \xrightarrow{d_0} V_{st}^1(D) \xrightarrow{d_1} ...$$

We want that $H^0(V_{st}^{\bullet}(D)) = V_{st}(D) = Fil^0(B_{st} \otimes_{K_0} D)^{N=0,\phi=1}$, and so we try to write it as a kernel of some map of modules, which is easy to spot:

$$V_{st}(D) = \ker\left(\delta : (B_{st} \otimes_{K_0} D)^{N=0,\phi=1} \to B_{dR} \otimes_K D_K \to B_{dR} \otimes_K D_K / Fil^0(B_{dR} \otimes_K D_K\right)$$

We therefore simply *define*:

Definition 4.13. The *fundamental complex* of a finite filtered $(\phi, N)-$module is given by

$$V_{st}^i(D) = \left\{ \begin{array}{cc} (B_{st} \otimes_{K_0} D)^{N=0,\phi=1}, & \text{if i =0} \\ B_{dR} \otimes_K D_K / Fil^0(B_{dR} \otimes_K D_K), & \text{if } i = 1 \\ 0 & \text{else} \end{array} \right\}$$

and $d_0 = \delta$.

It is important to note at this stage that V_{st}^0 does *not* depend on the filtration (and is therefore naturally a functor defined on the category of finite $(\phi, N)-$modules), whereas V_{st}^1 does crucially depend on it.

It is of course not clear at all why this definition should satisfy the nice formal properties we claimed - in order to establish long exact sequences on cohomology, we need exact sequences on the levels of chains in each degree. Will now verify this:

LEMMA 4.14. *The functor V_{st}^0 is exact.*

PROOF. (Sketch) We can define an analogous functor V_{cris}^0 which maps a finite $\phi-$module Δ given by

$$V_{cris}^0(\Delta) = (B_{cris} \otimes_{K_0} \Delta)^{\phi=1}$$

Writing F for the forgetful functor from (ϕ, N)-modules to $\phi-$modules, it is straightforward to show that we have an isomorphism of functors

$$V_{st}^0 \cong V_{cris}^0 \circ U$$

and that it is therefore sufficient to prove that V_{cris}^0 is exact.
If the residue field is closed, then the theorem Dieudonné-Manin shows that the category of finite $\phi-$modules is semisimple, this implies that the additive functor V_{cris} is exact.
The general case follows from this by the following two observations (expressed in the usual notation):

- $D \mapsto P_0 \otimes_{K_0} D$ is an exact functors from $\phi-$modules on K to $\phi-$modules on P.
- We have

$$V_{cris,k}^0(D) = (B_{cris} \otimes_{K_0} D)^{\phi=1} = (B_{cris} \otimes_{P_0} (P_0 \otimes_{K_0} D))^{\phi=1} = V_{cris,\overline{k}}^0(P_0 \otimes K_0 D)$$

□

We need a similar result for V_{st}^1, which is much more formal:

LEMMA 4.15. *The functor V_{st}^1 preserves short exact sequences of filtered $(\phi, N)-$modules.*

PROOF. This follows by simply writing out the short exact sequences defining V_{st}^1 vertically, and noticing that the horizontal sequences we obtain for those are obviously exact. □

We can use these statements to establish the following lemma:

LEMMA 4.16. *If*

$$0 \to D' \to D \to D'' \to 0$$

is a short exact sequence of finite filtered (ϕ, N) modules, then there is a long exact sequence

$$0 \to H^0(D') \to H^0(D) \to H^0(D'') \to$$
$$\to H^1(D') \to H^1(D) \to H^1(D'') \to 0 \to ...$$

PROOF. Exactness of V_{st}^i and naturality of δ gives a short exact sequence of complexes

$$0 \to V_{st}^{\bullet} D' \to V_{st}^{\bullet} D \to V_{st}^{\bullet} D'' \to 0$$

from which we get the desired long exact sequence by the usual snake-argument. □

We therefore see that for our purposes, it would be extremely desirable if there were a link between the vanishing of H^1 and admissibility. And indeed there is:

THEOREM 4.17. *A weakly admissible filtered $(\phi, N)-$module D is admissible if and only if*

$$H^1(D) = 0$$

PROOF. Let us first prove that admissibility implies vanishing of cohomology. Indeed, for an admissible module $D = D_{st}(V)$, we can compute the fundamental complex in a different way by using the equalities (using that semistable representations are de Rham):

$$B_{st} \otimes_{\mathbb{Q}_p} V \cong B_{st} \otimes_{K_0} D$$

$$B_{dR} \otimes_{\mathbb{Q}_p} V \cong B_{dR} \otimes_K D_K$$

In detail, we can write

$$V_{st}^0(D) = B_{cris}^{\phi=1} \otimes_{\mathbb{Q}_p} V$$

and

$$V_{st}^1(D) = B_{dR} \otimes_{\mathbb{Q}_p} V/Fil^0(B_{dR} \otimes_{\mathbb{Q}_p} V) = (B_{dR}/B_{dR}^+) \otimes_{\mathbb{Q}_p} V$$

and therefore compute that $H^1(D) = 0$ if and only if the map

$$B_{cris}^{\phi=1} \otimes_{\mathbb{Q}_p} V \to (B_{dR}/B_{dR}^+) \otimes_{\mathbb{Q}_p} V$$

is surjective. This follows immediately by tensoring the so-called *fundamental sequence* on the level of period rings with V (the complete proof is this sequence is rather technical, see proposition 1.3 in [**8**]):

LEMMA 4.18. *The natural sequence*

$$0 \to \mathbb{Q}_p \to B_{cris}^{\phi=1} \to B_{dR}/B_{dR}^+ \to 0$$

is exact

This shows that admissibility implies vanishing H^1.

