The Lubin-Tate Theory of Spectral Lie Algebras

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Abstract

We use equivariant discrete Morse theory to establish a general technique in poset topology and demonstrate its applicability by computing various equivariant properties of the partition complex and related posets in a uniform manner. Our technique gives new and purely combinatorial proofs of results on algebraic and topological André-Quillen homology. We then carry out a general study of the relation between monadic Koszul duality and unstable power operations. Finally, we combine our techniques to compute the operations which act on the homotopy groups $K(n)$-local Lie algebras over Lubin-Tate space.

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Exposition

Homotopy groups measure how spheres can map into a given space X. They constitute a powerful invariant $\pi_*(X)$ which detects equivalences between CW complexes. However, this invariant is not complete – for example, the spaces S^2 and $S^3 \times \mathbb{C}P^\infty$ have the same homotopy groups but are not homotopy equivalent.

One strategy to address this issue is to reveal more structure on $\pi_*(X)$. Whitehead observed in 1941 that any two maps $f : S^a \to X$, $g : S^b \to X$ from spheres into a space X give rise to a third map $[f, g] : S^{a+b-1} \to X$. This *Whitehead product* gives $\pi_{\geq 2}(X)$ the structure of a graded Lie algebra. On $\pi_1(X)$, it recovers the usual commutator. Even though the Lie algebra $\pi_{\geq 2}(X)$ can distinguish between the spaces S^2 and $S^3 \times \mathbb{C}P^\infty$, it is still far from a complete invariant.

Any space X has a so-called "arithmetic decomposition" into an infinite number of pieces: There is one component $X_\mathbb{Q}$ for the rational numbers and, for each prime p, a component $X_p^\wedge$ corresponding to the finite field $\mathbb{F}_p$. In 1969 [Qui69], Quillen used the Whitehead product to model the rational homotopy type $X_\mathbb{Q}$ of any (simply connected) space X by a differential graded Lie algebra over $\mathbb{Q}$. It is natural to ask whether or not similar "Lie invariants" exist for the remaining components $X_p^\wedge$ – to date, this question remains open.

Reminiscent of a prism separating a natural ray of light into its constituent pure colours, chromatic homotopy theory decomposes each space $X_p^\wedge$ even further. For every natural number h, one can define a space $\Phi_h X_p^\wedge$ called the "v_h-periodic component" of $X_p^\wedge$. Here Φ_h is the Bousfield-Kuhn functor.

$X_\mathbb{Q}$	$X_2^\wedge$	$X_3^\wedge$	$X_5^\wedge$	⋯
	$\Phi_1 X_2^\wedge$	$\Phi_1 X_3^\wedge$	$\Phi_1 X_5^\wedge$	⋯
	$\Phi_2 X_2^\wedge$	$\Phi_2 X_3^\wedge$	$\Phi_2 X_5^\wedge$	⋯
	$\Phi_3 X_2^\wedge$	$\Phi_3 X_3^\wedge$	$\Phi_3 X_5^\wedge$	⋯
	⋮	⋮	⋮	

Figure 1: The various periodic components of a space.

In 2012, Behrens and Rezk [BR15] generalised Quillen's construction and attached meaningful Lie invariants to the various components $\Phi_h X_{\mathbb{F}_p}$ of any space X. Their construction has been studied further in work by Heuts and Arone-Ching – we refer to [BR17] for a comprehensive survey.

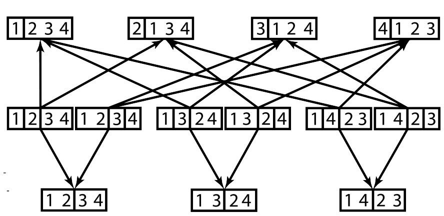


Figure 2: The partition complex Π_4.

The new invariants for $\Phi_h X_{\mathbb{F}_p}$ are Lie algebras in $K(h)$-local spectra and therefore defined using the *partition complex* $|\Pi_n|$. The k-simplices of this simplicial complex correspond to chains $[x_0 < \cdots < x_k]$ of finer and finer (proper nontrivial) partitions of the set $\{1, \ldots, n\}$ – the symmetric group Σ_n acts naturally.

Our work was spurred by the following complaint: unlike Quillen's rational Lie models, $K(h)$-local Lie algebras are computationally intractable since their homotopy groups are modules over the mysterious ring of $K(h)$-local homotopy groups of spheres.

We address this objection by using Lubin-Tate theory (often called Morava E-theory), a homotopically enhanced version of Lubin-Tate space from number theory. This is a coherently commutative ring spectrum E with a maximal ideal $\mathfrak{m}$ and an action by the (big) *Morava stabiliser group* $\mathbb{G}$, which is a Galois twist of $\mathcal{O}_D^\times$ for D the division algebra of Hasse invariant $\frac{1}{n}$ over $\mathbb{Q}_p$.

Writing $\mathrm{Mod}_{E,\mathbb{G}}^{Cpl(\mathfrak{m})}$ for the category of $K(h)$-local E-modules with $\mathbb{G}$-action (see Definition 4.1.1), work by Hopkins–Ravenel implies the existence of a descent equivalence $Sp_{K(h)} \xrightarrow{\simeq} \mathrm{Mod}_{E,\mathbb{G}}^{Cpl(\mathfrak{m})}$ to the aforementioned category of $K(h)$-local spectra (cf. [Mat17]). We therefore propose Lie algebras in $\mathrm{Mod}_{E,\mathbb{G}}^{Cpl(\mathfrak{m})}$ as modular analogues of Quillen's d.g. Lie algebras over the rational numbers $\mathbb{Q}$.

In this work, we compute the operations which act on the homotopy groups of Lie algebras in $\mathrm{Mod}_E^{Cpl(I)}$. [1] There is a Lie bracket, an action of a new ring $\mathcal{H}^{\mathrm{Lie}}$ of "Hecke operations", and a nonadditive operation θ, and these all interact in interesting ways.

In order to prove this theorem, we shall take a scenic route, unravel unexpected connections to other branches of combinatorics and geometry, and discover several substantially different results of independent interest along the way.

In Chapter 1, we give precise statements of our main results.

In Chapter 2, we introduce new combinatorial methods to study the equivariant topology of the partition complex and related spaces. More precisely, we use the equivariant discrete Morse theory of Forman [For98] and Freij [Fre09] to define an algorithm whose input is a finite lattice $\mathcal{P}$ with group action and a list $(F_1, \ldots, F_n)$ of functions and whose output is an algorithm collapsing a usually large subcomplex of $|\mathcal{P} - \{\hat{0}, \hat{1}\}|$ in an equivariant fashion, hence producing an equivalence to an indexed wedge of simpler spaces. This algorithm is of independent interest. The simplest nontrivial instance recovers a strengthened version of results by Björner-Walker [BW83], Kozlov [Koz98], and Welker [Wel90]. Using this machinery, we compute the simple homotopy type of the fixed point spaces $|\Pi_n|^H$ for general $H \subset \Sigma_n$ in terms of subgroup complexes, give a purely combinatorial proof of a strengthened version of Arone's formula [Aro15] for Young restrictions of $|\Pi_n|$, and offer a formula for the parabolic restrictions of Bruhat-Tits buildings. We introduce a general technique for the study of *strict* orbit spaces and apply it to deduce vanishing results for strict Young-quotients of $|\Pi_n|$. We fill these computations with more conceptual meaning by establishing a surprising connection between strict Young quotients of $|\Pi_n|$, commutative monoid spaces, and the algebraic André-Quillen homology of ordinary commutative rings. Using this, we recover a splitting of Goerss [Goe90] concerning the algebraic André-Quillen homology through entirely different combinatorial means. We then

[1] The interaction between our operations and the $\mathbb{G}$-action on the homotopy of Lie algebras in $\mathrm{Mod}_{E,\mathbb{G}}^{Cpl(\mathfrak{m})}$ is determined formally.

link our computations to topological André-Quillen homology – this will later allow us to study operations with many inputs acting on the homotopy of spectral Lie algebras (see Definition 4.1.11).

In Chapter 3, we introduce a general technique to intertwine unstable operations and monadic Koszul duality under certain Koszulness assumptions. More precisely, we study the relation between unary operations on algebras over a monad T and unary operations on coalgebras over the Koszul dual comonad $\mathrm{KD}(T)$. Along the way, we generalise and simplify the (completed) algebraic approximation functors of Barthel-Frankland [BF15] and Rezk [Rez09]. This chapter is complemented by Appendix D on p.133 in which we discuss the relation between Lurie's Koszul duality for monoidal ∞-categories and more classical instances of Koszul duality, namely the Yoneda product on Ext-groups and Ching's comultiplication on the Bar construction via tree grafting.

In Chapter 4, we use our technique from Chapter 3 to construct an additive action of the ring $\mathcal{H}^{\mathrm{Lie}}$ of "Hecke-like" operations (see Definition 4.3.1) on the homotopy groups of every $K(h)$-local Lie algebra over Lubin-Tate space. We then define a further operation θ and prove that it satisfies a congruence $\Psi(x) = [x, x] + 2 \cdot \theta(x)$ for some "Adams-like" operation $\Psi \in \mathcal{H}^{\mathrm{Lie}}$. We then compute the various the relations between these operations. Using the EHP sequence and our work in Chapter 2, we prove that, up to completion, we have produced a comprehensive list of operations and relations.

Our work has informed two subsequent collaborative projects (which are not included in this document):

In joint work with Arone, we set up an EHP sequence for strictly commutative monoid spaces and use it to decompose the strict Young quotients of the partition complex into atomic building blocks $\Sigma|\Pi_n|^{\diamond} \wedge_{\Sigma_n} (S^j)^{\wedge n}$ for j odd. Notably, defining the Hopf map in this sequence seems to require the use of point set models.

Together with Heuts (cf. [BH17]), we study Goodwillie towers on wedges (generalising [AK98]) and cofibres and use this to produce counterexamples which show that v_n-periodic Goodwillie towers are neither finite nor convergent on wedges and Moore spaces (this was unexpected as they *are* finite and convergent for spheres, as proven in [AM99]).

Chapter 1

Statement of Results

The body of our thesis is divided into three chapters: We first use combinatorics to study the partition complex, then establish a general framework for the comparison of topological and algebraic Koszul duality, and eventually compute the operations on the E-theory of spectral Lie algebras (see Definition 4.1.11). We outline the results of Chapters 2, 3, and 4 in the three self-contained sections of this introductory chapter and invite the reader to jump to whichever interests her or him the most. In the four parts of our Appendix, we first review completion, tensored ∞-categories, and graded algebraic theories, and then discuss the relation between ∞-categorical Koszul duality and more classical instances.

1.1 Discrete Morse Theory and the Partition Complex

The equivariant discrete Morse theory of Forman [For98] and Freij [Fre09] gives a systematic way of collapsing complicated simplicial complexes to points by iterating elementary collapses.

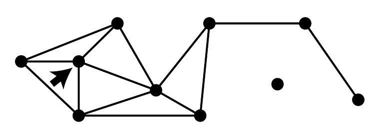


Figure 3: An elementary collapse.

We restrict attention to specific complexes: given a finite poset $\mathcal{P}$ with an order-preserving action of a finite group G, we can attach a G-simplicial complex, called the *order complex*, whose vertex set V is given by the elements of $\mathcal{P}$ and whose face set F consists of chains $[y_0 < \cdots < y_n]$ with $y_i \in \mathcal{P}$.

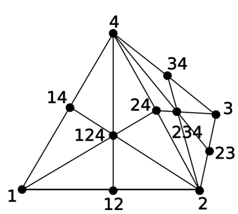


Figure 4: The representation sphere $S^{\sigma_4 - 2}$

Example 1.1.1. The symmetric group Σ_n acts in an evident way on the poset of proper nonempty subsets of $\{1, \ldots, n\}$. The order complex of this poset is the doubly desuspended representation sphere $S^{\sigma_n - 2}$ corresponding to the standard representation σ_n of Σ_n.

A finite G-poset $\mathcal{P}$ is called a *G-lattice* if any two elements have a meet and a join.

Given such a lattice, we define an algorithm that simplifies the order complex of the poset $\overline{\mathcal{P}} := \mathcal{P} - \{\hat{0}, \hat{1}\}$ obtained by removing top and bottom. In order to state our theorem, we introduce the following notion:

Definition 1.1.2. Let $\mathcal{P}$ be a finite G-lattice with face set $\mathcal{F}_{\mathcal{P}}$ containing nondegenerate chains $[y_0 < \cdots < y_n]$. A function $F : \mathcal{F}_{\mathcal{P}} \to \mathcal{P}$ is called an *orthogonality function* if

1. F is G-equivariant and increasing (i.e. $y \leq F(\sigma)$ for all chains $\sigma \in \mathcal{F}_{\mathcal{P}}$ containing a vertex y).
2. For any $\sigma = [y_0 < \cdots < y_m] \in \mathcal{F}_{\mathcal{P}}$ and $z > y_m$, the following subposet is discrete:

$$\{y_m < y < z \mid y \wedge F(\sigma) = y_m, \quad y \vee (F(\sigma) \wedge z) = z\}$$

Lists $\mathbf{F} = (F_1, \ldots, F_n)$ of orthogonality functions are examples of *orthogonality fans* (see Definition 2.2.8 for this more general notion), and there is a notion for when a chain $\sigma = [y_0 < \cdots < y_r]$ is orthogonal to $\mathbf{F}$, written $\sigma \perp \mathbf{F}$. We use discrete Morse theory to prove:

Theorem 2.2.10 (Complementary Collapse). *Let $\mathbf{F} = (F_1, \ldots, F_n)$ be an orthogonality fan on a finite $G-$lattice $\mathcal{P}$ with $F_1([\hat{0}]) \neq \hat{0}, \hat{1}$. Then there is a G-equivariant simple homotopy equivalence*

$$|\overline{\mathcal{P}}| \cong \bigvee_{[y_0 < \cdots < y_r] \perp \mathbf{F}} \Sigma^r |\overline{\mathcal{P}}_{(\hat{0}, y_0)}|^\diamond \wedge |\overline{\mathcal{P}}_{(y_0, y_1)}|^\diamond \wedge \cdots \wedge |\overline{\mathcal{P}}_{(y_{r-1}, y_r)}|^\diamond \wedge |\overline{\mathcal{P}}_{(y_r, \hat{1})}|^\diamond$$

Here $\overline{\mathcal{P}}_{(a,b)}$ denotes the subposet of elements z with $a < z < b$ and $X^\diamond$ stands for the *unreduced* suspension of a space X. An equivariant simple homotopy equivalence is an equivalence which can be obtained by iterated elementary expansions and collapses (see Definition 2.1.7). This is a special case of an equivariant homotopy equivalence. Applying our theorem to the *single* function F with $F(\hat{0}) = x$ for some fixed point x and $F(y) = \hat{1}$ for $y > \hat{0}$, we recover a common generalisation of results by Björner-Walker [BW83], Kozlov [Koz98], and Welker [Wel90].

Complementary collapse is a powerful tool, and we will now illustrate some of its applications. As before, we write Π_n for the partition complex on n elements. Given a subgroup $G \subset \Sigma_n$, it is natural to ask:

Question. *What is the $W_{\Sigma_n}(G) := N_{\Sigma_n}(G)/G$-equivariant simple homotopy type of $|\Pi_n|^G$?*

If G acts transitively on $\{1, \ldots, n\}$, then a result of Klass identifies $|\Pi_n|^G$ with the opposite of the poset of subgroups $\{H \subsetneq K \subsetneq G\}$ for H the stabiliser of $\{1\}$. For general subgroups, the question is more difficult. We introduce some notation. If X is a pointed H-space for $H \subset G$ a subgroup, then we can define the induced G-space as $\mathrm{Ind}_H^G(X) = G_+ \wedge_H X$. A subgroup $G \subset \Sigma_n$ is said to be *isotypical* if all its orbits are isomorphic G-sets. Complementary collapse can then be used to reduce the general case of the above question to the well-understood transitive setting:

Theorem 2.4.2. *If $G \subset \Sigma_n$ acts isotypically on $\{1, \dots, n\}$, we may assume after relabeling that G is a transitive subgroup of $\Sigma_d \xrightarrow{\Delta} \Sigma_d^{\times \frac{n}{d}} \subset \Sigma_n$ for $d \mid n$ and Δ the diagonal map.*
Then there is a $W_{\Sigma_n}(G) = N_{\Sigma_n}(G)/G$ -equivariant simple homotopy equivalence

$$|\Pi_n|^G \quad \xrightarrow{\simeq} \quad \mathrm{Ind}_{W_{\Sigma_d}(G)\times\Sigma_{\frac{n}{d}}}^{W_{\Sigma_n}(G)} \quad (|\Pi_d|^G)^\diamond \wedge |\Pi_{\frac{n}{d}}|^\diamond$$

Lemma 2.4.3. *If G acts non-isotypically, then $|\Pi_n|^G$ is $W_{\Sigma_n}(G)$-equivariantly collapsible.*

Our statements about fixed points can be combined with Proposition 6.2 in [ADL16] to express the fixed point spaces of $|\Pi_n|$ under p-subgroups in terms of Bruhat-Tits buildings. We recall:

Lemma (Arone-Dwyer-Lesh)**.** *If $G \subset \Sigma_n$ is a p-group, then $|\Pi_n|^G$ is $W_{\Sigma_n}(G)$-equivariantly contractible unless G is elementary abelian and acts freely.*

Corollary 2.5.19. *Let $\mathbb{F}_p^k \subset \Sigma_n$ be an elementary abelian p-group acting freely with ℓ orbits. Let $\mathrm{Aff}_{\mathbb{F}_p^k} = N_{\Sigma_{p^k}}(\mathbb{F}_p^k)$ be the affine group and write $\mathrm{Aff}_{\mathbb{F}_p^k \wr \Sigma_\ell} = N_{\Sigma_n}(\mathbb{F}_p^k)$. There is a simple equivalence of $\mathrm{Aff}_{\mathbb{F}_p^k \wr \Sigma_\ell}$-spaces*

$$|\Pi_n|^{\mathbb{F}_p^k} = \mathrm{Ind}_{\mathrm{Aff}_{\mathbb{F}_p^k} \times \Sigma_\ell}^{\mathrm{Aff}_{\mathbb{F}_p^k \wr \Sigma_\ell}} \left(|\,\mathrm{BT}(\mathbb{F}_p^k)|^\diamond \wedge |\Pi_\ell^\diamond| \right)$$

Here $\mathrm{BT}(\mathbb{F}_p^k)$ denotes the poset of proper nonempty subspaces of $\mathbb{F}_p^k$.

We invite the reader to observe Remark 2.4.10 clarifying the relation of Lemma 2.4.3, Theorem 2.4.2, and Corollary 2.5.19 to the work of Arone and Hausmann.

We can combine Corollary 2.5.19 with HKR character theory [HKR00] to compute the rationalised Morava E-theory of homotopy quotients of $|\Pi_n|$ by Young subgroups and hence understand the rationalised E-theory of free spectral Lie algebras on finitely many generators in degree zero (as modules). But one can do better. Complementary collapse gives a new, algorithmic, and purely combinatorial proof of a "simple homotopy" version of an equivalence of Arone [Aro15]:

Theorem 2.3.11. *Let $n = n_1 + \cdots + n_k$ and $g = \gcd(n_1, \dots, n_k)$. Then there is a $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$-equivariant simple homotopy equivalence*

$$|\Pi_n| \quad \longrightarrow \quad \bigvee_{\substack{d \mid g \\ B(\frac{n_1}{d}, \dots, \frac{n_k}{d})}} \mathrm{Ind}_{\Sigma_d}^{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}} (\Sigma^{-1} S^{(\frac{n}{d}-1)\sigma_d} \wedge |\Pi_d|^\diamond)$$

Here $B(m_1, \dots, m_k)$ denotes the set of Lyndon words (see 2.3.1) in k letters involving the i^{th} generator m_i times and S^{σ_d} denotes the standard representation sphere of Σ_d.

We can also give an asymmetric decomposition for Young restrictions of $|\Pi_n|$. We fix a Young subgroup $\Sigma_A \times \Sigma_{B_1} \times \cdots \times \Sigma_{B_k} \subset \Sigma_n$. Complementary collapse implies:

Theorem 2.3.15 (Breaking Symmetry).
There is a simple homotopy equivalence of $\Sigma_A \times \Sigma_{B_1} \times \cdots \times \Sigma_{B_k}$-spaces

$$|\Pi_n| \longrightarrow \bigvee_{\substack{A = A_1 \coprod \cdots \coprod A_r,\ A_i \neq \emptyset \\ f_i : A_i \hookrightarrow B \\ s.t.\ \operatorname{im}(f_{i+1}) \subset \operatorname{im}(f_i)}} \Sigma^{-1} S^{|A_1|} \wedge \cdots \wedge S^{|A_r|} \wedge |\Pi_B|^{\diamond}$$

Partition complexes can be thought of as Bruhat-Tits buildings over "the field with one element". In this heuristic picture, Young subgroups correspond to parabolic subgroups. Complementary collapse also has a nice consequence for Bruhat-Tits buildings. Let V be a finite-dimensional vector space over a finite field k. Fix a flag $\mathbf{A} = [A_0 < \cdots < A_r]$ with associated parabolic $P_{\mathbf{A}}$. Choose a complementary flag $\mathbf{B}$ with parabolic $P_{\mathbf{B}}$ and intersecting Levi $L_{\mathbf{AB}} = P_{\mathbf{A}} \cap P_{\mathbf{B}}$. Complementary collapse implies:

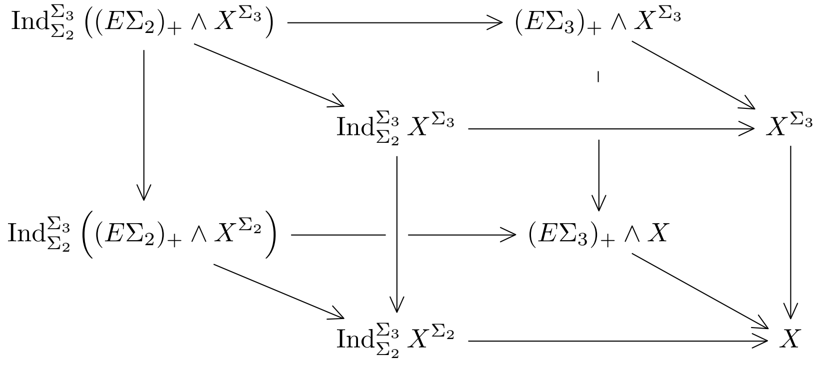
Lemma 2.3.17. *There is a $P_{\mathbf{A}}$-equivariant simple equivalence $|\operatorname{BT}(V)| \cong \operatorname{Ind}_{L_{\mathbf{AB}}}^{P_{\mathbf{A}}} \left(\Sigma^r \bigwedge_{i=0}^{r+1} |\operatorname{BT}(\operatorname{gr}^i(\mathbf{B}))|^{\diamond} \right)$. Here $\operatorname{gr}^i(\mathbf{B})$ denotes the i^{th} graded piece of the flag $\mathbf{B}$.*

Work of Arone raised the following question (cf. [Aro15]): What is the homology of $|\Pi_n|/\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$?