For the converse direction, assume $H^1(D) = 0$ and write $D' = D_{st}(V_{st}(D)) \subset D$, giving rise to a short exact sequence of finite filtered (ϕ, N)-modules

$$0 \to D' \to D \to D'' \to 0$$

By our previous work, it is enough to show that $D'' = 0$.
Using that $H^0(D) = H^0(D') = V$ and $H^1(D) = H^1(D') = 0$, the long exact sequence in cohomology is

$$0 \to V \to V \to H^0(D'') \to 0 \to 0 \to H^1(D'') \to 0 \to ...$$

From this we deduce that $H^0(D'') = H^1(D'') = 0$. We are therefore forced to ask:

QUESTION. *Can nonzero finite filtered $(\phi, N)-$modules have vanishing cohomology?*

The cohomology vanishes if and only if the composite map

$$V_{st}^0(D) = (B_{st} \otimes_{K_0} D)^{\phi=1, N=0} \to B_{st} \otimes_{K_0} D \to B_{dR} \otimes_K D_K/Fil^0(B_{dR} \otimes_K D_K)$$

is an isomorphism. It should not come as a surprise that this essentially never happens: This map is G_K-equivariant, and while B_{dR} has all of K as fixed field, the action on B_{st} fixes only the maximal unramified subextension K_0.

Since the actions on the two sides are strongly linked to these period rings, we expect to be able to find a contradiction by taking fixed points: The de Rham side should have "more" fixed points than the semistable side, and we therefore doubt that the $G-$equivariant map

$$B_{st} \otimes_{K_0} D \to B_{dR} \otimes_K D_K / Fil^0(B_{dR} \otimes_K D_K)$$

can induce a surjection on fixed points as vanishing H^0 and H^1 would imply.

Now there is one first trouble: K and K_0 might agree. This can be readily resolved by taking a nontrivial extension K' of K which is totally ramified over K_0 and arguing with $G_{K'}$ instead of G_K later. Write $e = [K' : K_0] > 1$.

We can now compute $G_{K'}$ invariants on the left hand side by using the obvious $G_{K'}$-equivariant isomorphism $B_{st} \otimes_{K_0} D \cong (B_{st})^r$, where $r = \dim_{K_0}(D)$ and therefore obtain

$$(B_{st} \otimes_{K_0} D)^{G_{K'}} \cong (B_{st}^{G_{K'}})^r = K_0^r$$

In particular, we see that it has K_0-dimension r.

So let us attack the other side. Once we choose a basis $\{d_1, ..., d_r\}$ for D_K which is *compatible with the filtration* [2], we can rewrite the right hand side more explicitly as

$$V_{st}^1 = \bigoplus_{j=1}^{r} (B_{dR}/B_{dR}^+) t^{-t_H(d_j)} \otimes d_j \cong \bigoplus_{j=1}^{r} (B_{dR}/Fil^{-t_H(d_j)}(B_{dR}))$$

This isomorphism is $G_{K'}$-equivariant.

One can show (it is not hard, but we skip the proof here) that

$$(B_{dR}/Fil^i B_{dR})^{G_{K'}} = \left\{ \begin{array}{ll} K' & \text{if } i \leq 0 \\ 0, & \text{else} \end{array} \right\}$$

We therefore see that

$$\dim_{K_0}(V_{st}^1)^{G_{K'}} = r \cdot (\text{the number of } j \text{ with } t_H(d_j) \geq 0)$$

If $t_H(d_j) \geq 0$ for all j, we have therefore produced a surjection from an r-dimensional space to an $e \cdot r$-dimensional space - a contradiction for $r > 0$.

The second problem that arises is that if some or all of the $t_H(d_j)$'s are negative, we cannot draw this conclusion.
The trick to solve this is to simply Tate-twist the entire situation an appropriate number of times: Choose r any integer larger than all the $t_H(d_j)$, and raneenter the argument at the beginning. We can twist the isomorphism r times and thus obtain a surjection

$$(B_{st} \otimes_{K_0} D)(r)^{G_{K'}} \to V_{st}^1(D)(r)^{G_{K'}}$$

[2] A basis of a filtered vector space is said to be *compatible with the filtration* if for each j, the basis vectors lying in the j^{th} but not in the $(j+1)^{st}$ piece project to give a basis of the j^{th} piece of the associated graded.

The space on the left has again K_0-dimension r (since the vector t representing the cyclotomic character in B_{st} is invertible), whereas the right hand side becomes

$$V_{st}^1(r) \cong \bigoplus_{j=1}^{r} (B_{dR}/Fil^{r-t_H(d_j)}(B_{dR}))$$

and therefore has K_0-dimension $e \cdot r$.
This implies $r \geq e \cdot r$ which shows that $r = 0$.

We can therefore give the following affirmative answer to the question raised above:

LEMMA 4.19. *If D is a finite filtered $(\phi, N)-$module with vanishing cohomology, then $D = 0$.*

This finishes the proof of 4.17. □

We can now reap the fruits of our labor, zoom out again, and easily deduce the following solution to the "short exact sequences problem", which previously hindered us from proving the existence of admissible filtrations and initially motivated our study of the fundamental complex:

THEOREM 4.20. *Given a short exact sequence*

$$0 \to D' \to D \to D'' \to 0$$

of finite filtered $(\phi, N)-$modules over K.
If D' and D'' are admissible, then D is admissible too.

PROOF. By 4.17, we know that $H^1(D') = H^1(D'') = 0$, and the long exact sequence (see 4.16) associated to the above short exact sequence then implies that $H^1(D) = 0$. By 4.12, D is weakly admissible, and therefore again by 4.17, we conclude that it is admissible. □

In the end of our previous section, we had reduced the existence of admissible filtrations to this claim, and we can therefore deduce:

THEOREM 4.21. *Every finite (ϕ, N) module over the p-adic field P has an admissible filtration.*

REMARK 4.22. This theorem also holds for K, i.e. without the assumption on the residue field being closed. We will not need this in order to prove our main theorem though.