We first study orbit spaces in general and introduce a new colimit decomposition for G-spaces and their approximations for G any finite group. Fix a subposet $\mathcal{C} \subset \operatorname{ccl}_G$ of the poset of conjugacy classes of G and write $\mathcal{D}$ for the opposite of the category of nondegenerate simplices of $\mathcal{C}$. Given a chain $(H_0 \subset \cdots \subset H_m)$ of subgroups, we set $N_G(\mathbf{H}) := \cap_i N_G(H_i)$ and $W_G(\mathbf{H}) := N_G(\mathbf{H})/H_0$.

Lemma 2.5.14. *The map* $\operatorname*{colim}_{[\mathbf{K}_0 \subset \cdots \subset \mathbf{K}_m] \in \mathcal{D}} \left(\bigvee_{\substack{H_0 \in \mathbf{K}_0 \\ \cdots \\ H_m \in \mathbf{K}_m \\ H_0 \subset \cdots \subset H_m}} EW_G(H_0, \ldots, H_m)_+ \wedge X^{H_m} \right) \to X$ *is a $\mathcal{C}$-approximation (in the sense of [AD01]).*

Example 1.1.3. For any Σ_3-space X and $\mathcal{C} = \operatorname{ccl}_{\Sigma_3}$, we write $\Sigma_2 = \Sigma_{12,3}$ and obtain a homotopy colimit



One can use this to deduce the following helpful result for maps between strict orbits:

Lemma 2.5.16. *Assume that $f : X \to Y$ is a map of G-spaces such that for all chains of p-subgroups $H_0 \subset \cdots \subset H_n \subset G$, the following map induces an isomorphism on $H_*(-, \mathbb{Z}_{(p)})$:*

$$(EW_G(H_0, \ldots, H_m)_+ \wedge X^{H_m}))_{/N_G(H_0,\ldots,H_m)} \longrightarrow (EW_G(H_0, \ldots, H_m)_+ \wedge Y^{H_m}))_{/N_G(H_0,\ldots,H_m)}$$

Then $f_{/G} : X_{/G} \to Y_{/G}$ induces an isomorphism on $H_*(-, \mathbb{Z}_{(p)})$.

We go back to the homology of $|\Pi_n|/\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$. By the decomposition result for Young-restrictions, it is enough to compute the homology of spaces $S|\Pi_n|^\diamond \wedge_{\Sigma_n} (S^j)^{\wedge n}$.

Our Lemma 2.5.16 for strict orbits can be combined with computations of Arone [Aro06] (cf. [ADL16]) for *homotopy* orbits to see that for p a prime, j even and $n \neq p^a, 2p^a$ for all a, *or* j odd and $n \neq p^a$ for all a, the homology group $H_*(|\Pi_n|^\diamond \wedge_{\Sigma_n} (S^j)^{\wedge n}, \mathbb{F}_p)$ vanishes.

A new conceptual insight is required to cover the remaining cases. We consider the pointed ∞-category $\mathbf{CMon}^{aug}$ of strictly commutative monoid spaces augmented over S^0, the monoid with two elements 0 and 1. Any space X gives rise to an augmented commutative monoid space $S^0 \vee \overline{X}$ by declaring that $a \cdot b = 0$ unless $a = 1$ or $b = 1$. This is the *trivial square zero extension* of S^0 by X. There is a natural notion of *André-Quillen homology*, denoted by AQ, for these commutative monoid spaces. The André-Quillen homology of trivial square zero extensions is intimately related to strict orbits of the partition complex:

Lemma 2.6.12. *If X is a well-pointed space, then $AQ(S^0 \vee \overline{X}) \cong \bigvee_{n\geq 1} \Sigma|\Pi_n|^\diamond \wedge_{\Sigma_n} X^{\wedge n}$.*

Given any ring R, we can apply the reduced R-valued chains functor $\tilde{C}_\bullet(-, R)$ to a commutative monoid space X and obtain a *simplicial commutative R-algebra*. Heuristically speaking, we base-change from $\mathbb{F}_1$ to R. This operation sends the trivial square-zero extension $S^0 \vee \overline{X}$ to the trivial square zero extension $R \oplus \tilde{C}_\bullet(X, R)$. Moreover, it intertwines AQ for strictly commutative monoid spaces with the usual AQ for simplicial commutative rings:

Lemma 2.6.10. *For any augmented monoid space X, we have an equivalence $\tilde{C}_\bullet(AQ(X), R) \cong AQ^R(\tilde{C}_\bullet(X, R))$.*

This shows that the R-valued homology of the space $\bigvee_{n\geq 1} \Sigma|\Pi_n|^\diamond \wedge_{\Sigma_n} X^{\wedge n}$ computes the algebraic André-Quillen homology of trivial square-zero extensions.

Our computation of Young restrictions of $|\Pi_n|$ via complementary collapse therefore gives a new proof of the Hilton-Milnor splitting (cf. Corollary 5.4.16) for the algebraic André-Quillen homology (due to Goerss [Goe90] over $\mathbb{F}_2$). We find it remarkable that our purely combinatorial technique has this nontrivial consequence in derived algebraic geometry.

1.2 Background on Operations

We will give a brief survey of the study of operations on highly structured objects and introduce necessary context and notation along the way. We encourage the impatient reader who is interested in our work concerning operations on Lie algebras to jump directly to the Section 1.2.3 and then proceed to Section 1.4. If the reader instead prefers to learn about our results relating algebraic and topological Koszul duality, we recommend jumping to the Section 1.2.2 and then proceed to Section 1.3.

1.2.1 Operations and Geometry

We begin with an elementary problem:

Question. *Are there 2 linearly independent vector fields on S^5?*

If this happens, then the map $\mathbb{R}P^5/\mathbb{R}P^2 \xrightarrow{p} S^5$ projecting off to the top cell must admit a section s. In order to find obstructions to this being possible, we can apply (reduced) singular cohomology $H^*(-, \mathbb{F}_2)$. If we think of this functor as landing in rings, then we cannot spot an obstruction. However, the cohomology of any space has more structure: it forms an unstable algebra over the *Steenrod algebra*. The top class $x_5 \in \tilde{H}^5(\mathbb{R}P^5/\mathbb{R}P^2, \mathbb{F}_2)$ can be expressed by applying the operation Sq^2 to the class $x_3 \in \tilde{H}^3(\mathbb{R}P^5/\mathbb{R}P^2, \mathbb{F}_2)$. This implies $p_*(x_5) = Sq^2 p_*(x_3) = 0$ and hence a section cannot exist. This technique can be pushed further to prove that if 2^m is the maximal power of 2 diving $n+1$, then there do not exist 2^m vector fields on S^n. This bound is *not* optimal.

In order to obtain the best possible bound (which is closely related to the Radon-Hurwitz number), a more sophisticated approach is necessary. In [Ada62], Adams made use of topological K-theory, a *generalised cohomology theory* which measures vector bundles on a space. Just like $\mathbb{F}_2$-cohomology, the K-theory of any space is not just a ring, but comes equipped with additional structure known as *Adams operations*.

Topological K-theory, once completed at a prime p, has a higher height analogue: Lubin-Tate theory. Operations on the E-cohomology of spaces have been studied by Ando, Rezk, and others, and will be crucial to our work. They also appear in Ando–Hopkins–Rezk's [AHR10] refinement of the Witten genus to a map of highly structured ring spectra in $MString \to tmf$.

1.2.2 Operations on (co)Algebras over (co)Monads

All of the above examples of operations fit into a general framework. Given a space X and an $\mathbb{E}_\infty$-ring spectrum R (e.g. $H\mathbb{F}_2$, K, or E) which is complete[1] with respect to some ideal $I \subset \pi_0(R)$ in the sense of Definition 5.1.2 in Appendix A, the mapping spectrum R^X naturally lies in the category $\mathrm{Alg}_{\mathrm{Comm}}(\mathrm{Mod}_R^{Cpl(I)})$ of commutative algebra objects in I-complete R-module spectra, i.e. I-complete $\mathbb{E}_\infty$-rings under R. We are therefore lead to study the operations which act naturally on the homotopy groups of all objects $A \in \mathrm{Alg}_{\mathrm{Comm}}(\mathrm{Mod}_R^{Cpl(I)})$.

Defining a (nonunital) commutative algebra object in $\mathrm{Mod}_R^{Cpl(I)}$ is equivalent to specifying an algebra over the monad $T(X) = \bigoplus_{n\geq 1} X^{\otimes n}_{h\Sigma_n}$ corresponding to the $\mathbb{E}_\infty$-operad (see Definition 4.1.10).

There are other monads which are of interest in topology: the symmetric power monad on connective module spectra over a field k building the free simplicial commutative k-algebra (or its extension to the non-connective setting), the monad corresponding to the $\mathbb{E}_n$-operad, and, most importantly for us, the monad corresponding to the spectral Lie operad (see Definition 4.1.9). In full generality, we can ask:

Question. *Given a monad T on the category $\mathrm{Mod}_R^{Cpl(I)}$, what are the operations which act naturally on the homotopy groups of all objects $A \in \mathrm{Alg}_T(\mathrm{Mod}_R^{Cpl(I)})$?*

(A dual question can be formulated about the cohomotopy of coalgebras over comonads.)

Any class $\alpha \in \pi_j(\mathrm{Free}_{\mathrm{Alg}_T}(\Sigma^{i_1}R + \cdots + \Sigma^{i_k}R))$ gives rise to a k-ary operation on the homotopy groups of T-algebras which takes k classes in degrees $i_1, \ldots, i_k$ and produces a single class in degree j as output.

Organising all the operations defined in this way in a coherent and tractable manner is a difficult task, and we will now give an incomplete list of the cases for which it has been carried out. The usual strategy is to give the homotopy groups three simultaneous pieces of structure: an algebra structure over some classical algebraic monad (e.g. the structure of an ordinary commutative ring or an ordinary Lie algebra), an additive action of some ring of unary operations, and a (usually very small) set of non-additive extra operations. These three structures then satisfy nontrivial relations which can be worked out by computing universal examples.

Example 1.2.1. The homotopy groups of $\mathbb{E}_\infty$-rings in the ∞-category $\mathrm{Mod}_{H\mathbb{F}_p}$ of $\mathbb{F}_p$-module spectra naturally form a (graded) commutative $\mathbb{F}_p$-algebra together with an unstable action of the Dyer-Lashof algebra, subject to certain axioms (cf. [BMMS86], Section III.1). The action on $\pi_* H\mathbb{F}_p^X$ for X a space recover the usual Steenrod operations.

Example 1.2.2. The homotopy groups of $\mathbb{E}_n$-rings in the ∞-category $\mathrm{Mod}_{H\mathbb{F}_p}$ support the structure of a

[1] Working in a completed setting is necessary in the case of Morava E-theory. Here completion with respect to the maximal ideal $I \subset \pi_0(E_h) = W(\overline{\mathbb{F}_p})[[u_1, \ldots, u_{h-1}]]$ is also known as $K(h)$-localisation.

restricted Poisson n-algebra which is acted on by the Dyer-Lashof algebra, again subject to certain rather involved relations. This has been worked out by Steinberger in [Ste86], relying heavily on Fred Cohen's computation in the case of $\mathbb{E}_n$-spaces (see [CLM76]).

Example 1.2.3. The homotopy groups of shifted spectral Lie algebras in connective modules in $\mathrm{Mod}_{H\mathbb{F}_2}$ have the structure of a shifted Lie algebra with an allowable action of the Dyer-Lashof-like algebra $\overline{R}$ introduced by Behrens [Beh12]. This was worked out by Antolín-Camarena in [AC15]. The relation $[Qx, y] = 0$ was first discovered by the author of this thesis and follows from a general transfer argument. [AC15] contains a different proof that does not use the transfer. The Jacobi identity was proven in joint work with Antolín-Camarena. A partial generalisation of the work of Behrens and Antolín-Camarena to odd primes has been obtained by Kjaer [Kja16], who computes the $\mathbb{F}_p$-homology of connective free spectral Lie algebras as $\mathbb{F}_p$-vector spaces (the composition structure of Dyer-Lashof-like operations remains open). In Section 4.1.3, we will indicate how some of the computational problems which are left open in [Kja16] and [AC15] can be solved.

1.2.3 On Rezk's work

We briefly review the operations on the homotopy of $K(n)$-local $\mathbb{E}_\infty$-rings under Morava E-theory. At height 1, these operations were initially studied by Adams in order to solve the vector fields problem, and computed in a $p-$complete setting by Bousfield [Bou96] and McClure [BMMS86]:

Example 1.2.4. Let K be p-adic K-theory. The homotopy groups of every $K(1)$-local $\mathbb{E}_\infty$-ring under K give rise to a $\mathbb{Z}/2\mathbb{Z}$*-graded* θ*-ring*, i.e. a strictly commutative $\mathbb{Z}/2\mathbb{Z}$-graded ring R together with a map of sets $\theta : R_0 \to R_0$ and group homomorphisms $\Psi_0 : R_0 \to R_0$, $\Psi_1 : R_1 \to R_1$, satisfying the following assertions:

- (R_0, θ_0) defines a θ-ring and $\Psi_0(x) = x^p + p\theta(x)$ for all $x \in R_0$.
- $\Psi_1(xy) = \Psi_0(x)\Psi_1(y)$ for all $x \in R_0$, $y \in R_1$.
- $\theta(xy) = \Psi_1(x)\Psi_1(y)$ for all $x, y \in R_1$.

This has been generalised by Rezk in [Rez09] to the setting of $K(h)$-local $\mathbb{E}_\infty$-rings under the Morava E-theory of some general height h. Using additive operations on degree 0, Rezk defines a weight-graded associative ring $\Gamma = \bigoplus_{k\geq 1}\Gamma[p^k]$ which contains E_0 as a non-central commutative subring and shows that the homotopy groups of the $\mathbb{E}_\infty$-rings in question have the structure of $\mathbb{Z}/2\mathbb{Z}$-graded Γ-algebras (with respect to a twisted tensor structure). There is also an additional non-additive operation θ and an additive Adams-like operation Ψ acting on degree 0 such that $\Psi(x) = x^p + p\theta(x)$ and such that all identities which would be forced if p were invertible hold true.

The ring Γ is a subject of active research - several important questions remain open, but there are several known structural results which we would like to recall. By definition, the E_0-linear dual $\Gamma[p^k]^\vee$ of the $(p^k)^{th}$ graded piece of Γ is given by the quotient of the cohomology of the symmetric groups $B\Sigma_{p^k}$ by the transfer ideal. Seminal work by Strickland [Str98] shows that these rings are free as E_0-modules, Gorenstein local, and that the associated formal scheme $\mathrm{Spf}(\Gamma[p^k]^\vee)$ gives the moduli of subgroup schemes of degree p^k inside the universal formal group $\mathbb{G}$ over Lubin-Tate space.

Warning. *The commutative cup product structure on the cohomology group $\Gamma[p^k]^\vee$ is (of course) not equal to the noncommutative product structure on Γ encoding the composition of operations.*

The ring Γ is free as an E_0-module. Its structure is far from being understood in terms of generators and relations at general heights, but work by Rezk [Rez08] and Zhu [Zhu14] give a complete description at height 2 and $p = 2$ or $p = 3$.

One of the major breakthroughs in the study of Γ, crucial to our later computations, was Rezk's topological proof that the graded ring Γ is *Koszul* (see [Rez12b], inspired by [AM99]). In [Rez12a], Rezk gives an identification of the Bar complex of Γ with the so-called *modular isogeny complex*. This complex captures flags of subgroups of the universal formal group $\mathbb{G}$ over Lubin-Tate space.

1.3 Intertwining Algebraic and Topological Koszul Duality

Our method to relate algebraic and topological Koszul duality is both abstract and technical. We will therefore limit ourselves to a brief overview here and invite the interested reader to jump straight to Chapter 3 for a detailed treatment.

Let R be an $\mathbb{E}_2$-ring for which $\pi_*(R)$ is Noetherian and fix an ideal $I \subset R_0$. We obtain an ∞-category $\mathrm{Mod}_R^{Cpl(I)}$ of I-complete left R-module spectra (see Definition 5.1.2 in Appendix A).

If $T \in \mathrm{Alg}^{aug}(\mathrm{End}(\mathrm{Mod}_R^{Cpl(I)}))$ is an augmented monad, Lurie's Koszul duality [Lur11b], a far-reaching generalisation of classical Moore duality, produces a Koszul dual comonad $\mathrm{KD}(T) = |\,\mathrm{Bar}_\bullet(1, T, 1)|$.

It is then natural to ask:

Question. *What is the relation between unary homotopy operations on T-algebras and unary cohomotopy operations on* $\mathrm{KD}(T)$*-coalgebras?*

We introduce a class of convenient monads (see Definition 3.2.19) for which operations on T-algebras are controlled by an augmented monad $\hat{\mathbb{T}}$ on the derived category $\mathcal{D}^-_{\geq 0}(\mathrm{Mod}_{R_*}^{Cpl(I)})$. We think of $\hat{\mathbb{T}}$ as an object living in the realm of algebra. In the case where T is the commutative monad axiomatising $K(h)$-local $\mathbb{E}_\infty$-rings under E-theory, our construction of the monad $\hat{\mathbb{T}}$ gives a substantially simplified treatment of the completed approximation functors of Barthel-Frankland [BF15] (see page 89). The augmented monad $\hat{\mathbb{T}}$ also has a Koszul dual comonad $\mathrm{KD}(\hat{\mathbb{T}})$, and the study of operations on $\mathrm{KD}(\hat{\mathbb{T}})$-coalgebras in terms of operations on $\hat{\mathbb{T}}$-algebras is located in the context of nonadditive derived functors in classical algebra as studied by Dold-Puppe [DP61], and many others.

In Definition 3.3.3, we single out the objects $M \in \mathrm{Mod}_R^{Cpl(I)}$ for which maps (of $\mathrm{KD}(\hat{\mathbb{T}})$-coalgebras) out of the free $\mathrm{KD}(\hat{\mathbb{T}})$-coalgebra on $\pi_*(M)$ in algebra lift naturally to corresponding maps (of $\mathrm{KD}(T)$-coalgebras) out of the free $\mathrm{KD}(T)$-coalgebra on M in topology. For this lifting, we use of the P_σ-construction from Section 4.2 in [Lur11a] to construct a hybrid between $\mathrm{Mod}_E^{Cpl(I)}$ and $\mathrm{Mod}_{E_*}^{Cpl(I)}$, which can be thought of as a modern version of the E_2-model structure. In Theorem 3.5.1, we see that the "Yoneda-composition" of the operations in algebra is compatible with composition of the lifted operations in topology.

We implement our technique for T the commutative monad on $\mathrm{Mod}_E^{Cpl(I)}$ in Chapter 4, but consider it likely that our method yields interesting results for other possible combinations of monad T and ring spectrum R (e.g. the ones discussed in Section 1.2) We remark that we also rely on Appendix D on p.133, which clarifies the relationship between Lurie's Koszul duality in monoidal ∞-categories, the Yoneda product on Ext-groups, and Ching's tree grafting.

1.4 Operations on Lie Algebras in $\mathrm{Mod}_E^{Cpl(I)}$

We fix a height h, a prime p, and write E for the corresponding Lubin-Tate spectrum with unique maximal ideal $I \subset \pi_0(E) \cong W(\overline{\mathbb{F}_p})[[u_1, \ldots, u_{h-1}]]$. We can ask:

Question. *What are the operations which act naturally on the homotopy groups of a $K(h)$-local Lie algebra in E-module spectra?*

Lie algebras in the ∞-category $\mathrm{Mod}_E^{Cpl(I)}$ of $K(h)$-local E-module spectra are defined as algebras over the monad $L(X) = \bigoplus_{n\geq 1} \mathbb{D}(\Sigma|\Pi_n|^\diamond) \otimes_{h\Sigma n} X^{\otimes n}$ acting on $\mathrm{Mod}_E^{Cpl(I)}$. The monad structure is defined as in [Chi05] (see Definition 4.1.11).

Just like in the case of $\mathbb{E}_\infty$-rings in $\mathrm{Mod}_E^{Cpl(I)}$ covered by Rezk, an explicit generators-and-relations description of the operations is out of reach. We will therefore settle for the next best result, namely an algorithmic description of the operations on Lie algebras *in terms of operations on $\mathbb{E}_\infty$-rings.*

For each i, Rezk introduces a weight-graded ring $\Delta^i = \bigoplus_w \Delta^i[w]$ of operations which act additively on the $(-i)^{th}$ degree of the cotangent space $\pi_*(A)/(\pi_*(A))^2$ of any nonunital $K(h)$-local $\mathbb{E}_\infty$-ring A under E. These rings are all (non-canonically) isomorphic to the ring Γ from above. Suspension yields a sequence of homomorphisms $\cdots \to \Delta^2 \to \Delta^1 \to \Delta^0 \to \Delta^{-1} \to \ldots.$ and there are canonical twisting morphisms $E_k \otimes_{E_0} \Delta^{-i} \otimes_{E_0} E_{-k} \longrightarrow \Delta^{-i-k}$.

One of the main facts which make the study of operations on the E-theory of $K(h)$-local $\mathbb{E}_\infty$-rings pleasant is that, up to scaling by a unit in E_*, additive operations preserve degree. This makes it possible to either restrict attention to the degree 0 part altogether or use a $\mathbb{Z}/2\mathbb{Z}$-graded framework like Rezk. The situation for Lie algebras is less convenient: up to scaling, additive operations of weight p^k shift degree down by k. We therefore phrase our results in terms of graded E_*-modules and abstain from a $\mathbb{Z}/2\mathbb{Z}$-graded approach.

1.4.1 Additive Unary Operations

We introduce the notion of a *power ring* P in Definition 3.1.7: For each $i, j \in \mathbb{Z}$ and $v \in \mathbb{N}$, there is an abelian group $P_i^j[v]$ which will record additive operations from degree i to degree j of weight v. There are composition maps $P_i^j[v] \otimes P_j^k[w] \to P_i^k[vw]$ which satisfy natural associativity conditions. A module over a power ring is a graded abelian group M_i with associative action maps $P_i^j[v] \otimes M_i \to M_j$.