We are now ready to present the final piece of the proof that weak admissibility implies admissibility:

5. Preservation of Admissibility under Manipulations of Filtrations

We have proven above that given any *weakly* admissible filtered $(\phi, N)-$module (D, Fil_{wa}) whose admissibility is to be established, there is another filtration Fil_a which makes D admissible. It is not clear however how this helps us with proving the admissibility of our original module. Rather than presenting Fontaine's idea immediately, let us dive into the proof and see how it arises rather naturally - note that we cannot abuse notation as much as before anymore as we need to keep track of multiple filtrations on the same module.

REMARK 4.23. Write $\mathcal{D} = B_{dR} \otimes_K D_K$, and note that it comes with two possible diagonal filtrations inherited from Fil_{wa} and Fil_a:

$$Fil^i_{wa}\mathcal{D} = \oplus_{j+k=i} Fil^j(B_{dR}) \otimes Fil^k_{wa} D$$

$$Fil^i_a\mathcal{D} = \oplus_{j+k=i} Fil^j(B_{dR}) \otimes Fil^k_a D$$

Then

$$V^0_{st}(D) = (B_{st} \otimes_{K_0} D)^{N=0,\phi=1}$$

and

$$V^1_{st}(D, Fil_a) = \mathcal{D}/Fil^0_a\mathcal{D}$$
$$V^1_{st}(D, Fil_{wa}) = \mathcal{D}/Fil^0_{wa}\mathcal{D}$$

Our aim is to show that $H^1(D, Fil_{wa}) = 0$. Writing $V_{wa} = V_{st}(D, Fil_{wa})$, this is just saying that the last map δ in the short exact sequence

$$0 \to V_{wa} \to V^0_{st}(D) \xrightarrow{\delta} (V^1_{st}(Fil_{wa}) = \mathcal{D}/Fil^0_{wa}\mathcal{D})$$

is a surjection.

The corresponding statement for the admissible filtration Fil_a is of course true, i.e.

$$0 \to V_a \to V^0_{st}(D) \to (V^1_{st}(Fil_a) = \mathcal{D}/Fil^0_a\mathcal{D}) \to 0$$

is exact.

We have the following diagram:

$$\begin{tikzcd} V^0_{st}(D) \arrow[r, "f"] & \mathcal{D} \arrow[r] \arrow[dr] & \mathcal{D}/Fil^0_a\mathcal{D} \\ & & \mathcal{D}/Fil^0_{wa}\mathcal{D} \end{tikzcd}$$

The principal strategy for producing preimages is therefore clear: Given some class $x \in \mathcal{D}/Fil^0_{wa}\mathcal{D}$, we choose $d \in \mathcal{D}$ projecting down to x, and then use surjectivity of the upper map to find $a \in V^0_{st}(D)$ with $f(a) = d + \tilde{d}$ for some $\tilde{d} \in Fil^0_a\mathcal{D}$. This is of course not enough, since $\tilde{d}$ could lie in any filtered piece of $Fil_{wa}\mathcal{D}$. However, we are now reduced to finding $\tilde{a} \in V^0_{st}(D)$ such that $f(\tilde{a}) = \tilde{d} + e$ for some $e \in Fil^0_{wa}\mathcal{D}$, since then $f(a - \tilde{a}) = d - e$.

Since any such $\tilde{a}$ would have to lie in

$$f^{-1}(Fil^0_a(\mathcal{D}) + Fil^0_{wa}(\mathcal{D}))$$

we see that in order to establish admissibility of (D, Fil_{wa}), it is sufficient to prove surjectivity of the following map:

$$f^{-1}(Fil^0_a(\mathcal{D}) + Fil^0_{wa}(\mathcal{D})) \xrightarrow{f} (Fil^0_a\mathcal{D} + Fil^0_{wa}\mathcal{D})/Fil^0_{wa}\mathcal{D}$$

The admissibility of $(D, Fil_a\mathcal{D})$ gives us a very good handle on the filtration $Fil_a\mathcal{D}$ purely in terms of the period ring B_{dR}.

We therefore embed $Fil^0_{wa}\mathcal{D}$ and hence $(Fil^0_a\mathcal{D} + Fil^0_{wa}\mathcal{D})$ into a filtered piece $Fil^s_a\mathcal{D}$, for $s \leq 0$. It is then sufficient to prove that the map

$$f^{-1}(Fil^s_a\mathcal{D}) \to Fil^s_a\mathcal{D}/Fil^0_{wa}\mathcal{D}$$

is surjective. By admissibility of (D, Fil_a) we can write the left hand side as:

$$f^{-1}(Fil_a^s\mathcal{D}) = Fil^s(B_{cris}^{\phi=1}) \otimes_{\mathbb{Q}_p} V_{st}(D, Fil_a)$$

where, as always, $Fil^s(B_{cris}) = B_{cris} \cap Fil^s(B_{dR})$.
For $n \geq 0$, write $B_n = B_{dR}^+/Fil^n(B_{dR}^+)$. Then $Fil^s(B_{cris}^{\phi=1})$ sits in a very nice short exact sequence - again a fact we shall only quote (see proposition 1.3. in [**2**]):

Lemma 4.24. *For all $s \leq -1$, there is a short exact sequence*

$$0 \to \mathbb{Q}_p \to Fil^s B_{cris}^{\phi=1} \to B_{-s}(s) \to 0$$

The quotient B_{-s} is the more complicated the more negative s is, and we will therefore choose s maximal.

We now begin an analysis of the right hand side: Choose a $P-$basis $d_1, ..., d_r$ of D compatible with *both* filtrations (checking the existence of such is an easy exercise). Write $t_H^a(d_i) = j_i^a$ and $t_H^{wa}(d_i) = j_i^{wa}$ for the index of the smallest filtered pieces containing these vectors. Then:

$$Fil_a^s\mathcal{D} = \bigoplus_{i=1}^r Fil^{s-j_i^a} B_{dR} \otimes d_j$$

$$Fil_{wa}^0\mathcal{D} = \bigoplus_{i=1}^r Fil^{-j_i^{wa}} B_{dR} \otimes d_j$$

We find that $s = \min\{j_i^a - j_i^{wa}\}$ and

$$Fil_a^s\mathcal{D}/Fil_{wa}^0\mathcal{D} = \bigoplus_{i=1}^r \left(Fil^{s-j_i^a} B_{dR}/Fil^{-j_i^{wa}} B_{dR}\right) \otimes d_j$$

Observe that $(Fil^{k-n}B_{dR})/(Fil^k B_{dR})$ is the free B_n-module $B_n(k-n)$. We therefore have:

$$Fil_a^s\mathcal{D}/Fil_{wa}^0\mathcal{D} = \bigoplus_{i=1}^r B_{j_i^a - j_i^{wa} - s}(s - j_i^a) \otimes d_j$$

Our main goal is to deduce is that *all* weakly admissible filtrations on a finite (ϕ, N)-module D are admissible given the fact that one of them is.
In order to make both sides of the map $f^{-1}(Fil_a^s\mathcal{D}) \to Fil_a^s\mathcal{D}/Fil_{wa}^0\mathcal{D}$ whose surjectivity we want to prove as tractable as possible, we want to keep the occuring $B_n's$ as simple as we can, and we therefore come up with the following aim:

Goal. *Define a notion of "similarity" between filtrations Fil_a, Fil_{wa} on a given finite $(\phi, N)-$module D for which:*

- $s = \min\{j_i^a - j_i^{wa}\} (\leq 0)$ *is as large as possible, this will make the domain simple*
- $\max_i(j_i^a - j_i^{wa} - s) = \max_i(j_i^a - j^{wa}) - \min_i(j_i^a - j^{wa})$ *is as small as possible, this will make the codomain simple*
- *any weakly admissible filtration is finitely many "similarity steps" away from an admissible one.*

Note that

$$\sum_i j_i^{wa} = t_H(Fil_{wa}) = t_N(D) = t_H(Fil_a) = \sum_i j_i^a$$

by weak admissibility.

Therefore, the best possible case $s = 0$ is clearly too restrictive.
The next best situation is $s = \max\{j_i^a - j_i^{wa}\} = -1$ and $\max_i(j_i^a - j_i^{wa} - s) = 2$, again by the equality of Hodge numbers. So let us assume this case.
There are now several configurations of the Hodge numbers which satisfy this claim, but the simplest is certainly:

Definition 4.25. We define two filtrations Fil_1, Fil_2 to be neighbours if there is a basis $d_1, ..., d_r$, compatible with both filtrations, such that

$$t_H^1(d_1) = t_H^2(d_1) + 1$$
$$t_H^1(d_2) + 1 = t_H^2(d_2)$$

and

$$t_H^1(d_i) = t_H^2(d_i)$$

for all $i > 2$.
Given two filtrations as above, we define their *distance* $d = d(Fil_1, Fil_2)$ to be the length of the shortest sequence of filtrations $Fil_1 = F_1, F_2, ..., F_d = Fil_2$ such that each one is a neighbour to its successor.

We then have the following very desirable theorem, whose proof is basic:

Lemma 4.26. *Two filtrations have finite distance if and only if their Hodge numbers agree. In particular, all weakly admissible filtrations on a given $(\phi, N)-$module are a finite distance away from each other.*

Our motivation for this definition was that we expect neighbouring weakly admissible filtrations to be most likely to "contaminate" each other with admissibility, and the last lemma shows that if this were indeed true, then we had finished the proof of Fontine's conjecture using the existence of admissible filtrations.
Words enough have been exchanged, let us at last see some proof:

Theorem 4.27. *Let D be a (ϕ, N)-module over P and assume Fil_{wa}, Fil_a are two neighbouring filtrations such that Fil_{wa} is weakly admissible and Fil_a is admissible.*
Then Fil_{wa} is admissible.

Proof. We choose our basis $d_1, ..., d_r$ such that $j_1^a = j_i^{wa} - 1, j_2^a = j_2^{wa} + 1$, and $j_k^a = j_k^{wa}$ for all other k. We have seen above that it is sufficient to prove the surjectivity of the following map:

$$f^{-1}(Fil_a^{-1}\mathcal{D}) \to Fil_a^{-1}\mathcal{D}/Fil_{wa}^0\mathcal{D}$$

By our previous arguments, we can rewrite the right hand side as a direct sum of a free B_2-module of rank 1 and a $\mathbb{C}-$vector space of dimension $n-2$.

$$M := (B_2(-1 - j_2^a) \otimes d_2) \oplus \bigoplus_{i=3}^{r} (\mathbb{C}(-1 - j_i^a) \otimes d_i)$$

using that $B_1 = \mathbb{C}$. It is part of our assumption that the map

$$f^{-1}(Fil_a^{-1}\mathcal{D}) \to Fil_a^{-1}\mathcal{D}/(Fil_{wa}^0\mathcal{D} + Fil_a^0\mathcal{D})$$

is surjective. In our case,

$$Fil_{wa}^0\mathcal{D} + Fil_a^0\mathcal{D} = \bigoplus_{i=1}^{r}(Fil^{-\max(j_i^a, j_i^{wa})}B_{dR}) \otimes d_j$$

and hence

$$Fil_a^{-1}\mathcal{D}/(Fil_{wa}^0\mathcal{D} + Fil_a^0\mathcal{D}) = (\mathbb{C}(-1 - j_2^a) \otimes d_2) \oplus \bigoplus_{i=3}^{r}(\mathbb{C}(-1 - j_i^a) \otimes d_i)$$