We introduce the power ring which acts on the homotopy of Lie algebras in $\mathrm{Mod}_E^{Cpl(I)}$:

Definition 4.3.1. *The power ring $\mathcal{H}^{\mathrm{Lie}}$ of Hecke operations on Lie algebras is given by*

$$(\mathcal{H}^{\mathrm{Lie}})_i^j[w] = \begin{cases} \mathrm{Ext}^a_{\Delta^i}(\overline{E}_0, \overline{E}_{-i+j+a}) & \text{if } w = p^a \\ 0 & \text{if } w \text{ is not a power of } p \end{cases}$$

The multiplication map $(H^{Lie})_i^j[p^a] \otimes (\mathcal{H}^{\mathrm{Lie}})_j^k[p^b] \to (\mathcal{H}^{\mathrm{Lie}})_i^k[p^{a+b}]$ is the composite:

$$(\mathrm{Ext}^a_{\Delta^i}(\overline{E}_0, \overline{E}_{-i+j+a})) \otimes (\mathrm{Ext}^b_{\Delta^j}(\overline{E}_0, \overline{E}_{-j+k+b})) \to (\mathrm{Ext}^a_{\Delta^i}(\overline{E}_0, \overline{E}_{-i+j+a})) \otimes (\mathrm{Ext}^b_{\Delta^{j+a}}(\overline{E}_0, \overline{E}_{-j+k+b}))$$

$$\to (\mathrm{Ext}^a_{\Delta^i}(\overline{E}_0, \overline{E}_{-i+j+a})) \otimes (\mathrm{Ext}^b_{\Delta^i}(\overline{E}_{-i+j+a}, \overline{E}_{-i+k+a+b})) \to (\mathrm{Ext}^{a+b}_{\Delta^i}(\overline{E}_0, \overline{E}_{-i+k+a+b}))$$

The first map uses the suspension $\Delta^{j+a} \to \Delta^j$, the second map uses the morphism $\mathrm{Ext}^*_{\Delta^\ell}(\overline{E}_0, \overline{E}_r) \to \mathrm{Ext}^*_{\Delta^{\ell-s}}(\overline{E}_s, \overline{E}_{r+s})$ coming from the twisting morphism, and the final map uses the Yoneda product.

Remark 1.4.1. We chose to call these operations "Hecke operations" since the ring Δ is related to the Hecke algebra for $GL_n(\mathbb{Z}_p)$ (see section 14 of [Rez06]).

In Chapter 3, we construct an additive action of the power ring $\mathcal{H}^{\mathrm{Lie}}$ on the homotopy of every $K(h)$-local Lie algebra under E. For this, we make use of Rezk's Koszul property of Γ ([Rez12b], cf. [BR15]) to produce maps $(\mathcal{H}^{\mathrm{Lie}})_i^j[w] \to \pi_j(\mathbb{D}(\Sigma|\Pi_n|^\diamond) \otimes_{h\Sigma_n} (\Sigma^i E)^{\otimes n})$ for i odd and extend our construction to i even by using the EHP sequence to conclude Theorem 4.2.19. To check that the composition of operations agrees with the multiplication in $\mathcal{H}^{\mathrm{Lie}}$, we implement the technology developed in Chapter 3 to "compose operations across spectral sequences" and thereby establish Theorem 4.3.2.

Generalising an argument of Rezk from elliptic curves to general heights in an evident way, one can compute the dimensions of the free E_0-modules which constitute the power ring $\mathcal{H}^{\mathrm{Lie}}$ in terms of the Hasse-Weil Zeta function counting points on the projective space $\mathbb{P}^{n-1}$:

$$\sum_{a=0}^{\infty} \mathrm{rk}(\mathcal{H}^{\mathrm{Lie}})_i^{i-a}[p^a] \cdot T^a = \frac{1}{\zeta(\mathbb{P}^{n-1}_{\mathbb{F}_p}, -T)} = (1+T)\cdot(1+pT)\cdot \cdots \cdot (1+p^{n-1}T)$$

As mentioned above, a generators-and-relations description of $\mathcal{H}^{\mathrm{Lie}}$ at general heights is currently out of reach. However, work by Priddy (see Theorem 2.5 of [Pri70]) gives a mechanical way to determine the structure of $\mathcal{H}^{\mathrm{Lie}}$ once a generators-and-relations description of the ring Δ has been found.

1.4.2 Nonadditive Unary Operations

After picking an orientation for E, we construct a nonadditive operation θ_{2n} starting in degree $2n$ and landing in degree $4n-1$ for every integer n (see Section 4.2.2). Along the way, we also specify an "Adams-like" additive operation $\Psi_{2n} \in (\mathcal{H}^{\mathrm{Lie}})_{2n}^{4n-1}[2]$ such that $\Psi_{2n}(x) = [x,x] + 2\cdot\theta_{2n}(x)$. Relying on results by Strickland [Str98] and Behrens-Rezk [BR15], we prove in Theorem 4.2.16 that free Lie algebras in $\mathrm{Mod}_E^{Cpl(I)}$ have completed-free homotopy and that all components of $\mathcal{H}^{\mathrm{Lie}}$ are free E_0-modules. We deduce several *divisibility properties* of $\mathcal{H}^{\mathrm{Lie}}$ which will be necessary in the formulation of our main result:

Proposition 4.3.18. *If $m, n \in \mathbb{Z}$ and $\lambda \in E_{2m} \cong (\mathcal{H}^{\mathrm{Lie}})_{2(n-m)}^{2n}[1]$ is any scalar, there is a unique element $\delta^{\lambda}_{2(n-m)} \in (\mathcal{H}^{\mathrm{Lie}})_{2(n-m)}^{4n-1}[2]$ satisfying $\Psi_{2n}\cdot\lambda - \lambda^2\cdot\Psi_{2(n-m)} = 2\cdot\delta^{\lambda}_{2(n-m)}$.*

Proposition 4.3.19. *If $m, n \in \mathbb{Z}$ and $\alpha \in (\mathcal{H}^{\mathrm{Lie}})_{m}^{2n}[p^k]$ is any operation with $k > 0$, there is a unique element $\epsilon^{\alpha}_{m} \in (\mathcal{H}^{\mathrm{Lie}})_{m}^{4n-1}[2p^a]$ satisfying $\Psi_{2n}\cdot\alpha = 2\cdot\epsilon^{\alpha}_{m}$*

1.4.3 Hecke Lie Algebras

After having defined all unary operations, we turn to operations with more than one input. We show that the homotopy groups of any Lie algebra in $\mathrm{Mod}_E^{Cpl(I)}$ form a shifted Lie algebra in E_* in the sense of Definition 4.4.1 . In Section 4.3, we then establish various relations between Lie bracket, θ-operations, and the Hecke operations. We axiomatise the resulting structure (for p an odd prime, it simplifies *substantially*):

Definition 4.4.2. *A Hecke Lie Algebra consists of a $\mathcal{H}^{\mathrm{Lie}}$-module M together with the structure of a shifted Lie algebra on the underlying E_*-module M_* and maps of sets $\theta_{2n} : M_{2n} \to M_{4n-1}$ such that:*

1. *$[x, \alpha(y)] = 0$ for all $\alpha \in (\mathcal{H}^{\mathrm{Lie}})_i^j[w]$ with $w > 1$ and all $x \in M_k$, $y \in M_i$.*
2. *$\Psi_{2n}(x) = [x,x] + 2\cdot\theta_{2n}(x)$ for any $x \in M_{2n}$.*

Additionally, we impose several additional identities which would all be forced in the torsion-free case:

3. *$[x, \theta_{2n}(y)] = [[x,y],y]$ for all $x \in M_m, y \in M_{2n}$.*
4. *$\theta_{2n}(x+y) = \theta_{2n}(x) + \theta_{2n}(y) - [x,y]$ for all $x, y \in M_{2n}$.*
5. *$\theta_{2n}(\lambda\cdot x) = \lambda^2\cdot\theta_{2(n-m)}(x) + \delta^{\lambda}_{2(n-m)}(x)$ for all $\lambda \in E_{2m}$, $x \in M_{2(n-m)}$.*
6. *$\theta_{2n}(\alpha(x)) = \epsilon^{\alpha}_{m}(x)$ for all $\alpha \in (\mathcal{H}^{\mathrm{Lie}})_{m}^{2n}[w]$ with $w > 1$, $x \in M_m$.*

There is an evident notion of "morphism of Hecke Lie algebras". We write $\mathrm{Lie}^{\mathcal{H}}$ for the resulting category of Hecke Lie algebras, and let $\mathrm{Free}_{\mathrm{Lie}^{\mathcal{H}}} : \mathrm{Mod}_{E_*} \to \mathrm{Lie}^{\mathcal{H}}$ denote the left adjoint to the forgetful functor.

Remark 1.4.2. For p an odd prime, the operation Ψ vanishes, $\theta_{2n}(x) = -\frac{1}{2}[x,x]$, the classes τ^a and δ^k vanish, and the relations (3) – (6) hold trivially. We can therefore ignore θ and Ψ.

1.4.4 The Main Computation

We can now proceed to state our main theorem on Lie algebras in $\mathrm{Mod}_E^{Cpl(I)}$. Writing $\mathrm{Free}_{\Sigma\,\mathrm{Lie}}$ for the monad which builds the free (shifted) Lie algebra on a module in $\mathrm{Mod}_E^{Cpl(I)}$, we have:

Theorem 4.4.4.

1. *The homotopy groups of any Lie algebra in $K(h)$-local E-module spectra naturally carry the structure of a Hecke Lie algebra.*

2. *Given a flat E-module spectrum M, the canonical map* $\mathrm{Free}_{\mathrm{Lie}^{\mathcal{H}}}(\pi_*(M)) \to \pi_*\left(\mathrm{Free}_{\Sigma\,\mathrm{Lie}}(L_{K(h)}(M))\right)$ *induces an isomorphism after completion.*

Part 2) of this theorem says that, up to completion, we have constructed all operations and all relations between them. We prove it in two steps. Relying on our discrete Morse theoretic computations, we establish:

Corollary 2.3.14. *Given spectra $X_1, \ldots, X_k$, every Lie word in k letters gives a map of spectral Lie algebras* $\mathrm{Free}_{\Sigma\,\mathrm{Lie}}(S^{1-|w|} \otimes X_1^{\otimes |w|_1} \otimes \ldots X_k^{\otimes |w|_k}) \xrightarrow{f_w} \mathrm{Free}_{\Sigma\,\mathrm{Lie}}(X_1 \oplus \ldots \oplus X_k)$. *Summing up all f_k for $w \in B_k$ yields an equivalence* $\bigoplus_{w\in B_k} \mathrm{Free}_{\Sigma\,\mathrm{Lie}}(S^{1-|w|} \otimes X_1^{\otimes |w|_1} \otimes \ldots \otimes X_k^{\otimes |w|_k}) \longrightarrow \mathrm{Free}_{\Sigma\,\mathrm{Lie}}(X_1 \oplus \ldots \oplus X_k)$.
Here B_k denotes the set of Lyndon words in k different letters (see Section 2.3.1) and $|w|_i$ the number of occurences of the letter i in the word w.

This reduces us to the case of free Lie algebras on one generator. Using the EHP-sequence, we further reduce to the case of one generator in odd degree. This case follows from Rezk's Koszulness of Γ. We also indicate how the main results by [AC15] and [Kja16] can be extended to the nonconnective setting (cf. Section 4.1.3).

Remark 1.4.3. At height 1, the rings Δ^i are polynomial on one generator of weight p. This implies that their Koszul dual is exterior on one generator of weight p. The first picture on the right illustrates the E-theory of a free $K(h)$-local Lie algebra on a generator in odd degree at height $h = 1$.
The second gives the corresponding picture when our generator lives in even degree and we work at the prime $p = 2$.

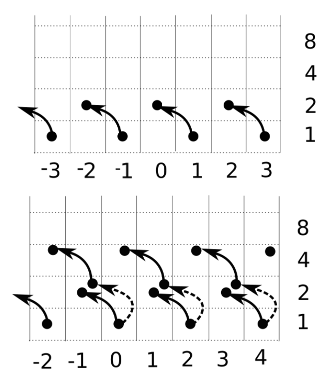


Figure 5: The free Hecke Lie algebra at height 1 on a single generator in odd and even degree respectively.

We observe from the rank formula above that the module $(\mathcal{H}^{\mathrm{Lie}})_i^j[p^a]$ vanishes whenever a exceeds the height h of E. From this, we deduce that if κ is an additive operation of weight n, then $\theta \circ \kappa = 0$.

Chapter 2

Discrete Morse Theory and the Partition Complex

In this chapter, we will use combinatorial methods to study the equivariant topology of the partition poset and uncover links to monoid spaces and to the algebraic André-Quillen homology of ordinary commutative rings.

2.1 Preliminaries

We will first recall various basic constructions and results.

2.1.1 Combinatorial Models for Spaces

In order to ensure the compatibility of our work with both the combinatorial and the homotopy-theoretical literature, we will recall the basic links between different combinatorial models for spaces.

In homotopy theory, we often model spaces as follows:

Definition 2.1.1. Let Δ be the category of nonempty finite linearly ordered sets. A *simplicial set* is a functor $\Delta^{op} \to \text{Set}$. We write **sSet** for the resulting category.

Example 2.1.2. The Yoneda embedding $i : \Delta \hookrightarrow \mathbf{sSet}$ sends the ordered set $[n]$ to the simplicial n-simplex.

In combinatorial topology, the following notion is commonly used:

Definition 2.1.3. A *simplicial complex* is given by a pair (V, F) consisting of a set V of *vertices* and a set $F \subset \mathcal{P}(V)$ of *finite* subsets of V, called *faces*, such that F is subset-closed and contains all singletons. A *morphism of simplicial complexes* $(V, F) \to (V', F')$ is a map of sets $V \to V'$ which sends subsets in F to subsets in F'. We write **sCpl** for the resulting category.

Example 2.1.4. Let Fin_+ be the category of nonempty finite sets. We can define a functor $\mathrm{Fin}_+ \to \mathbf{sCpl}$ by sending a set B to the simplicial complex $(B, \mathcal{P}(B))$ which models a simplex with B vertices. Here $\mathcal{P}(B)$ denotes the power set of B.

Yet another common model is given by CW complexes.

In order to link simplicial sets to simplicial complexes, we use the following gadget (cf. [RT03]):

Definition 2.1.5. The category **SymsSet** of *symmetric simplicial sets* is given by the category of functors $\mathrm{Fin}_+^{op} \to \mathrm{Set}$.

There is a natural diagram

$$\begin{array}{ccccccc}
\Delta & \xrightarrow{U} & \mathrm{Fin}_+ & \xrightarrow{F} & & & \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & \searrow & & & \\
\mathbf{sSet} & \xrightarrow{L} & \mathbf{SymsSet} & \xrightarrow{|-|} & \mathbf{CW} & \to & \mathbf{Top}
\end{array}$$

The vertical arrows are given by Yoneda embeddings, the functor U forgets the order, the functor L is the colimit-preserving extension making the diagram commute, the functor F sends a finite set B to the simplex on B vertices, and the functor $|-|$ is given by extending F in the natural way.

Every simplicial complex X gives a symmetric simplicial set $B \mapsto \mathrm{Map}_{sCpl}((B, \mathcal{P}(B)), X)$, and this assignment is in fact fully faithful.

Writing **Po** for the category of posets, there is a *nerve functor* $N_\bullet : \mathbf{Po} \to \mathbf{sSet}$ (defined by considering posets as categories) and an *order complex functor* $N : \mathbf{Po} \to \mathbf{sCpl}$ (defined by sending a poset P to the simplicial complex whose vertices are the elements of P and whose face set contains all subsets which are chains in P). These constructions are in fact compatible in the sense that the following diagram commutes:

$$\begin{array}{ccccccc}
\mathbf{Po} & \xrightarrow{N} & \mathbf{sCpl} & & & & \\
\downarrow{\scriptstyle N_\bullet} & & \downarrow & & & & \\
\mathbf{sSet} & \to & \mathbf{SymsSet} & \xrightarrow{|-|} & \mathbf{CW} & \to & \mathbf{Top}
\end{array}$$

We will abuse notation and denote all arrows landing in **CW** or in **Top** by $|-|$.

For the rest of this section, we fix a finite group G. We invite the reader to recall the notion of a G-CW complex from [Lüc89] – note that this is *not* the same as a G-object in CW complexes. We write $\mathbf{Po}^G$, $\mathbf{sCpl}^G$, $\mathbf{sSet}^G$, and $\mathbf{SymsSet}^G$ for the categories of objects with G-action in the undecorated versions of these respective categories.

One can then obtain an equivariant version of the above diagram:

$$\begin{array}{ccccccc}
\mathbf{Po}^G & \xrightarrow{N} & \mathbf{sCpl}^G & & & & \\
{\scriptstyle N_\bullet}\downarrow & & \downarrow & & & & \\
\mathbf{sSet}^G & \rightarrow & \mathbf{SymsSet}^G & \xrightarrow{|-|} & \mathbf{CW}^G & \rightarrow & \mathbf{Top}^G
\end{array}$$

Similar diagrams exist for pointed variants of the above categories.

2.1.2 Simple Equivariant Homotopy Theory

We briefly review the basic notions of simple homotopy theory in an equivariant setting. We begin by looking at simplicial complexes and recall a notion from [Koz15]:

Definition 2.1.6. An inclusion $(V, F) \subset (V', F')$ of G-complexes is called an *elementary G-collapse* if there is a $\sigma \in F'$ such that

1. There is exactly one face in F' which properly contains σ.

2. For every $g \in G$ with $g\sigma \neq \sigma$, there does not exist a simplex which simultaneously contains $g\sigma$ and σ.

3. F is obtained from F' by deleting all faces which contain $g\sigma$ for some $g \in G$.

There is a corresponding notion for G-CW complexes – our main reference is [Lüc89]. Write D^k for the k-dimensional disc.

Definition 2.1.7. An *elementary expansion* consists of a pushout of G-CW complexes

$$\begin{array}{ccc}
G/H \times D^{n-1} & \xrightarrow{f} & X \\
\big\downarrow & & \big\downarrow {\scriptstyle \iota} \\
G/H \times D^n & \longrightarrow & Y
\end{array}$$

such that the map $D^{n-1} \to D^n$ is given by including the lower hemisphere of the bounding sphere via $D^{n-1} \hookrightarrow S^n \hookrightarrow D^n$ and such that $f(G/H \times \partial D^{n-1}) \subset X_{n-2}$ and $f(G/H \times D^{n-1}) \subset X_{n-1}$. Here X_k denotes the k-skeleton of X.

Given such an elementary expansion $\iota : X \hookrightarrow Y$, we call any strong G-equivariant deformation retract $Y \to X$ an *elementary collapse*.

An elementary collapse between G-simplicial complexes induces an elementary collapse between their geometric realisations.

Definition 2.1.8. A G-simplicial complex (or G-CW complex) is said to be *collapsible* if it can be mapped to the point by a finite number of elementary collapses.

Definition 2.1.9. A G-map $f : X \to Y$ between G-CW complexes is a *simple homotopy equivalence* if it is G-homotopic to a finite composition of expansions and collapses.

We can ask:

Question. *Given an equivalence $G : X \to Y$ of G-CW complexes, when is it simple?*

The *Whitehead group* is an abelian group $Wh^G(Y)$ attached to any G-CW complex Y. The *Whitehead torsion* assigns an element $\tau^G(f) \in Wh^G(Y)$ to every map $f : X \to Y$ such that an equivalence f is simple if and only if $\tau^G(f) = 0$. We refer to [Lüc89] for a detailed treatment. One can use this formalism to establish:

Proposition 2.1.10. *Let A be a contractible sub-G-CW complex of a G-CW complex X. Then X/A carries a natural G-CW structure and the quotient map $X \to X/A$ is a simple homotopy equivalence.*

Proof. This follows from additivity of the Whitehead torsion (Theorem 4.8 in [Lüc89]). □

We now recall some further basic notions. We refer the reader to Sections 2.1.1. and 2.2.1 of [Aro15] for a more comprehensive overview. As before, we write $X^\diamond$ for the unreduced suspension of X and let $X * Y$ be the join of two spaces. The unreduced suspension of a G-CW complex and the join of two G-CW complexes inherit natural G-CW structures. The following Lemma is an easy strengthening of Lemma 2.5 in [Aro15]:

Lemma 2.1.11. *Given a well-pointed space X and a space Y, there is an equivalence $X * Y \cong X \wedge Y^\diamond$. If both X and Y are CW complexes, then this equivalence is simple.*

There are compatible notions of the join for simplicial complexes and posets.

The *star* and the *link* of a chain $\sigma = [x_0 < \cdots < x_k]$ in a poset $\mathcal{P}$ are given by

$$\mathrm{St}(\sigma) \cong \mathcal{P}_{(\hat{0},x_0)} * \{x_0\} * \mathcal{P}_{(x_0,x_1)} * \{x_1\} * \cdots * \{x_k\} * \mathcal{P}_{(x_k,\hat{1})}$$

$$\mathrm{Lk}(\sigma) \cong \mathcal{P}_{(\hat{0},x_0)} * \mathcal{P}_{(x_0,x_1)} * \cdots * \mathcal{P}_{(x_k,\hat{1})}$$

In Section 2.2.1 of [Aro15], Arone establishes several basic properties of stars and links in the partition complex Π_S on a set S. He calls a chain of partitions $\sigma = [x_0 < \cdots < x_k]$ *binary* if each class of x_i is the union of at most two classes of x_{i-1} for $0 \leq i \leq k$. For such chains, it is not hard to analyse the geometric realisation of the intervals and observe that $|\mathrm{Lk}(\sigma)| = S^{\sigma_{d_0}-2} * \cdots * S^{\sigma_{d_k}-2} * |\Pi_\ell|$. Here d_i is the number of classes in x_i which do not belong to x_{i-1} and ℓ is the number of equivalence classes of x_k.

2.1.3 Indexed Wedges

We recall the theory of indexed wedges from Section 2.2.3. of [HHR16]. Given a G-set J, we write $\mathcal{B}_J G$ for the category with $ob(\mathcal{B}_J G) = J$ and $Mor_{\mathcal{B}_J G}(j, j') = \{h \in G \mid h \cdot j = j'\}$ with the evident composition law. Given a functor of ordinary categories $X : \mathcal{B}_J G \to \mathbf{Top}_*$ with $j \mapsto X_j$, the wedge product $\bigvee_{j \in J} X_j$ picks up a G-action defined as $g \cdot (x \in X_j) := X_{(j \xrightarrow{g\cdot} gj)}(x) \in X_{gj}$.