Also the left hand side

$$f^{-1}(Fil_a^{-1}\mathcal{D}) = Fil^{-1}B_{cris}^{\phi=1} \otimes V_{st}(D, Fil_a)$$

can be rewritten: The short exact sequence for $Fil^{-s}B_{cris}$ gives

$$0 \to \mathbb{Q}_p \to Fil^{-1}B_{cris}^{\phi=1} \to \mathbb{C}(-1) \to 0$$

Define U to be the twist $U := Fil^{-1}B_{cris}^{\phi=1}(1)$ (we will give a better description later). We then have a natural map $U \to \mathbb{C}$. Setting $V = V_{st}(D, Fil_a)(-1)$, we obtain:

$$f^{-1}(Fil_a^{-1}\mathcal{D}) = U \otimes V$$

Let us now state the situation we are in more concisely. We have:

- A $\mathbb{Q}_p$−vector space $V = V_{st}(D, Fil_a)(-1)$ with basis $v_1, ..., v_r$ (its dimension is known by admissibility of Fil_a)
- A B_2-module $M = Fil_a^{-1}\mathcal{D}/Fil_{wa}^0\mathcal{D}$ of the form $B_2 \cdot w_2 \oplus \mathbb{C} \cdot w_3 \oplus ... \oplus \mathbb{C} \cdot w_r$
- A B_2-submodule tM such that M/tM is a $\mathbb{C}$−vector space of the form $\mathbb{C} \cdot \tilde{w}_2 \oplus \mathbb{C} \cdot \tilde{w}_3 \oplus ... \oplus \mathbb{C} \cdot \tilde{w}_r$ for which the projection $p : M \to M/tM$ is given by
$$\lambda_2 w_2 + ... + \lambda_r w_r \mapsto \theta(\lambda_2)\tilde{w}_2 + \lambda_3\tilde{w}_3 + ... + \lambda_r\tilde{w}_r$$
- A $\mathbb{Q}_p$-linear map $\xi : V \to M$
- An extension by scalars $\xi_{\mathbb{C}} : \mathbb{C} \otimes_{\mathbb{Q}_p} V \to M/tM$ which is surjective
- A compatible extension by scalars (using the B_2−module structure on M)
$$\xi_U : U \otimes_{\mathbb{Q}_p} V \to M$$
$$u \otimes v \mapsto u\xi(v)$$
with $\dim_{\mathbb{Q}_p}(\ker(\xi_U)) < \infty$ (this will be crucial later).

This setting is now very nice: since the projection p leaves all but one coefficient invariant, we only have to worry about what happens at the first basis vector b_2: Write $K = ker(p \circ \xi_U) \subset U \otimes_{\mathbb{Q}_p} V$ for the kernel of the composed map. In order to prove that $\xi_U : U \otimes_{\mathbb{Q}_p} V \to M$ is surjective, it is clearly sufficient to show that $\xi_K = \xi_U|_K : K \to Mt$ is onto. Define a new function by:

$$g : M \to Fil^1B_2$$

$$\sum t_i\tilde{w}_i \mapsto (t_2 - \theta(t_2))$$

Since any $y = \sum t_i\tilde{w}_i \in tM$ can be written as $y = (t_2 - \theta(t_2))w_2$, it is then enough to show that $g \circ \xi_K : K \to Fil^1B_2$ is surjective.

We can factor $(p \circ \xi_U) : U \otimes_{\mathbb{Q}_p} V \to M/tM$ through $\mathbb{C} \otimes_{\mathbb{Q}_p} V \to M/tM$ as

$$\sum_{i=1}^{r} u_i \otimes v_i \mapsto \sum_{i=1}^{r} \theta(u_i) \otimes v_i \mapsto \sum_{i=1}^{r} \theta(u_i)[\xi(v_i)]$$

But the second map is a linear map from an r to an $(r-1)$-dimensional $\mathbb{C}-$vector space, so has a $1-$dimensional kernel spanned by a vector $\sum_{i=1}^{r} \alpha_i \otimes v_i$, say.

We conclude that:

$$K = \left\{ \sum_{i=1}^{r} u_i \otimes v_i \;\mid\; \exists c \in \mathbb{C} \;\; \forall i : \theta(u_i) = c\alpha_i \right\} = \theta^{-1}(\mathbb{C} \cdot \alpha)$$

Write $\xi(v_i) = \lambda_{i2} w_2 + ... + \lambda_{ir} w_r \in M$ and set $\mu_i = \lambda_{i2} \in B_2$. Then, the map

$$\rho := g \circ \xi_K : K \to Fil^1 B_2$$

sends an element $y = \sum_{i=1}^{r} u_i \otimes v_i \in K$ with $\theta(u_i) = c\alpha_i$ to

$$\rho(y) = g\left(\sum_{i=1}^{r} u_i \xi(v_i)\right) = \sum_{i=1}^{r} \; (\mu_i u_i - \theta(\mu_i u_i))$$

But we also have

$$0 = p \circ \xi_K(y) = \sum_{i,j} \theta(u_i)\theta(\lambda_{ij})\tilde{w}_j$$

which implies that $\sum_{i=1}^{r} \theta(u_i)\theta(\mu_i) = c \cdot \sum_{i=1}^{r} \alpha_i \theta(\mu_i) = 0$. We therefore conclude:

$$\rho(y) = \sum_{i=1}^{r} u_i \mu_i$$

and

$$\sum_{i=1}^{r} \alpha_i \theta(\mu_i) = 0$$

We can of course use the chosen basis of V to obtain an isomorphism

$$\phi : U^r \to U \otimes_{\mathbb{Q}_p} V$$

and we find that

$$Y_\alpha := \phi^{-1}(K) = \{(u_1, ..., u_r) \in U^r \;\mid\; \exists c \in \mathbb{C} \;\; \forall i : \theta(u_i) = c\alpha_i\}$$

We then see that we have reduced the proof of the surjectivity of ρ, and hence of the admissibility of (D, Fil_{wa}), and hence of the admissibility of *all* weakly admissible modules to Colmez' fundamental lemma, which we shall state and prove in the following section. □

REMARK 4.28. The set U can in fact be expressed in a more conceptually pleasing manner:

We start off with the perfection $R(\mathcal{O}_{\mathbb{C}}))$ - note that this is *not* a discrete valuation ring, and we therefore need to be more careful.

We try to define a logarithm valued in B_{dR} by the usual formula

$$log([x]) = \sum_{i=1}^{\infty} \frac{([x]-1)^n}{n}$$

in B_{dR}, where $[x]$ denotes the Teichmüller lift $(x, 0, 0, ...)$.
In order to ensure convergence, we define

$$U_1^{\times} = \{x = (x_0, x_1, ..) \in R(\mathcal{O}_{\mathbb{C}}) \ \mid \ x_0 \in 1 + 2p\mathcal{O}_{\mathbb{C}}\}$$

One can easily check that the above logarithmic series then converges inside A_{cris} for $x \in U_1^{\times}$. We can extend this logarithm even to the larger subset

$$U^{\times} = \{x = (x_0, x_1, .. \in R(\mathcal{O}_{\mathbb{C}})) \ \mid \ v(x_0 - 1) > 0\}$$

by defining

$$\log(x) = \frac{1}{p^r} \log\left(x^{p^r}\right) \in B_{cris}^{+}$$

for any large enough r.

Note that $U_1^{\times} \subset R(\mathcal{O}_{\mathbb{C}})^{\times}$ is complete and separated in the $p-$adic topology, and therefore has a natural $\mathbb{Z}_p$-module structure. One can use this to prove:

$$U^{\times} = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} U_1^{\times}$$

We then have the following nice result:

Lemma 4.29. *The logarithm*

$$\log : U^{\times} \to B_{cris}^{+}$$

gives rise to an isomorphism

$$U^{\times} \cong U = Fil^{-1} B_{cros}^{\phi=1}(1)$$

We write $U_1 = \mathrm{im}(log|_{U_1^{\times}})$ and then have the following statement:

Lemma 4.30. *There is a natural short exact sequence*

$$0 \to \mathbb{Z}_p(1) \to U_1 \to 2p\mathcal{O}_{\mathbb{C}} \to 0$$

6. Colmez' Fundamental Lemma

The theorem which we have reduced all of our problems to through many nontrivial reduction steps is the following:

THEOREM 4.31. *Fix $r \geq 2$, and let*

$$\mu = (\mu_1, ..., \mu_r) \in B_2^r$$

and

$$\alpha = (\alpha_1, ..., \alpha_r) \in \mathbb{C}^r$$

be nonzero vectors with

$$\theta(\mu) \cdot \alpha = 0$$

Let

$$Y = \theta^{-1}(\langle \alpha \rangle) \subset U^r$$

be the preimage of the $\mathbb{C}$-vector space spanned by α. The map

$$U^r \to B_2$$

$$y \mapsto \mu \cdot y$$

restricts to a another map

$$\rho : Y \to Fil^1 B_2 \cong \mathbb{C}(1)$$

If ρ has kernel of finite $\mathbb{Q}_p-$dimension, then ρ is surjective.

PROOF. After rescaling, we may assume that $\alpha_i \in 2p\mathcal{O}_{\mathbb{C}}$.
Step 1) Functional Reformulation
The key functional-analytic insight of this proof is that it is often easier to produce well-behaved functions whose value at a fixed point λ is close to some prescribed value a than it is to produce approximate preimages of a for given functions.

In order to have nicer convergence properties later, we will work with $Y_1 = Y \cap \mathcal{O}_{\mathbb{C}}^r$ for now. The hope is that if ρ is indeed surjective, this should follow by the usual methods of $p-$adic analysis from $\rho(Y_1)$ being large.

We first *reinterpret* the elements $y = (y_1, ..., y_r)$ in the space Y_1 as the *values* that some function s assumes on a *completely fixed* tuple of points $(\lambda_1, ..., \lambda_r)$ in some new other space Z we will need to design. This means that we want $y_i = s(\lambda_i)$.
We will also require that $(s(\lambda_1), ..., s(\lambda_r))$ lies in Y_1 for all functions s - this will turn out to be a nontrivial condition on the point $(\lambda_1, ..., \lambda_r)$.
Given any point $(y_1, ..., y_r) \in Y_1$, we would then like to be able to express the value $\rho(y_1, ..., y_r)$ of ρ in terms of the value of s on some other point in our mysterious new space Z, and indeed once we claim that Z is not just a space, but a B_2-algebra, and that s lies in some collection S of morphisms of B_2-algebras from Z to B_2, we obtain:

$$\rho(y_1, ..., y_r) = \rho\left(s(\lambda_1), ..., s(\lambda_r)\right) = \sum_i \mu_i s(\lambda_i) = \sum_i s\left(\mu_i \lambda_i\right) = s\left(\sum \mu_i \lambda_i\right)$$

Setting $\lambda = \sum \mu_i \lambda_i \in Z$ and

$$\eta(s) = (s(\lambda_1), ..., s(\lambda_r)) \in U_1^r$$

we obtain a function

$$\eta : S \to U_1^r$$

satisfying

$$\rho(\eta(s)) = s(\lambda)$$

Since μ_i and λ_i are assumed to be fixed, this roughly means that we then only have to pick the right s in order to produce good values of ρ.