We can rewrite indexed wedge products as inductions once we chose representatives:

Proposition 2.1.12. *Given $X : \mathcal{B}_J G \to \mathbf{Top}_*$, there is an equivalence of G-spaces*

$$\bigvee_{[j] \in J/G} \mathrm{Ind}^G_{\mathrm{Stab}(j)} X_j \xrightarrow{\simeq} \bigvee_{j \in J} X_j$$

Proposition 2.1.13. *If $X \xrightarrow{\alpha} Y$ is a transformation of functors $\mathcal{B}_J G \to \mathbf{Top}_*$ such that all $j \in J$, the restriction $X_j \to Y_j$ is an equivalence of $\mathrm{Stab}(j)$-spaces, then the induced map $\bigvee_{j \in J} X_j \longrightarrow \bigvee_{j \in J} Y_j$ is an equivalence of G-spaces.*

Proof. Fix a subgroup $H \subset G$. Then $H \subset \mathrm{Stab}(j)$ for all $j \in J^H$ and hence:

$$\left(\bigvee_{j \in J} X_j \right)^H \simeq \bigvee_{j \in J^H} X_j^H \simeq \bigvee_{j \in J^H} Y_j^H \simeq \left(\bigvee_{j \in J} Y_j \right)^H$$

□

2.1.4 Removing Simplices

We start by observing the following basic fact:

Proposition 2.1.14. *Assume we are given a square*

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow & & \downarrow \\ C & \hookrightarrow & D \end{array}$$

of simplicial G-sets such that for each simplicial degree d, the map of sets $(B - A)_d \to (D - C)_d$ is bijective. Taking horizontal quotients induces an isomorphism of simplicial G-sets.

If $\sigma = [x_0 < \cdots < x_r]$ is a chain in some poset $\mathcal{P}$, we let $N_\bullet(\mathrm{St}(\sigma))^{\sigma \not\subset}$ be the simplicial subset of $N_\bullet(\mathrm{St}(\sigma))$ spanned by all simplices that *do not* contain the r-simplex σ. After geometric realisation, we can swap the order of the join-factors and obtain a $\mathrm{Stab}(x_0) \cap \cdots \cap \mathrm{Stab}(x_r)$-equivariant diagram

$$\begin{array}{ccc} |N_\bullet(\mathrm{Lk}(\sigma))| * \partial\Delta^r & \longrightarrow & |N_\bullet(\mathrm{Lk}(\sigma))| * \Delta^r \\ \cong\downarrow & & \cong\downarrow \\ |N_\bullet(\mathrm{St}(\sigma))^{\sigma\not<}| & \longrightarrow & |N_\bullet(\mathrm{St}(\sigma))| \end{array}$$

Lemma 2.1.15. *Let $\mathcal{P}$ be a G-poset with G-stable point x_0. Let S be a G-stable family of chains which do not contain $[x_0]$. Assume moreover that no two chains $\sigma, \sigma' \in S$ have a common refinement. Write $N(\mathcal{P})^{-S}$ for the simplicial subset containing precisely those simplices which do not contain any simplex in S.*
The following diagram induces isomorphisms on horizontal quotients:

$$\begin{array}{ccc} \bigvee_{\sigma\in S} N_\bullet(\mathrm{St}(\sigma))_+^{\sigma\not<} & \hookrightarrow & \bigvee_{\sigma\in S} N_\bullet(\mathrm{St}(\sigma))_+ \\ \downarrow & & \downarrow \\ N_\bullet(\mathcal{P})^{-S} & \hookrightarrow & N_\bullet(\mathcal{P}) \end{array}$$

The point $+$ is sent to x_0 and the other maps are induced by the evident inclusions.

Proof. By the "refinement condition", a d-simplex in $N_d(\mathcal{P}) - N_d(\mathcal{P})^{-S}$ must contain exactly one $\sigma \in S$ and therefore has a unique preimage lying in $N_d(\mathrm{St}(\sigma)) - N_d(\mathrm{St}(\sigma)^{\sigma\not<})$ under the vertical map. The statement follows from the preceding proposition. □

Corollary 2.1.16. *Under the above conditions, there is a homeomorphism of G-spaces*

$$|N_\bullet(\mathcal{P})|/|N_\bullet(\mathcal{P})^{-S}| \cong \bigvee_{\sigma=[y_0<\cdots<y_r]\in S} S^r * \mathrm{Lk}(\sigma) \cong \bigvee_{\sigma=[y_0<\cdots<y_r]\in S} \Sigma^r |\overline{\mathcal{P}}_{(\hat{0},y_0)}|^\diamond \wedge \cdots \wedge |\overline{\mathcal{P}}_{(y_r,\hat{1})}|^\diamond$$

2.2 Complementary Collapse

We will now present an algorithm which collapses large subcomplexes of order complexes attached to lattices and thereby produces equivariant equivalences to wedge sums of simpler complexes. We call this algorithm "complementary collapse". We fix a finite group G throughout this section.

2.2.1 A Reminder on Discrete Morse Theory

We briefly review the basics of discrete Morse theory for the reader's convenience.

Definition 2.2.1. A G-equivariant matching on a simplicial G-complex (V, F) with fixed point x consists of a partition σ of the face set F satisfying the following conditions:

- The partition is G-invariant, i.e. $\sigma \sim \tau$ implies $g\sigma \sim g\tau$ for all $g \in G$.
- Every equivalence class is either equal to $\{x\}$ or has the form $\{\sigma^-, \sigma^+\}$ with $\sigma^- < \sigma^+$ a face of codimension 1.

Such a matching is called *acyclic* if there *does not* exist a chain $\sigma_1^- < \sigma_1^+ > \sigma_2^- < \sigma_2^+ > \cdots > \sigma_n^- < \sigma_n^+ > \sigma_1^-$ with $n > 1$ and all σ_i distinct.

The following statement is due to Forman [For98] in the nonequivariant and to Freij [Fre09] in the equivariant case:

Theorem 2.2.2. *If $\sim$ is a G-equivariant acyclic matching on a simplicial G-complex $X = (V, F)$ with fixed point τ, then there is a G-equivariant collapse $|X| \simeq_G \{\tau\}$.*

2.2.2 Complementary Collapse against Points

A finite G-lattice $\mathcal{P}$ is a G-poset whose underlying poset is a finite lattice, which means that every two elements have a meet and a join. Such a lattice also possesses a least element $\hat{0}$ and a greatest element $\hat{1}$. Fix a finite G-lattice $\mathcal{P}$ and write $\overline{\mathcal{P}} = \mathcal{P} - \{\hat{0}, \hat{1}\}$.

Definition 2.2.3. The complement of a G-stable $x \in \overline{\mathcal{P}}$ is given by $x^\perp = \{y \in \overline{\mathcal{P}} \mid x \wedge y = \hat{0}, x \vee y = \hat{1}\}$.

Before stating "Complementary Collapse" in full generality, we will present a special case for the reader's convenience.

Theorem 2.2.4 (Complementary collapse against Points)**.** *Let $\mathcal{P}$ be a finite G-lattice containing a G-stable vertex $x \in \overline{\mathcal{P}}$. Then $N(\overline{\mathcal{P}})^{-x^\perp}$ is G-equivariantly collapsible.*

Remark 2.2.5. The contractibility of the space in the following theorem is originally due to Björner-Walker in the nonequivariant and Welker in the equivariant case. The collapsibility in the nonequivariant case also follows from work by Kozlov [Koz98] on nonevasiveness.

Using Corollary 2.1.16, Proposition 2.1.12, and the equivalence $(X * Y)^\diamond \simeq X^\diamond \wedge Y^\diamond$, we deduce:

Theorem 2.2.6 (Applied Complementary collapse against Points)**.** *Let $\mathcal{P}$ be a finite G-lattice and write $\overline{\mathcal{P}} = (\mathcal{P} - \{\hat{0}, \hat{1}\})$. Let $x \in \overline{\mathcal{P}}$ be a G-stable element for which $x^\perp$ is discrete.*

Then there are equivalences of G-spaces

$$|\overline{\mathcal{P}}| \longrightarrow \bigvee_{y \in x^{\perp}} |\overline{\mathcal{P}}_{<y}|^{\diamond} \wedge |\overline{\mathcal{P}}_{>y}|^{\diamond} \longrightarrow \bigvee_{[z] \in x^{\perp}/G} \mathrm{Ind}_{\mathrm{Stab}(z)}^{G} |\overline{\mathcal{P}}_{<z}|^{\diamond} \wedge |\mathcal{P}_{>z}|^{\diamond}$$

2.2.3 Orthogonality Fans

We will now prove a generalisation of "Complementary collapse against Points" in which the "reference simplex" x is allowed to vary across the poset. Once more, let $\mathcal{P}$ be a finite G-lattice. Write $\mathcal{F}_{\mathcal{P}}$ for the set of *nondegenerate* simplices.

Notation 2.2.7. Given a chain σ and an element z, we write $\sigma < z$ to indicate that all elements x in σ satisfy $x < z$. We write $[\sigma, z]$ for the chain obtained by adding z to σ. We let $\sigma^{<z}$ and $\sigma^{>z}$ denote the subchain of σ spanned by all elements x in σ which satisfy $x < z$ and $x > z$ respectively.

If $F : \mathcal{F}_{\mathcal{P}} \to \mathcal{P}$ is any function and $y \in \mathcal{P}$, we can define functions

$$F^{\leq y} : \mathcal{F}_{[\hat{0}, y]} \to [\hat{0}, y], \qquad F^{\geq y} : \mathcal{F}_{[y, \hat{1}]} \to [y, \hat{1}]$$

$$F^{\leq y}(\sigma) := F(\sigma) \wedge y, \qquad F^{\geq y}(\sigma) := F([\hat{0} < \sigma]) \vee y$$

Here $[\hat{0} < \sigma]$ denotes the chain obtained from σ by adding $\hat{0}$ as an initial vertex.

Definition 2.2.8. A list of functions $\mathbf{F} = (F_1, \ldots, F_r)$ from $\mathcal{F}_{\mathcal{P}}$ to $\mathcal{P}$ is called an *orthogonality fan* if $r = 0$ or

1. F_i is G-equivariant and increasing for all i (i.e. $y \leq F_i(\sigma)$ for all $\sigma \in \mathcal{F}_{\mathcal{P}}$, $y \in \sigma$.)
2. The subposet $F_1([\hat{0}])^{\perp}$ is discrete.
3. If $F_1([\hat{0}]) \neq \hat{1}$, then we have for any $y \in F_1([\hat{0}])^{\perp}$:
   The list $(F_2^{\leq y}, \ldots, F_r^{\leq y})$ is an orthogonality fan on the Stab_y-lattice $[\hat{0}, y]$.
   The list $(F_1^{\geq y}, F_2^{\geq y}, \ldots, F_r^{\geq y})$ is an orthogonality fan on the Stab_y-lattice $[y, \hat{1}]$.

Definition 2.2.9. A (possibly empty) chain σ in $\overline{\mathcal{P}}$ is *invisible* with respect to an orthogonality fan $\mathbf{F} = (F_1, \ldots, F_r)$ if either $r = 0$, or $F_1([\hat{0}]) = \hat{1}$, or there is a vertex $y \in [\sigma, \hat{1}]$ with

1. $y \perp F_1([\hat{0}])$
2. $\sigma^{>y}$ is $(F_1^{\geq y}, F_2^{\geq y}, \ldots, F_r^{\geq y})$–invisible
3. $\sigma^{<y}$ is $(F_2^{\leq y}, \ldots, F_r^{\leq y})$–invisible

An $\mathbf{F}$-invisible chain σ is said to be $\mathbf{F}$-orthogonal if it is minimally invisible, i.e. if none of its proper subchains are $\mathbf{F}$-invisible. We write $\sigma \perp \mathbf{F}$.

We can now state the main theorem of this section:

Theorem 2.2.10 (Complementary collapse against Fans)**.** *Let $\mathbf{F}$ be an orthogonality fan on a finite G-lattice $\mathcal{P}$ with $F_1([\hat{0}]) \neq \hat{0}, \hat{1}$. Let $N(\overline{\mathcal{P}})^{-\mathbf{F}^\perp}$ be the simplicial complex obtained from $N(\mathcal{P})$ by deleting all $\mathbf{F}$-invisible chains. Then $N(\overline{\mathcal{P}})^{-\mathbf{F}^\perp}$ collapses G-equivariantly to the point $F_1([\hat{0}])$.*

Theorem 2.2.11. *Under the above conditions, there is a simple homotopy equivalence*

$$|\overline{\mathcal{P}}| \cong \bigvee_{[y_0 < \cdots < y_r] \perp \mathbf{F}} \Sigma^r |\overline{\mathcal{P}}_{(\hat{0}, y_0)}|^\diamond \wedge |\overline{\mathcal{P}}_{(y_0, y_1)}|^\diamond \wedge \cdots \wedge |\overline{\mathcal{P}}_{(y_{r-1}, y_r)}|^\diamond \wedge |\overline{\mathcal{P}}_{(y_r, \hat{1})}|^\diamond$$

obtained by collapsing the subcomplex $N(\overline{\mathcal{P}})^{-\mathbf{F}^\perp}$.

Complementary collapse for fans can be used to prove the weaker and more concrete statements from above:

Proof of Theorems 2.2.4, 2.2.6. Let x be a G-stable object in a finite G-lattice $\overline{\mathcal{P}}$. Consider the function

$$F(\sigma) = \begin{cases} x & \text{if } \sigma = [\hat{0}] \\ \hat{1} & \text{else} \end{cases}$$

If $x^\perp$ is discrete, we can apply Complementary collapse for fans to the fan (F) consisting of *one* function.
A chain is F-invisible if it contains an object $y \perp x$.
A chain is F-orthogonal if it is of the form $\sigma = [y]$ for $y \perp x$.
One can check that the general proof presented in Section 2.2.4 in fact demonstrates Theorem 2.2.4 without the assumption that $x^\perp$ is discrete. □

Remark 2.2.12. One could use Theorem 2.2.6 and induction to prove the mere existence of an equivariant simple homotopy equivalence between left and right hand side of the last Theorem 2.2.11. However, the equivalence produced in this way would not be easily accessible due to their inductive definition. The equivalence asserted in this theorem on the other hand is obtained by collapsing a large subcomplex all at once – the involved maps are therefore entirely transparent.

The recursive axiom (3) in the definition of orthogonality fans might appear difficult to check. For this reason, we introduce the following notion:

Definition 2.2.13. Let G be a finite group and $\mathcal{P}$ a finite G-lattice with face set $\mathcal{F}_\mathcal{P}$.

A function $F : \mathcal{F}_\mathcal{P} \to \mathcal{P}$ is called an *orthogonality function* if

1. F is G-equivariant and increasing (i.e. $y \leq F(\sigma)$ for all $\sigma \in \mathcal{F}_\mathcal{P}$ and all $y \in \sigma$.)

2. For any $\sigma = [y_0 < \cdots < y_m] \in \mathcal{F}_\mathcal{P}$ and $z > y_m$, the following subposet is discrete:

$$\{y_m < y < z \mid y \wedge F(\sigma) = y_m, \quad (y \vee F(\sigma)) \wedge z = z\}$$

Orthogonality functions will provide us with many examples of orthogonality fans:

Lemma 2.2.14. *Let* $\mathbf{F} = (F_1, \ldots, F_r) : \mathcal{F}_\mathcal{P} \to \mathcal{P}$ *be a list of orthogonality functions. Then* $\mathbf{F}$ *is an orthogonality fan.*

Proof. The first axiom of orthogonality fans is evidently satisfied. The second follows by using that F_1 is an orthogonality function. To verify the third, we fix some nonzero $y \perp F_1([\hat{0}])$. Observe that $F_i^{\geq y}$ and $F_i^{\leq y}$ are again orthogonality functions for all i. By induction, this implies that $(F_1^{\geq y}, F_2^{\geq y}, \ldots, F_r^{\geq y})$ and $(F_2^{\leq y}, \ldots, F_r^{\leq y})$ are both orthogonality fans on the relevant lattices. The claim follows. □

2.2.4 Proof of Complementary Collapse against Fans

We will now prove Theorems 2.2.10 and 2.2.11. Let $\mathbf{F} = (F_1, \ldots, F_r)$ be an orthogonality fan on a G-lattice $\mathcal{P}$.

Definition 2.2.15. The *orthogonality tree* $\mathbf{T}_\mathbf{F}(\sigma)$ of a chain $\sigma = [y_0 < \cdots < y_k]$ in $\overline{\mathcal{P}}$ is an empty *or* planar rooted tree whose nodes $\mathbf{w}$ are labelled by pairs $(Z \in I_\mathbf{w})$ consisting of an interval $I_\mathbf{w} = [y_\alpha, y_\omega]$ in $\mathcal{P}$ and a $\mathrm{Stab}_{y_\alpha} \cap \mathrm{Stab}_{y_\omega}$– stable point $Z \in I_\mathbf{w}$, defined in the following recursive way:

1. If $r = 0$, then the tree is empty.

2. Otherwise, we create a root $\mathbf{v}$ of the tree $\mathbf{T}_\mathbf{F}(\sigma)$ and label it by $(F_1([\hat{0}]) \in [\hat{0}, \hat{1}])$.

   - If there does not exist a vertex of $[\sigma < \hat{1}]$ lying in $F_1([\hat{0}])^\perp$ or $F(\hat{0}) = \hat{1}$, we stop.
   - Otherwise assume y is the necessarily unique vertex of $[\sigma < \hat{1}]$ which lies in $F_1([\hat{0}])^\perp$.
     By recursion, the orthogonality fan $(F_2^{\leq y}, \ldots, F_r^{\leq y})$ on the $\mathrm{Stab}(y)$-lattice $[\hat{0}, y]$ gives rise to an orthogonality tree L for the chain $\sigma^{<y}$.
     Similarly, the orthogonality fan $(F_1^{\geq y}, F_2^{\geq y}, \ldots, F_r^{\geq y})$ on $[y, \hat{1}]$ gives rise to an orthogonality tree R for the chain $\sigma_{>y}$.

We create the labelled rooted planar tree $\mathbf{T}_\mathbf{F}(\sigma)$ by declaring the root of L to be the left child and the root of R to be the right child of $\mathbf{v}$

We call a vertex of the orthogonality tree a *leaf* if it has no children.

Fix an orthogonality fan $\mathbf{F}$ on a G-lattice $\mathcal{P}$.

Definition 2.2.16. Let σ be a chain and $\mathbf{w}$ a *leaf* of $\mathbf{T}_\mathbf{F}(\sigma)$ with label $(Z \in [y_\alpha, y_\omega])$.
The simplex σ is said to be $\mathbf{F}$-*invisible at* $\mathbf{w}$ if $Z \in \{y_\alpha, y_\omega\}$.

Proposition 2.2.17. *A chain σ is invisible for an orthogonality fan $\mathbf{F}$ (in the sense of Definition 2.2.9) if σ is $\mathbf{F}$-invisible at every leaf $\mathbf{w}$ of its orthogonality tree $\mathbf{T}_\mathbf{F}(\sigma)$ (in the sense of Definition 2.2.16).*

Proof. If $r = 0$ or we have $F_1([\hat{0}]) = \hat{1}$, then the equivalence is obvious, and we may therefore assume without restriction that $F_1([\hat{0}]) \neq \hat{1}$.
Suppose σ is $\mathbf{F}$-invisible. Then there is a vertex y in $[\sigma, \hat{1}]$ with $y \perp F_1([\hat{0}])$ such that $\sigma^{>y}$ is $(F_1^{\geq y}, \dots F_r^{\geq y})$-invisible and such that $\sigma^{<y}$ is $(F_2^{\leq y}, \dots F_r^{\leq y})$-invisible. By induction, the two orthogonality trees L and R used in the definition of $\mathbf{T}_\mathbf{F}(\sigma)$ therefore only have invisible leaves.
Assume conversely that every leaf of the orthogonality tree is invisible. We can deduce that $[\sigma, \hat{1}]$ contains some $y \perp F_1([\hat{0}])$. The left and right subtree L and R of $\mathbf{T}_\mathbf{F}(\sigma)$ contain only invisible vertices. By induction, this implies that $\sigma^{<y}$ and $\sigma^{>y}$ are $(F_2^{\leq y}, \dots F_r^{\leq y})$-invisible and $(F_1^{\geq y}, \dots F_r^{\geq y})$-invisible respectively. Hence σ is $\mathbf{F}$-invisible. ☐

The next two statements follow by similarly straightforward inductions:

Proposition 2.2.18. *If $\sigma \leq \tau$ is a subsimplex, then $\mathbf{T}_\mathbf{F}(\sigma) \leq \mathbf{T}_\mathbf{F}(\tau)$ is a (labelled) subtree.*

Proposition 2.2.19. *Fix a simplex σ with orthogonality tree $\mathbf{T}_\mathbf{F}(\sigma)$ and assume $\mathbf{w} \in \mathbf{T}_\mathbf{F}(\sigma)$ is a leaf with label $(Z \in [y_\alpha, y_\omega])$.*

- *Adding a vertex $x \in (y_\alpha, y_\omega)$ with $x \wedge Z \neq y_\alpha$ or $x \vee Z \neq y_\omega$ to σ gives rise to a simplex $\sigma^+ \geq \sigma$ with equal orthogonality tree.*
- *Removing a vertex $x \in (y_\alpha, y_\omega)$ from σ gives rise to a simplex $\sigma^- \leq \sigma$ with equal orthogonality tree.*

We can now prove complementary collapse for fans:

Proof of Theorem 2.2.10. We will define a G-equivariant perfect matching.
Fix a nondegenerate ℓ-simplex $\sigma = [y_0 < \cdots < y_\ell]$ in $N(\mathcal{Q})^{-\mathbf{F}^\perp}$ other than the zero-simplex $F_1([\hat{0}])$. Let

$\mathbf{w} = \mathbf{w}_\sigma$ be the leftmost *leaf* of the orthogonality tree $\mathbf{T_F}(\sigma)$ such that σ is *not* $\mathbf{F}$-invisible at $\mathbf{w}$. Write $(Z \in [y_\alpha, y_\omega])$ for the label of $\mathbf{w}$. Since σ is not $\mathbf{F}$-invisible at $\mathbf{w}$, we have strict inequalities $y_\alpha < Z < y_\omega$. Let $i \geq \alpha$ be the largest index with $y_i \wedge Z = y_\alpha$. Note that $i < \omega$ and observe that $y_i \vee Z < y_\omega$ as otherwise $\mathbf{w}$ would *not* be a leaf. Let $j \geq i$ be maximal with $y_j \leq y_i \vee Z$. We have $j < \omega$.

We call $(i, j, \mathbf{T}) = (i(\sigma), j(\sigma), \mathbf{T_F}(\sigma))$ the *structure triple* of σ.