We can now formulate a vague aim:

Goal. *Construct an B_2-algebra Z and a set S of algebra homomorphisms from Z to B_2 such that:*

- *There is an element $\lambda = \sum_i \mu_i \lambda_i \in Z$ for which*

$$\eta(s) := (s(\lambda_1), ..., s(\lambda_r)) \in Y_1 = Y \cap U_1^r$$

  for all $s \in S$
- *For many points $d \in \mathbb{C}(1)$, there is a function $s \in S$ such that $s(\lambda)$ is very close to c.*

Step 2) Construction of Z
Fontaine's and Colmez' construction is rather involved and goes as follows:

- Let $\mathcal{O}_{\mathcal{K}}$ be the ring of formal power series

$$a = \sum_{i=0}^{\infty} a_i T^i$$

  for which $a_i \in \mathcal{O}_{\mathbb{C}}$ tend p-adically to zero. Define $||a|| = \sup_i(a_i)$.
- Let $\mathcal{K}$ be its field of fractions obtained by inverting p - we obtain a norm by extending $|| - ||$.
- The algebraic closure $\overline{\mathcal{K}}$ then also comes with a norm obtained by mapping an element μ to the supremum of the norms of the coefficients in its minimal polynomial.
- Define $\mathcal{C}$ to be the completion of $\overline{\mathcal{K}}$ with respect to that norm.
- We let $Z = B_2(\mathcal{C})$ and $\mathcal{U}, \mathcal{U}_1 \subset Z$ be the rings constructed out of $\mathcal{C}$ in exactly the same way as B_2 was constructed out of $\mathbb{C}$.

We then have the following lemma:

Lemma 4.32. *Any element f in the set S of continuous homomorphisms in*

$$Hom^{cts}_{\mathcal{O}_{\mathbb{C}}}(\mathcal{O}_{\mathcal{C}}, \mathcal{O}_{\mathbb{C}})$$

extends uniquely to give a morphism

$$Z = B_2(\mathcal{C}) \to B_2$$

sending $\mathcal{U}$ to U and $\mathcal{U}_1$ to U_1. In fact, morphisms in S are determined by their behaviour on $\mathcal{O}_{\overline{\mathcal{K}}}$.

We can pick an element $s_0 \in S = Hom^{cts}_{\mathcal{O}_{\mathbb{C}}}(\mathcal{O}_{\mathcal{C}}, \mathcal{O}_{\mathbb{C}})$ sending the indeterminate T to zero.
Step 3) Approximation lemmata
One then proves the following approximation theorem with the usual methods of $p-$adic analysis:

THEOREM 4.33. *Let $\tau_0 \in \mathcal{O}_{\overline{\mathcal{K}}}$ be an element with $s_0(\tau_0) = 0$ and unit norm. Then there are finitely many "bad points" $c_1, \ \ldots, c_m \in \mathcal{O}_{\mathbb{C}}$ (with $m \leq deg(\tau_0)$) such that any element $c \in \mathcal{O}_{\mathbb{C}}$ which lies outside the "forbidden set"*

$$\bigcup_i (c_i + \mathfrak{m}_{\mathbb{C}})$$

appears as evaluation $s(\tau_0)$ of some function $s \in S$.

PROOF. (Outline) Finding the correct funcion s amounts to producing a morphism

$$s : \mathcal{O}_{\overline{\mathcal{K}}} \to \mathcal{O}_{\mathbb{C}}$$

of $\mathcal{O}_{\mathbb{C}}$-algebras (automatically continuous) such that $s(\tau_0) = c$.

We now use that $\overline{\mathcal{K}}$ is algebraic over $\mathcal{K}$ to pick a minimal polynomial

$$P(T, X) = X^n + a_{n-1}X^{n-1} + \ldots + a_0$$

for τ_0. Write its coefficients as $a_i = \sum b_{i,r}T^r$.

Claim: In order to find the required s for a given c, it is enough to show that $P(r, c) = 0$ for some $r \in \mathcal{O}_{\mathbb{C}}$.
Indeed, the sub $\mathcal{O}_{\mathcal{K}}$-algebra $\mathcal{O}_{\mathcal{K}}[\tau_0]$ generated by τ_0 can be written as

$$\mathcal{O}_{\mathcal{K}}[\tau_0] = \mathcal{O}_{\overline{\mathcal{K}}}[X]/(P)$$

The map $\mathcal{O}_{\overline{\mathcal{K}}}[X] \to \mathcal{O}_{\mathbb{C}}$ sending $f(T)(X)$ to $f(r)(\tau_0)$ then vanishes on (P) and therefore yields a map

$$\mathcal{O}_{\overline{\mathcal{K}}}[\tau_0] \to \mathcal{O}_{\mathbb{C}}$$

with kernel $q \subset \mathcal{O}_{\overline{\mathcal{K}}}[\tau_0]$, say. We can then find a prime $Q \subset \mathcal{O}_{\overline{\mathcal{K}}}$ lying over q, and one immediately concludes that $\mathcal{O}_{\overline{\mathcal{K}}}/Q = \mathcal{O}_{\mathbb{C}}$ (as it is an integral extension of the right hand side).
We can therefore define

$$s_{\tau_0} : \mathcal{O}_{\overline{\mathcal{K}}} \to \mathcal{O}_{\overline{\mathcal{K}}}/Q = \mathcal{O}_{\mathbb{C}}$$

and it is clear that $s_{\tau_0}(\tau_0) = c$

So let us find conditions on $c \in \mathcal{O}_{\mathbb{C}}$ which guarantee that the set

$$S_c = \{r \in \mathcal{O}_{\mathbb{C}} \quad | \quad P(r, c) = 0\}$$

is nonempty.
For each c, we obtain a power series $P(T, c) = \sum_j \alpha_j T^j$ with

$$\alpha_j = \sum_{i=0}^{n-1} b_{i,j}c^i \text{ for } j \geq 1$$

and

$$\alpha_0 = \sum_{i=0}^{n-1} b_{i,0}c^i + c^n$$

Claim: If $|\alpha_{j_0}| = 1$ for some $j_0 \geq 1$, then the polynomial $P(T, c)$ has a root in $\mathcal{O}_{\mathbb{C}}$. This follows from the basic theory of Newton polygons. Indeed, since the point $(j_0, 0)$ has to lie on it and since all other coefficients have nonnegative valuation, we see that at least one slope hast to be nonpositive - this implies the existence of a root in $\mathcal{O}_{\mathbb{C}}$ (compare chapter 6 in [**11**]).

We can now ensure the existence of such an index j_0 as follows: It can shown that the irreducibility of P and $b_{0,0} = 0$ (which follows from $s_0(\tau_0) = 0$) together imply that there is some $j_0 > 0$ and an $i \leq n-1$ such that b_{i,j_0} is a unity - we will not give this argument here.

In order to ensure that

$$|\alpha_{j_0}| = |\sum_{i=0}^{n-1} b_{i,j_0} c^i| = 1$$

it is therefore enough to claim that $|c - c_i| = 1$ for all the distinct roots $c_1, ..., c_m$ of

$$\sum_{i=0}^{n-1} b_{i,j_0} D^i = 0$$

The claim then follows. □

Unsurprisingly, this precise theorem for $\tau_0 \in \mathcal{O}_{\overline{\mathcal{K}}}$ can then be enhanced to an approximation statement for $\tau \in \mathcal{O}_{\mathcal{C}}$ an integer in the completion of $\overline{\mathcal{K}}$. The proof is straightforward, and we will therefore again skip it:

Corollary 4.34. *Let $\tau \in \mathcal{O}_{\mathcal{C}}$ be an element with $s_0(\tau) = 0$ and unit norm, and fix some $\epsilon > 0$.*
Then there are finitely many "bad points" $c_1, ..., c_m \in \mathcal{O}_{\mathbb{C}}$ such that any element $c \in \mathcal{O}_{\mathbb{C}}$ which lies outside the "forbidden set"

$$\bigcup_i (c_i + \mathfrak{m}_{\mathbb{C}})$$

$\epsilon-$approximately appears as evaluation $s(\tau)$ of some function $s \in S$, i.e.

$$|s(\tau) - c| < \epsilon$$

Step 5) Concluding the proof
Without loss of generality, we may assume that $\alpha_i \in 2p\mathcal{O}_{\mathbb{C}}$. We have a surjection

$$\theta : \mathcal{U}_1 \to 2p\mathcal{O}_{\mathcal{C}}$$

and can therefore find elements $\lambda_i \in \mathcal{U}_1$ with

$$\theta(\lambda_i) = \alpha_i T$$

We may furthermore assume without restriction that $s_0(\lambda_i) = 0$. Since each $s \in S$ maps $\mathcal{U}_1$ to U_1 and

$$\theta(s(\lambda_i)) = s(T)\alpha_i$$

for all i, we indeed get a well-defined evaluation map

$$\eta : S \to Y_1$$

Writing $\lambda = \sum_i \mu_i \lambda_i \in \mathcal{B}_2$, we then have

$$\rho(\eta(s)) = s(\lambda)$$

We *cannot* have $\lambda = 0$ as otherwise one could easily deduce that ρ had infinite-dimensional kernel.
Since $\theta(\lambda) = 0$, we have $\lambda \in \mathbb{C}(1)$.

These two facts imply that we can write $\lambda = a\tau t \in B_2$ with t the usual cyclotomic element, $\tau \in \mathcal{O}_{\mathcal{C}}$ and $a \in \mathbb{C}$ with norm 1. We then have:

LEMMA 4.35. *In order to prove that ρ is surjective, it is sufficient to show that the induced map*

$$Y_2 := Y_1 \cap \rho^{-1}(pa\mathcal{O}_{\mathbb{C}}(1)) \to pa\mathcal{O}_{\mathbb{C}(1)}/p^2 a\mathcal{O}_{\mathbb{C}}(1))$$

is surjective

PROOF. This follows immediately from the fact that Y_2 is complete and separated for its $p-$adic topology. □

So let us fix an element $x \in \mathcal{O}_{\mathbb{C}}$ and try to find $y \in Y_2$ with

$$\rho(y) - (pat)x \in p^2 at\mathcal{O}_{\mathbb{C}}$$

But with our previous results, this is now easy: We use 4.34 for $\epsilon = \frac{1}{p^2}$ and τ to give us a forbidden set

$$\bigcup_i (c_i + \mathfrak{m}_{\mathbb{C}})$$

We choose any c_0 outside this set. Then both c_0 and $c_0 + px$ lie outside, and we can therefore choose functions s_{c_0}, s_{c_0+pd} such that:

$$s_{c_0}(\tau) - c_0 \in p^2\mathcal{O}_{\mathbb{C}}$$
$$s_{c_0+px}(\tau) - (c_0 + px) \in p^2\mathcal{O}_{\mathbb{C}}$$

We then set $z = \eta(s_{c_0} - s_{c_0+pd})$ and compute

$$\eta(z) = (s_{c_0} - s_{c_0+pd})(\lambda) = at(s_{c_0} - s_{c_0+pd})(\tau) \equiv (atp)x \mod p^2 at\mathcal{O}_{\mathbb{C}}$$

This finishes the proof of Colmez' fundamental lemma □

Since Colmez' fundamental lemma was the final stage of our long chain of reductions, we have now finally proven:

The Main Theorem

THEOREM 4.36. *(Colmez, Fontaine)*
A filtered $(\phi, N)-$module over a $p-$adic field K is weakly admissible if and only if it is admissible.

□

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