The element $y_{j+1} \wedge (y_i \vee Z)$ is larger than y_α since it contains $y_{j+1} \wedge Z$ and smaller than y_ω as it is contained in $y_i \vee Z < y_\omega$.

If $y_j < (y_i \vee Z) \wedge y_{j+1}$, we match σ and $\sigma^+ := [\dots < y_\alpha < \dots < y_j < ((y_i \vee Z) \wedge y_{j+1}) < y_{j+1} < \dots < y_\omega < \dots]$.
If $y_j = (y_i \vee Z) \wedge y_{j+1}$, we pair up σ with $\sigma^- := [\dots < y_\alpha < \dots < y_{j-1} < y_{j+1} < \dots < y_\omega < \dots]$.

In the first case, we consider the orthogonality tree of σ^+. Since $y_\alpha < ((y_i \vee Z) \wedge y_{j+1}) \wedge Z$, we know by Lemma 2.2.19 that σ and σ^+ have the same orthogonality tree. Hence $\mathbf{w}$ is also the first non-invisible node of the orthogonality tree for σ^+ and it is also labelled by $(Z \in [y_\alpha, y_\omega])$. We can now observe that $i(\sigma) = i(\sigma^+)$, $j(\sigma^+) = j(\sigma) + 1$, and hence $(\sigma^+)^- = \sigma$.

In the second case, we first observe that $j > i$ as otherwise we would have $y_i = (y_i \vee Z) \wedge y_{i+1}$ which is absurd as $y_\alpha = y_i \wedge Z$ and $y_\alpha < ((y_i \vee Z) \wedge y_{i+1}) \wedge Z$. We hence remove a vertex in the open interval (y_α, y_ω) and again conclude by Lemma 2.2.19 that σ and σ^- the first r nodes of the orthogonality trees of σ and σ^- are equal . We therefore have $i(\sigma) = i(\sigma^-)$ and $j(\sigma) = j(\sigma^-) - 1$, $\mathbf{T}(\sigma) = \mathbf{T}(\sigma^-)$. We conclude that $(\sigma^-)^+ = \sigma$. We have therefore defined a complete matching with fixed point $F(\hat{0})$, and it is evidently G-equivariant.

To see that the matching is acyclic, assume for the sake of contradiction that we are given a cycle of distinct non-degenerate simplices

$$\sigma_1 < \sigma_1^+ > \quad \sigma_2 = d_{t_1}(\sigma_1^+) < \quad \sigma_2^+ > \quad \sigma_3 = d_{t_2}(\sigma_2^+) < \quad \dots \quad \sigma_N^+ > \quad \sigma_1 = d_{t_N}(\sigma_N^+)$$

Let $(i_s, j_s, \mathbf{T}_s)$ be the structure triple of σ_s. We have observed above that the triple attached to σ_s^+ is $(i_s, j_s + 1, \mathbf{T}_s)$. By Proposition 2.2.18, we have $\mathbf{T}_{s+1} \leq \mathbf{T}_s$. Since the above is a cycle, the orthogonality tree $\mathbf{T}_s$ must be constant equal to $\mathbf{T}$, say. Let $\mathbf{w}$ be the first nonorthogonal node of $\mathbf{T}$ with label $(Z \in [y_\alpha, y_\omega])$.

We now want to examine how i and j change as s increases. Fix s. By definition, the number i_{s+1} is the largest number with $\alpha < i_{s+1} (< \omega)$ such that the i_{s+1}^{st} vertex of $\sigma_{s+1} = d_{t_s}(\sigma_s^+)$ intersects Z in y_α.

Since $\sigma_{s+1} \neq \sigma_s$, we know that $t_s \neq j_s + 1$.
If $t_s \leq i_s$ then $i_{s+1} = i(d_{k_s}(\sigma_s^+)) = i_s - 1$.
If $t_s > i_s$ and $t_s \neq j_s + 2$, then σ_{s+1} is an upper simplex in the matching, a contradiction.
If $t_s = j_s + 2$, then $(i_{s+1}, j_{s+1}) = (i_s, j_s + 1)$.

The function $(i_s, j_s, \mathbf{T}_s)$ hence cannot visit the same point twice and so the above cycle cannot exist. □

In order to deduce Theorem 2.2.11 from Theorem 2.2.10, we need the following Lemma:

Lemma 2.2.20. *Every* $\mathbf{F}$*-invisible chain* σ *contains a unique orthogonal chain.*

Proof. Let σ be an $\mathbf{F}$-invisible chain containing two orthogonal subchains τ_1, τ_2.
If $F_1([\hat{0}]) = \hat{1}$ or $r = 0$, then every chain is invisible and so both τ_i are equal to the empty chain.
Otherwise pick a vertex y in $[\sigma, \hat{1}]$ with $y \perp F_1([\hat{0}])$. Clearly τ_1 and τ_2 must also contain y as otherwise they would not be invisible. We obtain chains $\tau_i^{>y}$ in $\sigma^{>y}$ which must be $(F_1^{\geq y}, \ldots, F_r^{\geq y})$-orthogonal. By induction, this implies $\tau_1^{>y} = \tau_2^{>y}$. Similarly, we conclude that $\tau_1^{<y} = \tau_2^{<y}$. The claim follows. □

Proof of Theorem 2.2.11. Combine Theorem 2.2.10, Proposition 2.2.20, and Corollary 2.1.16. □

2.3 Young Restrictions

In this section, we will use the full strength of complementary collapse against orthogonality fans to study the Young restrictions of the partition complex and the parabolic restrictions of Bruhat-Tits buildings.

2.3.1 Lyndon words

Definition 2.3.1. A word w in the alphabet $c_1, \ldots, c_k$ is said to be a *(weak) Lyndon word* if it is (weakly) smaller than any of its cyclic rotations in the lexicographic order with $c_1 < \cdots < c_k$. Write $B(n_1, \ldots, n_k)$ (or $B^w(n_1, \ldots, n_k)$) for the set of (weak) Lyndon words which involve the letter c_i precisely n_i times.

A Lyndon word w of length $\ell > 1$ can be written uniquely as $w = u \cdot v$ with $u < v$ both Lyndon words and v as long as possible - this is called the *standard factorisation*. Given any two Lyndon words $u < v$, the word $w = uv$ is again a Lyndon word. The factorisation $w = u \cdot v$ is standard if and only if *either* u is a letter *or* it has standard factorisation $u = xy$ with $y \geq v$. We refer to [Mel92] for a detailed exposition.

There is a unique injection ϕ from the set of Lyndon words in $c_1, \ldots, c_k$ to the free Lie algebra $\mathrm{Lie}[c_1, \ldots, c_k]$ over $\mathbb{Z}$ on generators $c_1, \ldots, c_k$ satisfying $\phi(w) = [\phi(u), \phi(v)]$ for each standard factorisation $w = u \cdot v$. Elements in its image shall be called *basic monomials*. They form a basis. For notational convenience, we will sometimes abuse notation and confuse Lyndon words with their image under ϕ.

Any weak Lyndon word w can be written uniquely as $w = u^d$ with u a Lyndon word and d chosen as large as possible. We call d the *period* of w. This gives a natural identification $B^w(n_1, \ldots, n_k) = \coprod_{d|n_i} B(\frac{n_1}{d}, \ldots, \frac{n_k}{d})$.

Now let S be a finite set and assume $g : S \to \{1, \ldots, k\}$ is a map whose fibre C_i over i has size n_i.

Definition 2.3.2. An (S, g)-*labeling* of a weak Lyndon word $w = u^d \in B^w(n_1, \ldots, n_k)$ consists of a bijection f_s between $C_s = g^{-1}(s)$ and the set of occurrences of the letter c_s in w for each s. We consider two labelings to be equivalent if one can be obtained from the other by permuting the various copies of u. We can represent the labeling by a single function f from S to the letters of w.

Write $B^w_{(S,g)}(n_1, \ldots, n_k)$ for the $\Sigma_{C_1} \times \cdots \times \Sigma_{C_k}$-set of labelled weak Lyndon words in $B^w(n_1, \ldots, n_k)$.

Labelled Lyndon words give multilinear Lie monomials (cf. Definition 3.4 in [Aro15]):

Definition 2.3.3. Let w be a strict Lyndon word of multi-degree $(n_1, \ldots, n_k)$ together with an (S, g)-label given by $f = \{f_i\}$. The (S, g)-*resolution* of w, denoted $\tilde{w}^{(S,g)} \in \mathrm{Lie}[S]$, is the Lie monomial obtained from w by replacing all occurrences of the letter c_i with their preimages under f_i in S for all i.

If $w = u^d$ is a *weak* Lyndon word of period d with an (S, g)-label $\{f_i\}$, we carry out exactly the same procedure d times to obtain d different words in $\mathrm{Lie}[S]$ such that each letter in S is used exactly once.

There is a standard choice for (S, g): take $S = \{1, \ldots, n\}$ for $n = \sum n_i$ and use the unique order-preserving function $g : \{1, \ldots, n\} \to \{1, \ldots, k\}$ whose fibres have the required size, i.e. such that $|g^{-1}(i)| = n_i$. A resolution for this standard choice will be called the *standard resolution* and written as $\tilde{w}$.

2.3.2 Young Restrictions of the Partition Complex

Let Π^+_S be the lattice of partitions of the finite set S, ordered under refinement, so that $\Pi_S = \overline{\Pi^+_S}$. Fix a map g from S to $\{1, \ldots, k\}$ and write $C_i = g^{-1}(i)$. Let $\Sigma_g = \Sigma_{C_1} \times \cdots \times \Sigma_{C_k}$ be the associated Young subgroup.

Words on chains. Given a chain of partitions $\sigma = [y_0 < \cdots < y_m] \in \mathcal{F}_{\Pi^+_n}$, we can attach a word $w_K = w_K(\sigma) \in F\langle c_1, \ldots, c_k \rangle$ in the free group on k generators to every equivalence class K of y_m as follows:

- If $m = 0$, we attach the word $c_1^{|K \cap C_1|} \ldots c_k^{|K \cap C_k|}$ to K.
- If $m > 0$, we first use the chain $[y_0 < \cdots < y_{m-1}]$ to attach a word to every equivalence class in y_{m-1}. We then let w_K be the product of all words attached to y_{m-1}-classes which are contained in K, multiplied in ascending lexicographical order (where $c_1 < \cdots < c_k$).

Example 2.3.4. For $S = \{1, \ldots, 6\}$, $k = 3$, and $C_1 = \{1, 2\}$, $C_2 = \{3, 4, 5\}$, $C_3 = \{6\}$, we send the chain $[\{1|23|4|5|6\} < \{1|23|45|6\} < \{123|456\}]$ to the words $c_1^2 c_2, \ \ c_2^2 c_3$.

An orthogonality fan $\mathbf{F}(S,g)$ on the Σ_g-lattice Π_S^+

Given a chain $\sigma = [y_0 < \cdots < y_m]$, we use the above algorithm to attach a word in the free group $F\langle c_1, \ldots, c_k\rangle$ to every class in y_m and record them in ascending lexicographical order as $w_a < w_{b_1} < \cdots < w_{b_s}$. Write A for the set of y_m-classes whose associated word is w_a. Define $B_1, \ldots, B_s$ in a similar manner.

We define $F_1(\sigma) = F_1(S,g)(\sigma)$ to be the partition obtained from y_m by merging all equivalence classes in $B_1 \cup \cdots \cup B_s$ (i.e. all classes K whose label w_K *is not* minimal). We define $F_2(\sigma) = F_2(S,g)(\sigma)$ to be the partition obtained from y_m by merging all equivalence classes K of σ whose label w_K *is* minimal.

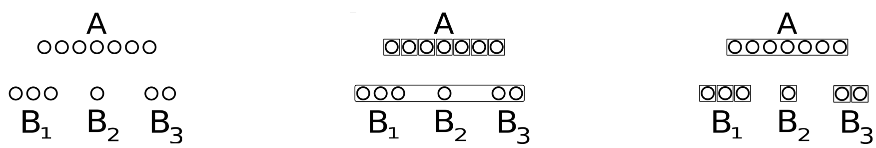


Figure 6: The bullets in the leftmost picture represent the equivalence classes of y_m. The middle picture represents the partition $F_1(\sigma)$ and the rightmost picture represents $F_2(\sigma)$.

Theorem 2.3.5. *The pair $\mathbf{F} = \mathbf{F}(S,g) = (F_1(S,g), F_2(S,g))$ is an orthogonality fan on the Σ_g-lattice $\Pi_S^+ f$.*

Proof of Theorem 2.3.5. By Lemma 2.2.14, it suffices to check that F_1 and F_2 are *orthogonality functions* in the sense of Definition 2.2.13. The functions F_1, F_2 are clearly increasing and equivariant.
To check axiom (3) of an orthogonality function, let $\sigma = [y_0 < \cdots < y_m]$ be a chain of partitions in Π_S^+ and let $z > y_m$. As before, we attach words $w_a < w_{b_1} < \cdots < w_{b_s}$ to the classes of y_m and write A for the family of y_m-classes whose associated word is the minimal w_a and B_i for the collection of classes whose associated word is the nonminimal w_{b_i}. Let $A' \subset A$ be the collection of y_m-classes which are merged with a class in $B = \cup B_i$ in z. Similarly, write $B' \subset \cup B_i$ for the family of classes which are merged with a class in A in z.

For the function F_1, we observe a natural injection

$$\{A' \xrightarrow{f} B' \mid \forall a \in A' : \quad a \simeq_z f(a)\} \hookleftarrow \{y_m < y < z \quad | \quad y \wedge F_1(\sigma) = y_m, \quad (y \vee F_1(\sigma)) \wedge z = z\}$$

obtained by sending a function f to the finest partition y_f containing y_m which merges a and $f(a)$ for all a.

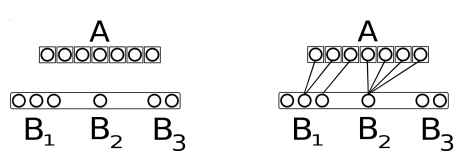


Figure 7: $F_1(\sigma)$ (left) and the partition y_f corresponding to a function $f : A \to B$ in the case $z = \hat{1}$ (right).

The subposet is therefore discrete and F_1 an orthogonality function. For F_2, we observe the injection

$$\{B' \xrightarrow{g} A' \mid \forall b \in B' : \quad b \simeq_z g(b)\} \hookrightarrow \{y_m < y < z \quad \mid \quad y \wedge F_2(\sigma) = y_m, \quad (y \vee F_2(\sigma)) \wedge z = z\}$$

and conclude that the right hand side is discrete as well. □

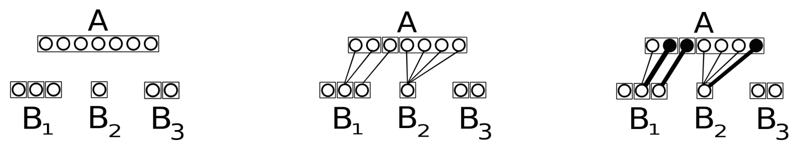


Figure 8: Here $z = y_f$ from the previous figure. We have illustrated $F_2(\sigma)$ on the left, $F_2(\sigma) \wedge z$ in the middle, and the partition y_g corresponding to some function $g : B' \to A$ on the right.

We conclude from complementary collapse:

Theorem 2.3.6. *There is a Σ_P-equivariant equivalence*

$$|\Pi_S| \xrightarrow{\simeq} \bigvee_{[y_0 < \cdots < y_r] \perp (F_1, F_2)} \Sigma^{-1}(\Sigma|(\Pi_S)_{(\hat{0}, y_0)}|^\diamond) \wedge \cdots \wedge (\Sigma|(\Pi_S)_{(y_{r-1}, y_r)}|^\diamond) \wedge |(\Pi_S)_{(y_r, \hat{1})}|^\diamond$$

Orthogonal Chains from Labelled Lyndon Words

We will now bring this last result into a more convenient form. For this, we want to find a tractable description of the set of (F_1, F_2)-orthogonal chains which indexes the wedge sum in Theorem 2.3.6. We freely use the terminology related to Lyndon words introduced in Section 2.3.1.

We will now describe an algorithm which attaches a chain of partitions to a weak labelled Lyndon word. We begin with an informal description.

Assume we are given a weak Lyndon word $w = x_1 x_2 x_3 \dots x_n$ in letters which are ordered as $\dots < c_{k-1} < c_k$.
If $n = 1, 2$ or all letters are equal, we stop and assign the empty chain of partitions.
Otherwise, we pick the smallest index α such that the letter c_α occurs in w.
We define an increasing chain of partitions $y_0 < y_1 < ... < y_t$ of the set $\{1, \dots n\}$ by merging $\{a-i, ..., a\}$ in the partition y_ℓ whenever $x_{a-i} = \dots = x_{a-1} = c_\alpha$, $x_a \neq c_\alpha$, and $i \leq t+1$. We proceed until we have defined a partition y_t for which all classes have the form $\{a-i, \dots, a\}$ with $x_{a-i} = \dots = x_{a-1} = c_\alpha$, $x_a \neq c_\alpha$, $i \geq 0$.
We then form a new weak Lyndon word by replacing each occurrence of $c_\alpha^i c_j$ with $j \neq \alpha$ by a new letter called $c_{\alpha^i j}$. We order these letters lexicographically as $\dots < c_{\alpha^2 k} < \dots < c_{\alpha(k-1)} < c_{\alpha k} < \dots < c_{k-1} < c_k$ and thus obtain a shorter weak Lyndon word whose letters correspond to the classes of y_t. We proceed

in this manner and eventually obtain a chain of partitions $y_0 < y_1 < \ldots < y_r$ which is orthogonal to $\mathbf{F} = (F_1(S,g), F_2(S,g))$. One can think of this process as "bracketing" the letters of w.

Example 2.3.7. When starting with the Lyndon word $xxyxxzyz$, our algorithm gives

$$x(xy)x(xz)yz < (xxy)(xxz)yz < (xxyxxz)yz < (xxyxxzy)z$$

More formally, let $S \to \{1, \ldots, k\}$ be a surjection with fibres C_i of size n_i. We will now show:

Lemma 2.3.8. *There is a natural $\Sigma_{C_1} \times \cdots \times \Sigma_{C_k}$-equivariant bijection $B^w_{(S,g)}(n_1, \ldots, n_k) \xrightarrow{\simeq} \mathbf{F}^\perp(S,g)$.*

We proceed in two steps. At first, we observe that we can form new labelled weak Lyndon words by rebracketing:

Lemma 2.3.9. *Let $k > 1$ and $n_i = |C_i| > 0$ for all i. There is a $\Sigma_{C_1} \times \cdots \times \Sigma_{C_k}$-equivariant bijection*

$$F : B^w_{(S,g)}(n_1, \ldots, n_k) \xrightarrow{\cong} \coprod_{\substack{y \perp F_1([\hat{0}]) \\ [z_0 < \cdots < z_m] \perp F_2|_{(\hat{0},y)}}} B^w_{(S_y,g_y)}(\ldots, n_{1^2 2}, \ldots, n_{1^2 k}, n_{12}, \ldots, n_{1k}, n_2, \ldots, n_k)$$

where S_y is the set of classes of y, and $g_y : S_y \to \{\ldots, 1^2 2, \ldots, 1^2 k, 12, \ldots, 1k, 2, \ldots, k\}$ is the "evident type function", and $n_{1^a i} = |g_y^{-1}(1^a i)|$.

Proof. Assume that we are given a weak Lyndon word $w = u^d \in B^w_{(S,g)}(n_1, \ldots, n_k)$ labelled by a function f. We form a new weak Lyndon word $F(w) \in B^w(\ldots, n_{1^2 2}, \ldots, n_{1^2 k}, n_{12}, \ldots, n_{1r}, n_2, \ldots, n_k)$ by replacing subwords of the form $c_1^a c_i$ with $i \neq 1$ by a single new letter $c_{1^a i}$ *in a way that accounts for all copies of* c_1. Write $n_{1^a i}$ for the number of times $c_{1^a i}$ occurred. We obtain a partition y of $\{1, \ldots, n\}$ which merges all sets $\{d_1, \ldots, d_a \in C_1, e \in C_i\}$, $i \neq 1$, for which $f(d_1) \ldots f(d_a) f(e)$ is a subword of w. We observe that $y \perp F_1([\hat{0}])$. We define z_t to be the partition which merges all sets $\{d_1, \ldots, d_a \in C_1, e \in C_i \mid a \leq t+1\}$, $i \neq 1$, for which $f(d_1) \ldots f(d_a) f(e)$ is a subword of w. We observe $[z_0 < \cdots < z_m] \in F_2^\perp|_{(\hat{0},y)}$ for some m chosen maximally. The word $F(w)$ is naturally labelled by (S_y, g_y) for $g_y : S_y \to \{\ldots, 1^2 2, \ldots, 1^2 k, 12, \ldots, 1k, 2, \ldots, k\}$ the "evident type function". We observe that all these choices are well-defined, i.e. do not depend on which function f we chose to represent our initial (S,g)-labeling. We have thus produced the asserted bijection. □

Lemma 2.3.10. *Let $k > 1$ and $n_i = |C_i| > 0$ for all i. There is a $\Sigma_{C_1} \times \cdots \times \Sigma_{C_k}$-equivariant bijection of orthogonal chains*

$$\mathbf{F}(S,g)^\perp \xrightarrow{\cong} \coprod_{\substack{y \perp F_1([\hat{0}]) \\ [z_0 < \cdots < z_m] \perp F_2|_{(\hat{0},y)}}} \mathbf{F}(S_y, g_y)^\perp$$

Here S_y is the set of equivalence classes of y and $g_y : S_y \to \{\ldots, 1^2 2\ldots, 1^2 k, 12\ldots, 1k, 2, \ldots, k\}$ is the "evident type function".

Proof. This follows straightforwardly from the definition of orthogonality with respect to $\mathbf{F}$. □

Proof of Lemma 2.3.8. We combine Lemma 2.3.10 with Lemma 2.3.9 and induct on the number $\sum n_i$. □

These results give an algorithm for producing chains of partitions in $\mathbf{F}^{\perp}$ from labelled weak Lyndon words. We can now reexamine Theorem 2.3.6. Take $S = \{1, \ldots n\}$ and let $g : S \to \{1, \ldots k\}$ be the unique order-preserving function g whose fibre over i has size n_i. The analysis of intervals in binary chains explained in Section 2.1.2 allows us to read off a new and purely combinatorial proof of Arone's formula from Theorem 2.3.6:

Theorem 2.3.11. *Let $n = n_1 + \cdots + n_k$. Then there is a $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$-equivariant simple homotopy equivalence*

$$|\Pi_n| \quad \longrightarrow \quad \bigvee_{d \ | \ \gcd(n_1,\ldots,n_k)} \left(\bigvee_{B(\frac{n_1}{d},\ldots,\frac{n_k}{d})} \mathrm{Ind}_{\Sigma_d}^{\Sigma_{n_1}\times\cdots\times\Sigma_{n_k}} \left(\Sigma^{-1}(S^d)^{\wedge \frac{n}{d}-1} \wedge |\Pi_d|^{\diamond} \right) \right)$$

Proof. By Lemma 2.3.8, we can rewrite the indexing set of the wedge sum appearing in Theorem 2.3.6 as $B^w_{(S,g)}(n_1, \ldots, n_k)$. Given a weak Lyndon word $w = u^d$, the group $\Sigma_{n_1} \times \ldots \times \Sigma_{n_k}$ acts transitively on the set of all (S, g)-labels of w. Each orbit has a canonical representative where the n_i occurrences of the letter c_i in w are labelled in "increasing order". We deduce that the quotient of $B^w_{(S,g)}(n_1, \ldots, n_k)$ by the action of the group $\Sigma_{n_1} \times \ldots \times \Sigma_{n_k}$ can be identified with $B^w(n_1, \ldots, n_k) = \coprod_{d|n_i} B(\frac{n_1}{d}, \ldots, \frac{n_k}{d})$.

Assume that $w = u^d$ is a weak Lyndon word whose letters are labelled "in increasing order" as above. The various different copies of the word u partition the set $S = \{1, \ldots, n\}$ into *disjoint* subsets $S_1, \ldots, S_d$ via the labeling. The stabiliser of w in $B^w_{(S,g)}(n_1, \ldots, n_k)$ is equivalent to Σ_d.

Restriction gives an $(S_i, g|_{S_i})$-labeling of the word u for each i. Using our above procedure from Lemma 2.3.8, we obtain an associated chain of partitions $[y^i_0 < \ldots < y^i_{r-1}]$ of the set S_i. The chain $[y_0 < y_1 < \ldots < y_r]$ associated to the labelled weak Lyndon word $w = u^d$ is then given by $[\coprod_i y^i_0 < \coprod_i y^i_1 < \ldots < \coprod_i y^i_{r-1} < y_r]$, where y_r denotes the partition of S into the various sets S_i. Here we used evident notation for the disjoint union of partitions on the disjoint union of different sets.

We analyse the Σ_d-space $\Sigma^{-1}(\Sigma|(\Pi_S)_{(\hat{0},y_0)}|^{\diamond} \wedge \cdots \wedge \Sigma|(\Pi_S)_{(y_{r-1},y_r)}|^{\diamond} \wedge |(\Pi_S)_{(y_r,\hat{1})}|^{\diamond})$ corresponding to the labelled weak Lyndon word $w = u^d$ in Theorem 2.3.6. The space $|(\Pi_S)_{(y_r,\hat{1})}|^{\diamond}$ is evidently equivalent to $|\Pi_d|^{\diamond}$. In order to describe the space $|(\Pi_S)_{(y_{j-1},y_j)}|^{\diamond}$, we note that each class in y^i_j is either a class in y^i_{j-1} or obtained by merging precisely two classes in y^i_{j-1} (the chain is binary). For $j = 1, \dots, r$, we write t_j for the number of classes in y^i_j that are obtained by merging two classes in y^i_{j-1} (which is independent of i by symmetry). We use the convention that y^i_{-1} is the discrete and y^i_r the indiscrete partition of S_i. We deduce an equivariant equivalence $\Sigma|(\Pi_S)_{(y_{j-1},y_j)}|^\diamond \simeq (S^{t_j})^{\wedge d}$. Since $\sum_j t_j = \frac{n}{d} - 1$, the smash product $\Sigma^{-1}(\Sigma|(\Pi_S)_{(\hat{0},y_0)}|^\diamond \wedge \cdots \wedge \Sigma|(\Pi_S)_{(y_{r-1},y_r)}|^\diamond)$ is therefore indeed equivalent to $\Sigma^{-1}(S^{\frac{n}{d}-1})^{\wedge d}$. □

Remark 2.3.12. Given an (S, g)-labelled weak Lyndon word $w = u^d$ as in the proof above inducing a partition $S = S_1 \coprod \dots \coprod S_d$, we can in fact factor the collapse map as

$$\Sigma|\Pi_S|^\diamond \longrightarrow (\bigwedge_{i=1}^{d} \Sigma|\Pi_{S_i}|^\diamond) \wedge \Sigma|\Pi_d|^\diamond \longrightarrow \bigwedge_{i=1}^{d} S^{\frac{n}{d}-1} \wedge \Sigma|\Pi_d|^\diamond$$

where the first map is given by tree ungrafting in the sense of Ching (see p.145 in Appendix D) and the second map uses the chains $\sigma^i = [y^i_0 < \dots < y^i_{r-1}]$ to define collapse maps $\Sigma|\Pi_{S_i}|^\diamond \to S^{\frac{n}{d}-1}$ to the (suspended) links of the various chains σ_i.

Free Lie Algebras on Many Generators

We will now use our methods to describe free Lie algebras on direct sums of spectra $X_1, \dots, X_k$ in terms of free Lie algebras on a single spectrum.

We start by producing maps corresponding to Lyndon words. For this, assume that we are given a *strict* Lyndon word $w \in B(|w|_1, \dots, |w|_k)$ of length $|w|$. Let $S = \{1, \dots, |w|\}$ be endowed with the unique order-preserving function g to $\{1, \dots, k\}$ whose fibre over i has size $|w|_i$. We can assemble the collapse maps in Theorem 2.3.11 corresponding to the various (S, g)-labels of w to obtain a single $\Sigma_{|w|_1} \times \dots \times \Sigma_{|w|_k}$-equivariant collapse map

$$c_w : \Sigma|\Pi_{|w|}|^\diamond \to \mathrm{Ind}_1^{\Sigma_{|w|_1} \times \dots \times \Sigma_{|w|_k}}(S^{|w|-1})$$

In a second step, we apply the functor $\mathrm{Ind}^{\Sigma_{|w|}}_{\Sigma_{|w|_1} \times \dots \times \Sigma_{|w|_k}} \left(\mathbb{D}(-) \otimes (X_1^{\otimes |w|_1} \otimes \dots \otimes X_k^{\otimes |w|_k}) \right)$ to c_w and obtain an arrow

$$\mathrm{Ind}^{\Sigma_{|w|}}_{\Sigma_{|w|_1} \times \dots \times \Sigma_{|w|_k}} \left(\mathrm{Ind}_1^{\Sigma_{|w|_1} \times \dots \times \Sigma_{|w|_k}} S^{1-|w|} \otimes (X_1^{\otimes |w|_1} \otimes \dots \otimes X_k^{\otimes |w|_k}) \right)$$
$$\longrightarrow \mathrm{Ind}^{\Sigma_{|w|}}_{\Sigma_{|w|_1} \times \dots \times \Sigma_{|w|_k}} \left(\mathbb{D}(\Sigma|\Pi_{|w|}|^\diamond) \otimes (X_1^{\otimes |w|_1} \otimes \dots \otimes X_k^{\otimes |w|_k}) \right)$$

By the projection formula, this map is equivalent to a map

$$\mathrm{Ind}_1^{\Sigma_{|w|}} \left((S^{1-|w|}) \otimes (X_1^{\otimes |w|_1} \otimes \dots \otimes X_k^{\otimes |w|_k}) \right) \to \left(\mathbb{D}(\Sigma|\Pi_{|w|}|^\diamond) \otimes \mathrm{Ind}^{\Sigma_{|w|}}_{\Sigma_{|w|_1} \times \dots \times \Sigma_{|w|_k}} (X_1^{\otimes |w|_1} \otimes \dots \otimes X_k^{\otimes |w|_k}) \right)$$

Using the "binomial formula", we observe that the last (naïve) Σ_w-spectrum includes equivariantly into $\mathbb{D}(\Sigma|\Pi_{|w|}|^\diamond) \otimes (X_1 \oplus \ldots \oplus X_k)^{\otimes |w|}$. Applying $(-)_{h\Sigma_{|w|}}$ to the composite yields a map f_w given by

$$(S^{1-|w|}) \otimes (X_1^{\otimes |w|_1} \otimes \ldots \otimes X_k^{\otimes |w|_k}) \longrightarrow \mathbb{D}(\Sigma|\Pi_{|w|}|^\diamond) \underset{h\Sigma_{|w|_1} \times \ldots \times \Sigma_{|w|_k}}{\otimes} X_1^{\otimes |w|_1} \otimes \ldots \otimes X_k^{\otimes |w|_k}$$

$$\longrightarrow \mathbb{D}(\Sigma|\Pi_{|w|}|^\diamond) \underset{h\Sigma_{|w|}}{\otimes} (X_1 \oplus \ldots \oplus X_k)^{\otimes |w|} \to \mathrm{Free}_{\Sigma\,\mathrm{Lie}}(X_1 \oplus \ldots \oplus X_k)$$

Remark 2.3.13. For $X_i = S^{j_i}$ a sphere for all i, a lengthy but straightforward induction can be used to verify that the class f_w indeed classifies the k-ary operation corresponding to the Lie word w.

Corollary 2.3.14. *Inducing up the maps f_w and summing over all Lyndon words yields an equivalence*

$$\bigoplus_{w \in B_k} \mathrm{Free}_{\Sigma\,\mathrm{Lie}}(S^{1-|w|} \otimes X_1^{\otimes |w|_1} \otimes \ldots \otimes X_k^{\otimes |w|_k}) \longrightarrow \mathrm{Free}_{\Sigma\,\mathrm{Lie}}(X_1 \oplus \ldots \oplus X_k)$$

Proof. The spectrum $\mathrm{Free}_{\Sigma\,\mathrm{Lie}}(X_1 \oplus \ldots \oplus X_k)$ is naturally $\mathbb{Z}^k$-graded. Since the map f_k sends the spectrum $S^{1-|w|} \otimes X_1^{\otimes |w|_1} \otimes \ldots X_k^{\otimes |w|_k}$ into the summand of degree $(|w|_1, \ldots, |w|_k)$, the d^{th} piece of the summand corresponding to a word w on the left is sent to the summand of degree $(d|w|_1, \ldots, d|w|_k)$ on the right. It therefore suffices to check that for all degrees $(n_1, \ldots, n_k)$ with $n = \sum_i n_i$, the map

$$\bigoplus_{\substack{d | n_1, \ldots, n_k \\ w \in B(\frac{n_1}{d}, \ldots, \frac{n_k}{d})}} \mathbb{D}(\Sigma|\Pi_d|^\diamond) \underset{h\Sigma_d}{\otimes} (S^{1-\frac{n}{d}} \otimes X_1^{\otimes \frac{n_1}{d}} \otimes \ldots \otimes X_k^{\otimes \frac{n_k}{d}})^{\otimes d} \longrightarrow \mathbb{D}(\Sigma|\Pi_n|^\diamond) \underset{h\Sigma_{n_1} \times \ldots \times \Sigma_{n_k}}{\otimes} X_1^{\otimes n_1} \otimes \ldots \otimes X_k^{\otimes n_k}$$

gives an equivalence of spectra. This in turn follows by combining Remark 2.3.12 with Theorem 2.3.11. □

2.3.3 Breaking Symmetry

Complementary collapse can also be used to give an asymmetric version of Arone's splitting. For this, fix a partition $S = A \cup B_1 \cup \cdots \cup B_k$ corresponding to a map $g : S \to \{1, \ldots, k+1\}$. Let $x = F_1([\hat{0}])$ be the partition which identifies all points in $B = \cup_i B_i$. Let

$$F_1'(\sigma) = \begin{cases} x & \text{if } \sigma = [\hat{0}] \\ \hat{1} & \text{else} \end{cases}$$

Let $F_2' = F_2(S, g)$ be as in the previous section for the orbits ordered as $A < B_1 < \cdots < B_k$.

An easy special case of our argument in the last section shows that F_1' is an orthogonality function. We have already checked this for $F_2' = F_2(S, g)$. We deduce from complementary collapse:

Theorem 2.3.15 (Breaking Symmetry)**.** *There is a simple $\Sigma_A \times \Sigma_{B_1} \times \cdots \times \Sigma_{B_k}$-equivariant homotopy equivalence*

$$|\Pi_n| \longrightarrow \bigvee_{\substack{A = A_1 \coprod \cdots \coprod A_r,\ A_i \neq \emptyset \\ f_i : A_i \hookrightarrow B \\ s.t.\ \mathrm{im}(f_{i+1}) \subset \mathrm{im}(f_i)}} \Sigma^{-1} S^{|A_1|} \wedge \cdots \wedge S^{|A_\ell|} \wedge |\Pi_B|^{\diamond}$$

Proof. This is essentially a special case of the above – we will paraphrase it here for the reader's convenience. Choosing a chain $[z_1 < \cdots < z_\ell < y]$ orthogonal to $\mathbf{F}'$ amounts to the following data:

1. First, we have to chose a function $f : A \to B$ corresponding to a partition $y = y_f \perp x$.
2. In a second step, we choose a set A_1 containing one f-preimage for each point in $f(A) \subset B$. We obtain a partition $z_1 = z_{A_1}$ by identifying each point in A_1 with its image under f.
3. In a third step, we chose a set $A_2 \subset A - A_1$ containing one f-preimage for each point in $f(A - A_1)$. We obtain a partition z_2 taking z_1 and merging all points in A_2 with their image under f.

Proceeding in this way, we obtain a chain $[z_1 < \cdots < z_\ell < y] \perp \mathbf{F}'$, and it is easy to see that every chain occurs precisely once. ☐

Remark 2.3.16. Symmetry breaking can also be used to give a neat inductive proof of the Hilton-Milnor splitting in Theorem 2.3.11. The technical disadvantage of this approach is that it is hard to describe the involved collapse maps.

2.3.4 Parabolic Restrictions of Bruhat-Tits Buildings

Let V be a finite-dimensional vector space over a finite field k. Fix a flag $\mathbf{A} = [A_0 < \cdots < A_r]$ with associated parabolic subgroup $P_{\mathbf{A}}$. We define an orthogonality fan (F) of length 1 by $F([B_0 < \cdots < B_i]) = A_{r-i} \vee B_i$. A flag $[C_0 < \cdots < C_r]$ is then F-orthogonal if

$$C_0 \wedge A_r = 0, \quad C_0 \vee A_r = V$$

$$C_1 \wedge (C_0 \vee A_{r-1}) = C_0, \quad C_1 \vee (C_0 \vee A_{r-1}) = V$$

$$\cdots$$

These conditions are clearly equivalent to $C_0 \perp A_r,\ \ C_1 \perp A_{r-1},\ \ \ldots\ \ , C_r \perp A_0$. Choosing a flag $\mathbf{B}$ complementary to $\mathbf{A}$ with parabolic $P_{\mathbf{B}}$ and intersecting Levi $L_{\mathbf{AB}} = P_{\mathbf{A}} \cap P_{\mathbf{B}}$, we deduce from complementary collapse:

Lemma 2.3.17. *There is a $P_{\mathbf{A}}$-equivariant simple homotopy equivalence*

$$|\operatorname{BT}(V)| \cong \operatorname{Ind}_{L_{\mathbf{AB}}}^{P_{\mathbf{A}}}(\Sigma^r \bigwedge_{i=0}^{r+1} |\operatorname{BT}(\operatorname{gr}^i(\mathbf{B}))|^{\diamond})$$

2.4 Fixed Points

Given a subgroup $G \subset \Sigma_n$, the subspace $|\Pi_n|^G$ of *G-fixed points* carries a natural action of the *Weyl group* $W_{\Sigma_n}(G) = N_{\Sigma_n}(G)/G$. In this section, we will provide an answer to the following question:

Question. *What is the $W_{\Sigma_n}(G)$-equivariant simple homotopy type of $|\Pi_n|^G$?*

In order to present our analysis, we single out a special class of subgroups:

Definition 2.4.1. A subgroup $G \subset \Sigma_n$ is said to be *isotypical* if all G-orbits are equivariantly isomorphic.

Our theorem reduces the general case of the question raised above to the transitive situation (note Remark 2.4.10):

Theorem 2.4.2. *If $G \subset \Sigma_n$ acts isotypically on $\{1, \ldots, n\}$, we may assume after relabeling that G is a transitive subgroup of $\Sigma_d \xrightarrow{\Delta} \Sigma_d^{\times \frac{n}{d}} \subset \Sigma_n$ for $d \mid n$ and Δ the diagonal map.*
Then there is a $W_{\Sigma_n}(G) = N_{\Sigma_n}(G)/G$ -equivariant simple homotopy equivalence

$$|\Pi_n|^G \quad \xrightarrow{\simeq} \quad \operatorname{Ind}_{W_{\Sigma_d}(G) \times \Sigma_{\frac{n}{d}}}^{W_{\Sigma_n}(G)} \quad (|\Pi_d|^G)^{\diamond} \wedge |\Pi_{\frac{n}{d}}|^{\diamond}$$

Lemma 2.4.3. *If G acts non-isotypically, then $|\Pi_n|^G$ is $W_{\Sigma_n}(G)$-equivariantly collapsible.*

Remark 2.4.4. We could also view $|\Pi_n|^G$ as a $N_{\Sigma_n}(G)$-space or a $C_{\Sigma_d}(G)^{\frac{n}{d}}$-space. In this case, the above result implies equivalences

$$|\Pi_n|^G \cong \operatorname{Ind}_{N_{\Sigma_d}(G) \times \Sigma_{\frac{n}{d}}}^{N_{\Sigma_n}(G)} \quad (|\Pi_d|^G)^{\diamond} \wedge |\Pi_{\frac{n}{d}}|^{\diamond}, \qquad |\Pi_n|^G \cong \operatorname{Ind}_{C_{\Sigma_d}(G) \times \Sigma_{\frac{n}{d}}}^{C_{\Sigma_d}(G)^{\frac{n}{d}}} \quad (|\Pi_d|^G)^{\diamond} \wedge |\Pi_{\frac{n}{d}}|^{\diamond}$$

Proof of Theorem 2.4.2 and Lemma 2.4.3. Without restriction, we assume that G is nontrivial.
Write $x \in \Pi_n$ for the partition of $\{1, \ldots, n\}$ into G-orbits. Since the action is *not* transitive and G is nontrivial, we know that $x \neq \hat{0}, \hat{1}$.

Claim. *The group G is isotypical if and only if $x^{\perp} \neq \emptyset$.*

Proof of Claim. Let $y \in x^{\perp}$ be a partition with corresponding equivalence relation $\simeq_y$ on $\{1, \ldots, n\}$. Given two elements a, b with $a \simeq_y b$, we observe that for each $c \in \operatorname{Orb}_G(a)$, there is a $d \in \operatorname{Orb}_G(b)$ with $c \simeq_y d$, and that this d must be unique as $x \wedge y = \hat{0}$. We obtain an G-equivariant function $\mathrm{Orb}_G(a) \to \mathrm{Orb}_G(b)$ which by symmetry is bijective. Since $y \vee x = \hat{1}$, this implies that all orbits are isomorphic G-sets.

For the converse, assume that the action is isotypical with orbits $O_1, \ldots, O_k$. We pick G-equivariant isomorphisms $f_i : O_i \to O_{i+1}$ and observe that the finest partition y of $\{1, \ldots, n\}$ with $(a \in O_i) \simeq_y (f_i(a) \in O_{i+1})$ for all a and all i satisfies $y \perp x$. □

Claim. *The action of $W_{\Sigma_n}(G)$ on $x^{\perp}$ is transitive.*

Proof of Claim. Given $y, z \in x^{\perp}$, we define $\sigma \in \Sigma_n$ by setting $\sigma(i) = j$ if and only if i, j lie in the same G-orbit and there is an $g \in G$ with $i \simeq_y g(1)$ and $j \simeq_z g(1)$. A simple argument shows that this gives a permutation σ in $N_{\Sigma_n}(G)$ with $\sigma(i) \simeq_z \sigma(j)$ if and only if $i \simeq_y j$, i.e. $\sigma \cdot z = y$. □

We can now deduce Theorem 2.4.2 and Lemma 2.4.3. The Lemma follows immediately from Theorem 2.2.6 since $x^{\perp} = \emptyset$. In order to prove Theorem 2.4.2, we take $z \in x^{\perp}$ to be the G-invariant partition with $i \simeq_z (i + kd)$ for all numbers i, k. The transitivity of the $W_{\Sigma_n}(G)$-action on $x^{\perp}$ and Theorem 2.2.6 together imply the existence of a $W_{\Sigma_n}(G)$-equivariant equivalence $|\Pi_n|^G \cong \mathrm{Ind}_{\mathrm{Stab}(z)}^{W_{\Sigma_n}(G)} \ (|\Pi_{n,<z}|^G)^{\diamond} \wedge (|\Pi_{n,>z}|^G)^{\diamond}$. The result then follows by observing that $\mathrm{Stab}(z) = W_{\Sigma_d}(G) \times \Sigma_{\frac{n}{d}}$, $\Pi^G_{n,<z} \cong \Pi_{\frac{n}{d}}$, and $\Pi^G_{n,>z} \cong \Pi^G_d$. □

We are therefore reduced to the study of fixed points under transitive subgroups G of Σ_n. This case is the subject of the following lemma of Klass:

Lemma 2.4.7. *If $G \subset \Sigma_n$ is a transitive subgroup and H the stabiliser of 1, then the poset of G-invariant partitions of $\{1, \ldots, n\}$ is isomorphic to the opposite of the poset of subgroups $H \subseteq K \subseteq G$.*

We recall a crucial result from [ADL16]:

Lemma 2.4.8 (Arone-Dwyer-Lesh)**.** *If $G \subset \Sigma_n$ is a p-group, then $|\Pi_n|^G$ is $W_{\Sigma_n}(G)$-equivariantly contractible unless G is elementary abelian and acts freely.*

As already mentioned in the introduction, we can combine our results about fixed points with this Lemma to compute the fixed point spaces of the partition complex $|\Pi_n|$ under general p-subgroups in terms of Bruhat-Tits buildings:

Corollary 2.4.9. *Let $\mathbb{F}_p^k \subset \Sigma_n$ be an elementary abelian p-group acting freely with ℓ orbits. Let $\mathrm{Aff}_{\mathbb{F}_p^k} = N_{\Sigma_{p^k}}(\mathbb{F}_p^k)$ be the affine group and write $\mathrm{Aff}_{\mathbb{F}_p^k \wr \Sigma_\ell} = N_{\Sigma_n}(\mathbb{F}_p^k)$. There is a simple equivalence of $\mathrm{Aff}_{\mathbb{F}_p^k \wr \Sigma_\ell}$-spaces*

$$|\Pi_n|^{\mathbb{F}_p^k} = \mathrm{Ind}_{\mathrm{Aff}_{\mathbb{F}_p^k} \times \Sigma_\ell}^{\mathrm{Aff}_{\mathbb{F}_p^k \wr \Sigma_\ell}} \left(|\,\mathrm{BT}(\mathbb{F}_p^k)|^{\diamond} \wedge |\Pi_\ell^{\diamond}| \right)$$

This can be used to compute the rationalised Morava E-theory of $|\Pi_n|^{\diamond}_{h\Sigma_{n_1}\times\cdots\times\Sigma_{n_k}}$ using HKR character theory without appealing to the above general computation of the Young restrictions.

Remark 2.4.10. Our proof of Theorem 2.4.2 and Corollary 2.5.19 were originally discovered in the spring of 2014 and presented at a presentation in Bonn in May 2015. Arone independently discovered a substantially different and more complicated proof of the "non-simple" version of this statement which was made public in August of 2015 in [Aro15]. The "contractible version" of Lemma 2.4.3 was also observed independently by M. Hausmann.

2.5 A Colimit Decomposition for G-Spaces

Using the idea of "dual fracture cubes", we give a computationally useful colimit formula for G-spaces and deduce a criterion for when a map $f : X \to Y$ of G-spaces induces an isomorphism on $H_*((-)_{/G}, \mathbb{Z}_{(p)})$. Decompositions on classifying spaces and G-spaces have been studied by several people before us (see for example [Dwy97] and [JSo01]). Fracture cubes in the form of homotopy limits have also been studied by many authors (recent examples include [ACB14] and [Gla15]).

2.5.1 G-spaces and their Approximations

Let G be a finite group. We write $\mathcal{S}_*^G$ for the ∞-category of pointed G-spaces and let $\mathcal{O}_{G,*}$ be the full subcategory spanned by all orbits $(G/H)_+$. These form a family of compact projective generators for $\mathcal{S}_*^G$. The poset of conjugacy classes of G will be denoted by ccl_G. We write $[H] \leq [K]$ if H is subconjugate to K.

Definition 2.5.1. The *iterated normaliser* of a chain of subgroups $\mathbf{H} = (H_0 \subset \cdots \subset H_m)$ is given by $N_G(\mathbf{H}) := \cap_i N_G(H_i)$. The *iterated Weyl group* is defined as $W_G(\mathbf{H}) := N_G(\mathbf{H})/H_0$.

Notation 2.5.2. We write ${}^gH = g^{-1}Hg$ for $g \in G$ and $H \subset G$ a subgroup.

Definition 2.5.3. We define the *strict orbit functor* $(-)_{/G} : S_*^G \longrightarrow S_*$ by *left Kan extending* the constant functor with value S^0 from $\mathcal{O}_{G,*}$ to S_*^G along the natural inclusion $j : \mathcal{O}_{G,*} \to S_*^G$.

Let $\mathcal{C} \subset \mathrm{ccl}_G$ be a collection of conjugacy classes. Write $\mathcal{O}_{\mathcal{C},*} \hookrightarrow \mathcal{O}_{G,*}$ for the full subcategory of all orbits of the form $(G/H)_+$ for $[H] \in \mathcal{C}$. We now recall the classical theory of $\mathcal{C}$-approximations (cf. [AD01]) in our setting:

Definition 2.5.4. The ∞*-category* $\mathcal{S}_*^G(\mathcal{C})$ *of pointed spaces with isotropy in* $\mathcal{C}$ is the smallest full subcategory of $\mathcal{S}_*^G$ which is closed under small colimits and contains $\mathcal{O}_{\mathcal{C},*}$.

The inclusion $\iota : \mathcal{S}_*^G(\mathcal{C}) \hookrightarrow \mathcal{S}_*^G$ admits a right adjoint $R_{\mathcal{C}}$ since it preserves small colimits.

Definition 2.5.5. The counit $(-)_{\mathcal{C}} = \iota \circ R_{\mathcal{C}} \to \mathrm{id}_{\mathcal{S}_*^G}$ is called the *$\mathcal{C}$-approximation functor*. An arrow $Y \to X$ is called a *$\mathcal{C}$-approximation* to X if it is equivalent to $X_{\mathcal{C}} \to X$ in $(\mathcal{S}_*^G)_{/X}$.

Proposition 2.5.6. *The functor $R_{\mathcal{C}}$ is given by $\mathrm{Lan}_{\mathcal{S}_*^G(\mathcal{C})}^{\mathcal{S}_*^G}(\mathrm{id}|_{\mathcal{S}_*^G(\mathcal{C})}) \cong \mathrm{Lan}_{\mathcal{O}_{\mathcal{C},*}}^{\mathcal{S}_*^G}(\mathcal{O}_{\mathcal{C},*} \hookrightarrow \mathcal{S}_*^G(\mathcal{C}))$. In particular, the unit of the above adjunction is given by the identity functor.*

Proof. The first identification is standard. The second follows from $\mathrm{id}|_{\mathcal{S}_*^G(\mathcal{C})} \cong \mathrm{Lan}_{\mathcal{O}_{\mathcal{C},*}}^{\mathcal{S}_*^G(\mathcal{C})}\left(\mathcal{O}_{\mathcal{C},*} \hookrightarrow \mathcal{S}_*^G(\mathcal{C})\right)$. □

The following two propositions are straightforward:

Proposition 2.5.7. *Since G is finite, the functors $R_{\mathcal{C}}$ and $(-)_{\mathcal{C}}$ commute with small colimits.*

Proposition 2.5.8. *If $\mathcal{C} \subset \mathcal{C}'$, then $\mathcal{S}_*^G(\mathcal{C}) \subset \mathcal{S}_*^G(\mathcal{C}')$.*

Proposition 2.5.9. *If $X \in \mathcal{S}_*^G(\mathcal{C})$ is a pointed space with isotropy in $\mathcal{C}$ and the conjugacy class $[H]$ is not contained in any element of $\mathcal{C}$, then $X_{[H]} \cong *$ and $X^H \cong *$.*

Proof. Since the functors $(-)_{[H]}$ and $(-)^H$ commute with small colimits, it is enough to prove the claim for $X = (G/K)_+$ where $K \in \mathcal{C}$. Here, it follows from the observation that for any $H' \in [H]$, we have

$$\mathrm{Map}_{\mathcal{S}_*^G}((G/H')_+, (G/K)_+) \cong ((G/K)^{H'})_+ \cong \{g \in G \mid g^{-1}H'g \subset K\}_+ \cong *$$

□

The following is very close to Proposition 2.3. in [AD01]:

Proposition 2.5.10. *An arrow $Y \to X$ is a $\mathcal{C}$-approximation if and only if $Y^H \to X^H$ is an equivalence for all $[H] \in \mathcal{C}$ and Y lies in $\mathcal{S}_*^G(\mathcal{C})$.*

Proof. The "only if"-direction is clear. For the "if"-direction, we apply $(-)_{\mathcal{C}} \to \mathrm{id}$ to $Y \to X$ and obtain

$$\begin{array}{ccc} Y_{\mathcal{C}} & \longrightarrow & X_{\mathcal{C}} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

The top arrow is an equivalence by the first condition and Proposition 2.5.6, the left arrow is an equivalence by the second assumption. □

If $\mathcal{C} = [K]$ a single conjugacy class, the following Lemma gives an explicit characterisation of $X_{\mathcal{C}}$:

Lemma 2.5.11 (Arone-Dwyer-Lesh)**.** *The following map is a $\mathcal{C}$-approximation to X:*

$$\bigvee_{H \in [K]} (EW_H)_+ \wedge X^H \cong \mathrm{Ind}_{N_K}^G((EW_K)_+ \wedge X^K) \to X$$

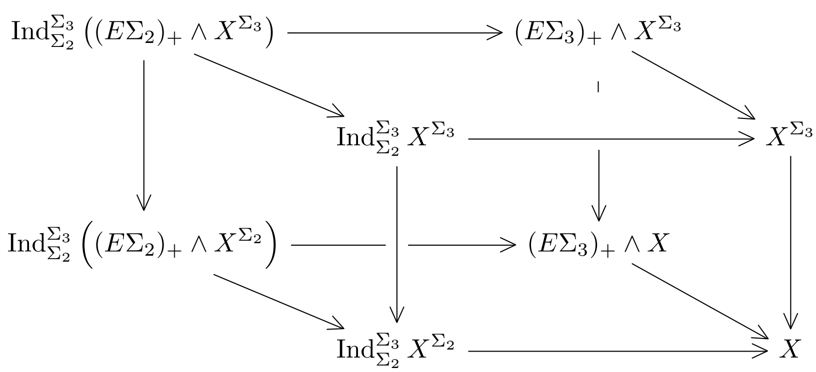
2.5.2 A Normaliser Decomposition for Spaces

We give a new colimit formula for the approximation of G-spaces and thereby generalise Lemma 2.5.11 to general $\mathcal{C}$. Our formula is nontrivial even for $\mathcal{C} = \mathrm{ccl}_G$ the collection of all conjugacy classes. We begin with two easy cases:

Example 2.5.12. For a Σ_2-space X and $\mathcal{C} = \mathrm{ccl}_{\Sigma_2}$, we obtain a homotopy pushout

$$\begin{array}{ccc} (E\Sigma_2)_+ \wedge X^{\Sigma_2} & \longrightarrow & X^{\Sigma_2} \\ \downarrow & & \downarrow \\ (E\Sigma_2)_+ \wedge X & \longrightarrow & X \end{array}$$

Example 2.5.13. For a Σ_3-space X and $\mathcal{C} = \mathrm{ccl}_{\Sigma_3}$, we write $\Sigma_2 = \Sigma_{12,3}$ and obtain a homotopy colimit



In general, fix a subposet $\mathcal{C} \subset \mathrm{ccl}_G$ and write $\mathcal{D}$ for the opposite of the category of nondegenerate simplices of $\mathcal{C}$. Objects of $\mathcal{D}$ are chains $[\mathbf{H}] = ([H_0] < \cdots < [H_m])$ of *strict* inclusions of conjugacy classes in $\mathcal{C}$. There is a unique morphism $[\mathbf{H}] \to [\mathbf{K}]$ if $[\mathbf{K}]$ can be obtained from $[\mathbf{H}]$ by removing some of the conjugacy classes. Given a pointed G-space X, we define a functor $\beta_X : \mathcal{D} \longrightarrow \mathcal{S}_*^G$ by the formula

$$[\mathbf{H}] = ([H_0] \subset \cdots \subset [H_m]) \longmapsto ((\ldots (X)_{[H_m]}) \ldots)_{[H_0]}$$

Lemma 2.5.14. *The natural map $\operatorname{colim}_{[\mathbf{H}] \in \mathcal{D}} \beta_X([\mathbf{H}]) \to X$ is a $\mathcal{C}$-approximation to X.*

Lemma 2.5.15. *The value of β_X on a chain $[\mathbf{K}] = \Big([K_0] \subset \cdots \subset [K_m]\Big)$ is given by*

$$\beta_X([\mathbf{K}]) \cong \bigvee_{\substack{H_0 \in [\mathbf{K}_0] \\ \vdots \\ H_m \in [\mathbf{K}_m] \\ H_0 \subset \cdots \subset H_m}} \Big(EW_G(H_0, \ldots, H_m)\Big)_+ \wedge X^{H_m}$$

$$\cong \bigvee_{\substack{g_{m-1} \in N_G(K_m)\backslash G/N_G(K_{m-1}) \\ g_{m-2} \in N_G({}^{g_{m-1}}K_{m-1}, K_m)\backslash G/N_G(K_{m-2}) \\ \vdots \\ g_0 \in N_G({}^{g_1}K_1, \ldots, {}^{g_{m-1}}K_{m-1}, K_m)\backslash G/N_G(K_0) \\ ({}^{g_0}K_0) \subset ({}^{g_1}K_1) \subset \cdots \subset K_m}} \operatorname{Ind}^G_{N_G({}^{g_0}K_0, {}^{g_1}K_1, \ldots, K_m)} \Big(EW_G({}^{g_0}K_0,\ {}^{g_1}K_1, \ldots, K_m)_+ \wedge X^{K_m}\Big)$$

Lemma 2.5.16. *Let $f : X \to Y$ be a map of G-spaces such that for all chains of p-subgroups $H_0 \subset \ldots \subset H_n \subset G$, the following map induces an isomorphism on $H_*(-, \mathbb{Z}_{(p)})$:*

$$(EW_G(H_0, \ldots, H_m)_+ \wedge X^{H_m}))_{/N_G(H_0, \ldots, H_m)} \longrightarrow (EW_G(H_0, \ldots, H_m)_+ \wedge Y^{H_m}))_{/N_G(H_0, \ldots, H_m)}$$

Then $f_{/G} : X_{/G} \to Y_{/G}$ induces an isomorphism on $H_*(-, \mathbb{Z}_{(p)})$.

Proof of Lemma 2.5.14. Since the domain of $\operatorname{colim}_{[\mathbf{H}] \in \mathcal{D}} \beta_X([\mathbf{H}]) \to X$ lies in $S^G_*(\mathcal{C})$ (combine Proposition 2.5.7 with Proposition 2.5.8), it is enough to check that the map induces an isomorphism on K-fixed points for any $[K] \in \mathcal{C}$ (by Proposition 2.5.10). Given such K, write $\mathcal{D}'$ for the full subcategory of $\mathcal{D}$ spanned by all $([H_0] < \cdots < [H_m])$ with $[K] \leq [H_0]$. We observe that $\mathcal{D}'$ has no outgoing morphisms to objects in $\mathcal{D} \backslash \mathcal{D}'$. Write $\mathcal{D}''$ for the full subcategory spanned by all chains $([K] < [H_1] < \cdots < [H_m])$ starting with $[K]$. Observe that there is a functor $\tau : \mathcal{D}' \longrightarrow \mathcal{D}''$ given by $([H_0] < \cdots < [H_m]) \mapsto ([K] \leq [H_0] < \cdots < [H_m])$. We now prove that all arrows in the following diagram are equivalences:

$$
\begin{array}{ccc}
\beta_X|_{\mathcal{D}''}([K]) & \xrightarrow{\simeq} & X^K \\
\simeq \uparrow & (5) & \simeq \uparrow \\
\operatorname*{colim}_{[\mathbf{H}]\in\mathcal{D}''} (\beta_X|_{\mathcal{D}''}([\mathbf{H}]))^K & \longrightarrow & X^K \\
\simeq \uparrow & (4) & \simeq \uparrow \\
\operatorname*{colim}_{[\mathbf{H}]\in\mathcal{D}'} (\beta_X|_{\mathcal{D}''} \circ \tau([\mathbf{H}]))^K & \longrightarrow & X^K \\
\simeq \downarrow & (3) & \simeq \downarrow \\
\operatorname*{colim}_{[\mathbf{H}]\in\mathcal{D}'} \beta_X|_{\mathcal{D}'}([\mathbf{H}])^K & \longrightarrow & X^K \\
\simeq \downarrow & (2) & \simeq \downarrow \\
\operatorname*{colim}_{[\mathbf{H}]\in\mathcal{D}} \beta_X([\mathbf{H}])^K & \longrightarrow & X^K \\
\simeq \downarrow & (1) & \simeq \downarrow \\
\left(\operatorname*{colim}_{[\mathbf{H}]\in\mathcal{D}} \beta_X([\mathbf{H}])\right)^K & \longrightarrow & X^K
\end{array}
$$

The left vertical arrows in square (1) are equivalences since K is finite.

The left vertical arrow in (2) is an equivalence: for any $[\mathbf{H}] = \Big([H_0] < \cdots < [H_m]\Big) \in \mathcal{D}\backslash\mathcal{D}'$, the G-space $\beta([\mathbf{H}])$ has isotropy in $[H_0]$. Since $[K] \nleq [H_0]$, we use Proposition 2.5.9 to conclude $\beta_X([\mathbf{H}])^K$ is contractible. The canonical morphism $\mathrm{Lan}_{\mathcal{D}'}^{\mathcal{D}} \beta_X^K|_{\mathcal{D}'} \to \beta_X^K$ is therefore an equivalence.

The left vertical arrow in square (3) is an equivalence since $[K]$-approximation preserves $(-)^K$-fixed points.

The left vertical arrow in square (4) is an equivalence since the functor τ is cofinal (indeed, $\mathcal{D}' \times_{\mathcal{D}''} (\mathcal{D}'')_{d/}$ has an initial object (d, id_d) for all $d \in \mathcal{D}''$).

Finally, the left vertical arrow in square (5) is an equivalence since $[K]$ is a terminal object in the category $\mathcal{D}''$.

□

We can now prove our general formula:

Proof of Lemma 2.5.15. : We proceed by induction on the length m of the chain.

The base case $m = 0$ is just Proposition 2.5.11.

Assume now that $m > 0$ and the statement holds true for $m - 1$. Then:

$$\Big(\Big(\ldots\Big(X\Big)_{[K_m]}\Big)\ldots\Big)_{[K_0]} \cong \bigvee_{H_0\in[\mathbf{K}_0]} EW_G(H_0)_+ \wedge \Bigg(\bigvee_{\substack{H_1\in[\mathbf{K}_1] \\ \cdots \\ H_m\in[\mathbf{K}_m] \\ H_1\subset\cdots\subset H_m}} \Big(EW_G(H_1,\ldots,H_m)_+ \wedge X^{H_m}\Big)\Bigg)^{H_0}$$

$$\cong \bigvee_{H_0 \in [\mathbf{K}_0]} \left(EW_G(H_0)_+ \wedge \bigvee_{\substack{H_1 \in [\mathbf{K}_1] \\ \vdots \\ H_m \in [\mathbf{K}_m] \\ H_1 \subset \cdots \subset H_m \\ H_0 \subset N_G(H_1, \ldots, H_m)}} \left(EW_G(H_1, \ldots, H_m)_+^{H_0} \wedge (X^{H_m})^{H_0} \right) \right)$$

If $H_0 \not\subset H_1$, then $EW_G(H_1, \ldots, H_m)_+^{H_0}$ is contractible and the summand can be ignored. Hence:

$$\left(\left(\ldots \left(X \right)_{[H_m]} \right) \ldots \right)_{[H_0]} \cong \left(\bigvee_{\substack{H_0 \in [\mathbf{K}_0] \\ H_1 \in [\mathbf{K}_1] \\ \vdots \\ H_m \in [\mathbf{K}_m] \\ H_0 \subset H_1 \subset \cdots \subset H_m}} EW_G(H_0) \wedge EW_G(H_1, \ldots, H_m)_+ \wedge X^{H_m} \right)$$

Given a subgroup $K \subset N_G(H_0, \ldots, H_m)$, the space

$$(EW_G(H_0))_+^K \wedge (EW_G(H_1, \ldots H_m))_+^K$$

is contractible unless $K \subset H_0$, in which case it is S^0. We have an equivalence of $N_G(H_0, \ldots, H_m)$-spaces

$$EW_G(H_0) \wedge EW_G(H_1, \ldots H_m) \cong EW_G(H_0, \ldots, H_m)$$

and the first asserted formula for the functor β follows.

For the second expression, we first recall that we can rewrite a wedge $\bigvee_{j \in J} A_j$ indexed by a G-set J as $\bigvee_{[j] \in J/G} \operatorname{Ind}_{\mathrm{Stab}(j)}^G A_j$.
For us, the G-action on $J := \left\{ H_0 \in [\mathbf{K}_0], \ldots, H_m \in [\mathbf{K}_m] \;\middle|\; H_0 \subset \cdots \subset H_m \right\}$ has orbits

$$J/G = \left\{ H_0 \in [\mathbf{K}_0], \ldots, H_{m-1} \in [\mathbf{K}_{m-1}] \;\middle|\; H_0 \subset \cdots \subset H_{m-1} \subset K_m \right\} / _{N_G(K_m)}$$

$$= \cdots = \left\{ \substack{g_{m-1} \in N_G(K_m) \backslash G / N_G(K_{m-1}) \\ g_{m-2} \in N_G({}^{g_{m-1}}K_{m-1}, K_m) \backslash G / N_G(K_{m-2}) \\ \vdots \\ g_0 \in N_G({}^{g_1}K_1, \ldots, {}^{g_{m-1}}K_{m-1}, K_m) \backslash G / N_G(K_0) \\ ({}^{g_0}K_0) \subset ({}^{g_1}K_1) \subset \cdots \subset K_m} \right\}$$

The stabiliser of a given chain $(H_0 \subset \cdots \subset H_m) \in J$ is evidently given by $N_G(H_0, \ldots, H_m)$. □

Proof of Theorem 2.5.16. Let $\mathcal{C} = \overline{\mathcal{S}}_p \subset \mathrm{ccl}_G$ be the family of conjugacy classes of p-subgroups. Write $\mathcal{D}$ for the category of nondegenerate simplices of this poset . Consider:

$$\begin{array}{ccc}
H_*(X_{/G}, \mathbb{Z}_{(p)}) & \longrightarrow & H_*(Y_{/G}, \mathbb{Z}_{(p)}) \\
\cong \downarrow & (1) & \cong \downarrow \\
H^G_*(X, \underline{\mathbb{Z}}_{(p)}) & \longrightarrow & H^G_*(Y, \underline{\mathbb{Z}}_{(p)}) \\
\cong \downarrow & (2) & \cong \downarrow \\
H^G_*(X_{\overline{\mathcal{S}}_p}, \underline{\mathbb{Z}}_{(p)}) & \longrightarrow & H^G_*(Y_{\overline{\mathcal{S}}_p}, \underline{\mathbb{Z}}_{(p)}) \\
\cong \downarrow & (3) & \cong \downarrow \\
H_*((X_{\overline{\mathcal{S}}_p})_{/G}, \mathbb{Z}_{(p)}) & \longrightarrow & H_*((X_{\overline{\mathcal{S}}_p})_{/G}, \mathbb{Z}_{(p)}) \\
\cong \downarrow & (4) & \cong \downarrow \\
H_*\left(\left(\operatorname*{colim}_{[\mathbf{H}]\in\mathcal{D}} \beta_X([\mathbf{H}])\right)_{/G}, \ \mathbb{Z}_{(p)}\right) & \longrightarrow & H_*\left(\left(\operatorname*{colim}_{[\mathbf{H}]\in\mathcal{D}} \beta_Y([\mathbf{H}])\right)_{/G}, \ \mathbb{Z}_{(p)}\right) \\
\cong \downarrow & (5) & \cong \downarrow \\
H_*\left(\operatorname*{colim}_{[\mathbf{H}]\in\mathcal{D}} \left(\beta_X([\mathbf{H}])_{/G}\right), \ \mathbb{Z}_{(p)}\right) & \longrightarrow & H_*\left(\operatorname*{colim}_{[\mathbf{H}]\in\mathcal{D}} \left(\beta_Y([\mathbf{H}])_{/G}\right), \ \mathbb{Z}_{(p)}\right) \\
\cong \downarrow & (6) & \cong \downarrow \\
\pi_*\left(\operatorname*{colim}_{[\mathbf{H}]\in\mathcal{D}} \left(\Sigma^\infty \beta_X([\mathbf{H}])_{/G} \otimes H\mathbb{Z}_{(p)}\right)\right) & \longrightarrow & \pi_*\left(\operatorname*{colim}_{[\mathbf{H}]\in\mathcal{D}} \left(\Sigma^\infty \beta_Y([\mathbf{H}])_{/G} \otimes H\mathbb{Z}_{(p)}\right)\right)
\end{array}$$

The vertical arrows in square

- (1) and (3) are equivalences by elementary properties of Bredon homology.
- (2) are equivalences because the constant Mackey functor $\underline{\mathbb{Z}}_{(p)}$ has the p-transfer property and is hence projective relative to p-subgroups by Lemma 3.8 of [ADL16].
- (4) are equivalences by Lemma 2.5.14.
- (5) are equivalences because for G finite, strict orbits and homotopy colimits commute.
- (6) are equivalences because the smash product commutes with (homotopy) colimits.

The assumptions imply that $\Sigma^\infty \beta_X([\mathbf{H}])_{/G} \otimes H\mathbb{Z}_{(p)} \longrightarrow \Sigma^\infty \beta_Y([\mathbf{H}])_{/G} \otimes H\mathbb{Z}_{(p)}$ is an equivalence for all $[\mathbf{H}] \in ob(\mathcal{D})$ and the claim follows. $\square$

Lemma 2.5.16 can be used to deduce certain statements on strict orbits from their corresponding statements on homotopy orbits. As an example, we start with the following result of Arone [Aro06]:

Theorem 2.5.17. *Let p be an odd prime. If j is odd and $n \neq p^a$ or j is even and $n \neq p^a, 2p^a$ for all a, we have $H_*(|\Pi_n|^\diamond \wedge_{h\Sigma_n} (S^j)^{\wedge n}, \mathbb{F}_p) = 0$*

We can use our machinery to prove the analogous statement for *strict* coinvariants:

Theorem 2.5.18. *Let p be an odd prime. If j is odd and $n \neq p^a$ or j is even and $n \neq p^a, 2p^a$ for all a, we have $H_*(|\Pi_n|^\diamond \wedge_{\Sigma_n} (S^j)^{\wedge n}, \mathbb{F}_p) = 0$.*

Proof. We apply Lemma 2.5.16 to the map $|\Pi_n|^\diamond \wedge_{\Sigma_n} (S^j)^{\wedge n} \to *$. It suffices to check that

$$H_*\left(EW_{\Sigma_n}(H_0, \dots, H_m)_+ \wedge (|\Pi_n|^\diamond \underset{\Sigma_n}{\wedge} (S^j)^{\wedge n})^{H_m}_{/N_{\Sigma_n}(H_0,\dots,H_m)}, \mathbb{Z}_{(p)}\right) = 0$$

for all chains of p-subgroups $\mathbf{H} = (H_0 \subset \cdots \subset H_m)$. If H_m is not of the form $\Delta_k(\mathbb{F}_p^\ell)$, then this is clear by the contractibility result of Arone-Dwyer-Lesh. If $\mathbf{H} = \Delta_k(\mathbf{K})$ for some flag $\mathbf{K} = (K_0 \subset \cdots \subset K_m = \mathbb{F}_p^\ell)$ with associated parabolic $P_{\mathbf{K}}$, we use Theorem 2.4.2 to write:

$$H_*\left(EW_{\Sigma_n}(H_0, \dots, H_m)_+ \wedge (|\Pi_n|^\diamond \wedge (S^j)^{\wedge n})^{H_m}_{/N_{\Sigma_n}(H_0,\dots,H_m)}, \mathbb{F}_p\right)$$

$$\cong H_*\left(\left(E\left(\frac{\mathbb{F}_p^\ell \rtimes P_{\mathbf{K}}}{K_0}\right)_+ \wedge \Sigma S|\operatorname{BT}(\mathbb{F}_p^\ell)|\right)_{\mathbb{F}_p^\ell \rtimes P_{\mathbf{K}}}, \mathbb{F}_p\right) \quad \otimes \quad H_*\left(\left(|\Pi_k|^\diamond \wedge (S^j)^{\wedge k}\right)_{h\Sigma_k}, \mathbb{F}_p\right)$$

This tensor product vanishes by applying Theorem 2.5.17 to the right factor. □

Remark 2.5.19. We can combine Lemma 2.5.14 with our Corollary and Lemma 2.4.8(Arone-Dwyer-Lesh) to obtain an interesting colimit decomposition of the approximation of $|\Pi_n|$ relative to p-subgroups.

2.6 Strict Quotients and Commutative Monoid Spaces

Work of Arone [Aro15] raised the following question:

Question. *What is the $\mathbb{F}_p$-homology of the strict quotient of $|\Pi_n|$ by a Young subgroup $\Sigma_{n_1}\times\cdots\times\Sigma_{n_k}\subset\Sigma_n$?*

In this section, we will uncover the conceptual significance of quotient spaces of this form, establish a surprising link to algebraic André-Quillen homology, and thereby give an answer to the above question for $p=2$.

2.6.1 Commutative Monoid Spaces and Simplicial Commutative Monoids

We start with the following definition:

Definition 2.6.1. A *commutative monoid space* is a (compactly generated Hausdorff) space R together with two distinguished points $0,1\in R$ and an associative and commutative multiplication $R\times R\to R$ such that 1 acts as a unit and $x\cdot 0=0\cdot x=0$ for all $x\in X$. Equivalently, the pointed space $(R,0)$ is a (unital) commutative algebra object in the symmetric monoidal category $(\mathbf{Top}_*,\wedge,S^0)$ of pointed (compactly generated Hausdorff) spaces, i.e. an algebra for the monad $T(X)=\bigvee_{n\geq 0}X^{\wedge n}_{\Sigma_n}$.

We write **CMon** for the category of commutative monoid spaces. This category and variants thereof have been studied by many before us (see for example [Kuh04]).

Theorem 2.6.2 (Schwänzl - Vogt [SV91])**.** *The category* **CMon** *of commutative monoid spaces with* 0 *admits the structure of a cofibrantly generated model category where a map f is a fibration or weak equivalences if and only if the underlying map of pointed spaces has this property.*

We say a commutative monoid space R is *well-pointed* if its underlying pointed space $(R,0)$ has the corresponding property. We now introduce some variants.

The category of *augmented commutative monoid spaces* is the overcategory $\mathbf{CMon}^{aug}:=\mathbf{CMon}_{/S^0}$, where S^0 denotes the commutative monoid space with two elements 0 and 1. It inherits a model category structure such that the forgetful functor preserves fibrations, cofibrations, and weak equivalences.

The category $\mathbf{CMon}^{nu}$ consists of algebras for the monad $T^{>0}(X)=\bigvee_{n>0}X^{\wedge n}_{\Sigma_n}$. Once more, it is endowed with a model category structure whose fibrations and weak equivalences are defined on the level of spaces.

The *augmentation ideal functor* $I_{(-)}:\mathbf{CMon}^{aug}\to\mathbf{CMon}^{nu}$ takes an augmented monoid space $A\to S^0$ and assigns the preimage of the base point $0\in S^0$.

Every nonunital commutative monoid space X gives rise to a unital augmented monoid space $S^0 \vee X$ by adding a disjoint unit 1.

Warning. *The functors*

$$\mathbf{CMon}^{aug} \underset{S^0\vee(-)}{\overset{I}{\rightleftarrows}} \mathbf{CMon}^{nu}$$

do not assemble to an equivalence.

We can also start with the symmetric monoidal model category $(\mathbf{sSet}_*, \wedge, S^0)$ of pointed simplicial sets and follow Quillen to define a model structure on its category **SCM** of commutative algebra objects. Weak equivalences and fibrations are again defined using the forgetful functor to $\mathbf{sSet}_*$. Objects are just Quillen's *simplicial commutative monoids* (with 0 and 1). Again, there is an augmented version $\mathbf{SCM}^{aug}$ and a nonunital variant $\mathbf{SCM}^{nu}$.

2.6.2 Extension of Scalars

Given a ring R and a commutative monoid space X, we will produce a simplicial commutative R-algebra $R \otimes X$. Heuristically speaking, we "extend scalars" from $\mathbb{F}_1$ to R.

Fix an ordinary ring R. Consider the Quillen adjunction $\mathbf{sSet}_* \underset{U}{\overset{\tilde{F}_R}{\rightleftarrows}} \mathbf{sMod}_R$. Here $\tilde{F}_R(* \to X) = \operatorname{coker}(\mathrm{Free}_{\mathrm{Mod}_R}(*) \to \mathrm{Free}_{\mathrm{Mod}_R}(X))$ is the reduced levelwise free R-module construction, and U forgets. This adjunction is monoidal. We therefore obtain a Quillen adjunction $\mathbf{SCM} \leftrightarrows \mathbf{SCR}_R$, which can be promoted to a Quillen adjunction $\mathbf{SCM}^{aug} \leftrightarrows \mathbf{SCR}_R^{aug}$.

Definition 2.6.3. The functor $R \otimes (-) : \mathbf{CMon}^{aug} \to \mathbf{SCM}^{aug} \to \mathbf{SCR}_R^{aug}$ given by composing singular chains $Sing_\bullet$ and $\tilde{F}_R$ is called *extension of scalars to R.*

Proposition 2.6.4. *The functor $R \otimes (-)$ preserves weak equivalences and homotopy colimits.*

Proof. If $M \to N$ is a weak equivalence of monoid spaces, then it is a weak equivalence of underlying spaces. The morphisms

$$Sing_\bullet(M) \to Sing_\bullet(N), \qquad (\tilde{F}_R \circ Sing_\bullet)(M) \to (\tilde{F}_R \circ Sing_\bullet)(N)$$

are weak equivalences of simplicial sets and hence also weak equivalences of simplicial commutative monoids and simplicial R-algebras respectively.

To see the second claim, we observe that the functor $Sing_\bullet : \mathbf{CMon} \to \mathbf{SCM}$ is the right half of a Quillen *equivalence*, which implies that its right derived functor $RSing_\bullet$ preserves homotopy colimits. However, since every space is fibrant, we know that every monoid space is fibrant and hence $Sing_\bullet$ computes its own right derived functor. It must therefore preserve homotopy colimits.

The functor $\tilde{F}_R$ is left Quillen, and so its left-derived functor $L\tilde{F}_R$ preserves homotopy colimits. If $Q \to id$ is a cofibrant replacement, we use that $\tilde{F}_R$ preserves weak equivalences to see that $L\tilde{F}_R(X) \cong \tilde{F}_R(QX) \xrightarrow{\simeq} \tilde{F}_R(X)$ is a weak equivalence. Hence, given a diagram $D : I \to \mathbf{CMon}$, we compute:

$$\tilde{F}_R(\underset{I}{\text{hocolim}}\ D) \xleftarrow{\simeq} L\tilde{F}_R(\underset{I}{\text{hocolim}}\ D) \cong \underset{I}{\text{hocolim}}\ ((L\tilde{F}_R) \circ D) \xrightarrow{\simeq} \underset{I}{\text{hocolim}}\ (\tilde{F}_R(D))$$

In the last step, we used that the pointwise equivalence of diagrams $\tilde{F}_R \circ Q \circ D \to \tilde{F}_R \circ D$ induces an equivalence on homotopy colimits. □

2.6.3 André-Quillen Homology for Commutative Monoid Spaces

We follow Quillen's general approach in the context of our commutative monoid spaces. We begin with the following definition:

Definition 2.6.5. The *indecomposables functor* $\mathbf{V} : \mathbf{CMon}^{nu} \to \mathbf{Top}_*$ assigns to a nonunital commutative monoid space A the quotient space $\mathbf{V}(A) = A/A \cdot A$. Here $A \cdot A \subset A$ is the space of all elements which can be decomposed into a product of two elements in A.

We can also form square-zero extensions:

Definition 2.6.6. Given a space X, we write $\overline{X}$ for the nonunital commutative monoid space obtained by declaring that $x \cdot y = 0$ for all points x, y in X.

Algebraic and topological square-zero extensions interact well as $R \otimes (S^0 \vee \overline{X})$ is just the trivial square zero extension $R \oplus \tilde{C}_\bullet(X, R)$ of R by the simplicial module $\tilde{C}_\bullet(X, R)$ of reduced $R-$valued singular chains on X. The above functors in fact determine a Quillen adjunction $\mathbf{CMon}^{nu} \underset{\overline{(-)}}{\overset{\mathbf{V}}{\rightleftarrows}} \mathbf{Top}_*$.

Definition 2.6.7. The *André-Quillen chains* $AQ(A)$ of a *nonunital* commutative monoid space $A \in \mathbf{CMon}^{nu}$ are given by the value of the left derived functor $L(\mathbf{V})(A) \in Ho(\mathbf{Top}_*)$. The *André-Quillen chains* of an *augmented* commutative monoid space are given by $AQ(LI_A)$, where I_A denotes the augmentation ideal (i.e. the fibre over 0). The *André-Quillen homology* of an augmented monoid space is given by the homotopy groups $H^Q_*(A) := \pi_*(AQ(A))$.

We can give a formula for the André-Quillen chains of a commutative monoid space:

Proposition 2.6.8. *If $A \in \mathbf{CMon}^{nu}$ is a nonunital commutative monoid space, then the André-Quillen chains of A are given by* $AQ(A) \cong \text{hocolim}_{\Delta^{op}}(\text{Bar}_\bullet(1, \mathbf{T}^{>0}, A))$.

Proof. The augmented simplicial $\mathbf{T}^{>0}$-algebra $\mathrm{Bar}_\bullet(\mathbf{T}^{>0}, \mathbf{T}^{>0}, A) \to A$ has a contracting homotopy. Hence $\mathrm{hocolim}_{\Delta^{op}}(\mathrm{Bar}_\bullet(\mathbf{T}^{>0}, \mathbf{T}^{>0}, A)) \cong A$. The left derived functor $L\mathbf{V}$ preserves homotopy colimits. Hence:

$$AQ(A) = L\mathbf{V}(A) \cong \operatorname*{hocolim}_{\Delta^{op}} L\mathbf{V}(\mathrm{Bar}_\bullet(\mathbf{T}^{>0}, \mathbf{T}^{>0}, A)) \cong \operatorname*{hocolim}_{\Delta^{op}} (\mathrm{Bar}_\bullet(1, \mathbf{T}^{>0}, A))$$

□

André-Quillen chains for commutative monoid spaces behaves well under "base-change" to ordinary rings. Fix an ordinary ring R and write $\mathbf{V} : \mathbf{SCR}_R^{aug} \to \mathbf{sMod}_R$ for the "indecomposables functor" defined on nonunital simplicial commutative R-algebras. Recall:

Definition 2.6.9. The André-Quillen chains of a nonunital simplicial commutative R-algebra S are defined as $AQ^R(S) := L\mathbf{V}(S) \in Ho(\mathbf{sMod}_R)$.

Write $\mathbf{Sym}_R^{>0}(X) = \bigoplus_{n>0} X_{\Sigma n}^{\otimes_R n}$ for the nonunital commutative algebra monad on $\mathbf{sMod}_R$. We can give an explicit formula for the André-Quillen chains as $AQ^R(S) = \mathrm{hocolim}_{\Delta^{op}} \mathrm{Bar}_\bullet(1, \mathbf{Sym}_R^{>0}, S)$. Reduced R-valued chains $\tilde{C}_\bullet(-, R)$ interact well with symmetric powers in the sense that there is an identification $\tilde{C}_\bullet(\mathbf{T}^{>0}(X), R) \cong \mathbf{Sym}_R^{>0}(\tilde{C}_\bullet(X, R))$. André-Quillen chains therefore intertwine with extension of scalars:

Lemma 2.6.10. *There is a commutative square*

$$\begin{array}{ccc} Ho(\mathbf{CMon}^{nu}) & \xrightarrow{AQ} & Ho(\mathbf{Top}_*) \\ {\scriptstyle R\otimes(-)}\downarrow & & \downarrow{\scriptstyle \tilde{C}_\bullet(-,R)} \\ Ho(\mathbf{CAlg}_R^{nu}) & \xrightarrow{AQ^R} & Ho(\mathbf{sMod}_R) \end{array}$$

Proof. Given a nonunital commutative monoid space A, we compute

$$\tilde{C}_\bullet(AQ(A), R) \cong (\tilde{C}_\bullet(-, R) \circ L\mathbf{V})(A) \simeq \operatorname*{hocolim}_{\Delta^{op}}(\tilde{C}_\bullet(\mathrm{Bar}_\bullet(1, \mathbf{T}^{>0}, A), R))$$

$$\simeq \operatorname*{hocolim}_{\Delta^{op}}(\mathrm{Bar}_\bullet(1, \mathbf{Sym}_R^{>0}, \tilde{C}_\bullet(A, R)) \cong AQ^R(R \otimes A)$$

□

Definition 2.6.11. The *strict nonunital commutative operad* $\mathbf{O}_{\mathrm{Comm}}^{nu}$ on the model category $\mathbf{Top}_*$ of pointed spaces has $(\mathbf{O}_{\mathrm{Comm}}^{nu})_{\underline{n}} = S^0$ for all $n > 0$, $(\mathbf{O}_{\mathrm{Comm}}^{nu})_{\underline{0}} = 0$, and all structure maps are the identity.

By an elementary combinatorial argument, one can compute the (strict) operadic bar construction to obtain a Koszul dual cooperad $\mathbf{O}_{\mathrm{coLie}} = \mathrm{Bar}(\mathbf{O}_{\mathrm{Comm}}^{nu})$ with

$$\mathbf{O}_{\mathrm{coLie}}(n) = |\,\mathrm{Bar}_\bullet(\mathbf{O}_{\mathrm{Comm}}^{nu})(n)| = \Sigma|\Pi_n|^\diamond$$

for all n. The cooperad structure was defined by Ching [Chi05] using tree crafting and by Salvatore [Sal98].

Any symmetric sequence $\{P_n \in \mathbf{Top}_*^{\Sigma_n}\}$ defines an endofunctor $S_P(X) = \bigvee_{n\geq 0} P_n \wedge_{\Sigma_n} X^{\wedge n}$. Here we take strict orbits. We define a functor C_{coLie} by $C_{\text{coLie}}(X) = S_{\mathbf{O}_{\text{coLie}}}(X) = \bigvee_{n\geq 1} \Sigma|\Pi_n|^{\diamond} \wedge_{\Sigma_n} X^{\wedge n}$.

We now get back to our initial problem of computing strict orbits of the partition complex under the action of Young subgroups. The quotient space $\Sigma|\Pi_n|^{\diamond}/_{\Sigma_{n_1}\times\cdots\times\Sigma_{n_k}}$ naturally appears as the piece of multi-degree $(n_1,\ldots,n_k)$ in $C_{\text{coLie}}((S^0)^{\vee k}) = \bigvee_n \Sigma|\Pi_n|^{\diamond} \wedge_{\Sigma_n} ((S^0)^{\vee k})^{\wedge n} = \bigvee_{n_1,\ldots,n_k} \Sigma|\Pi_{n_1+\cdots+n_k}|^{\diamond}/_{\Sigma_{n_1}\times\cdots\times\Sigma_{n_k}}$.

Hence we need to study the value of the functor C_{coLie} on *wedges of spheres*. Using Theorem 2.3.11, we can split up C_{coLie} evaluated on wedges of spaces $X_1,\ldots,X_k$ and obtain an equivalence of pointed spaces (this strategy is a variant of the computation of Goerss [Goe90]):

$$\bigvee_{\substack{\ell_1,\ldots,\ell_k \\ w\in B(\ell_1,\ldots,\ell_k)}} C_{\text{coLie}}(S^{\ell_1+\cdots+\ell_k-1} \wedge X_1^{\wedge \ell_1} \wedge \cdots \wedge X_k^{\wedge \ell_k}) \;\simeq\; C_{\text{coLie}}(X_1 \vee \cdots \vee X_k)$$

Here the degree d piece in a summand of signature $(\ell_1,\ldots,\ell_k)$ on the left has multi-degree $(d\ell_1,\ldots,d\ell_k)$ on the right. Taking $X_i = S^0$ again, this equivalence gives $C_{\text{coLie}}((S^0)^{\vee k}) \cong \bigvee_{\substack{\ell_1,\ldots,\ell_k \\ w\in B(\ell_1,\ldots,\ell_k)}} C_{\text{coLie}}(S^{\ell_1+\cdots+\ell_k-1})$. Restricting to some multi-degree $(n_1,\ldots,n_k)$ with $\sum_i n_i = n$, we recover a suspened version of Proposition 10.1 in [Aro15]. The functor C_{coLie} is closely related to *square zero extensions* by the following crucial observation:

Lemma 2.6.12. *If X is a well-pointed space, then $AQ(S^0 \vee \overline{X}) \cong C_{\text{coLie}}(X)$.*

Proof. We have $\mathbf{T}^{>0} = S_{\mathbf{O}^{nu}_{\text{Comm}}}$ and hence

$$AQ(S^0 \vee \overline{X}) = \underset{\Delta^{op}}{\text{hocolim}}\ \text{Bar}_\bullet(1, S_{\mathbf{O}^{nu}_{\text{Comm}}}, \overline{X}) = \underset{\Delta^{op}}{\text{hocolim}}\ S_{\text{Bar}_\bullet(\mathbf{O}^{nu}_{\text{Comm}})}(\overline{X}) = \bigvee_{n\geq 1} \Sigma|\Pi_n|^{\diamond} \underset{\Sigma_n}{\wedge} X^{\wedge n} = C_{\text{coLie}}(X)$$

□

We can now establish a surprising link between the reduced homology of strict quotients of the partition complex and a familiar invariant in derived algebraic geometry:

Theorem 2.6.13. *If X is a space and R is a ring, then*

$$\tilde{H}_*\Big(C_{\text{coLie}}(X), R\Big) = \tilde{H}_*\Big(\bigvee_{d\geq 1} \Sigma|\Pi_d|^{\diamond} \underset{\Sigma_d}{\wedge} X^{\wedge d}, R\Big) \cong AQ^R_*\Big(R \oplus \tilde{C}_\bullet(X,R)\Big)$$

Here $R \oplus \tilde{C}_\bullet(X,R)$ denotes the trivial square zero extension of R by $\tilde{C}_\bullet(X,R)$.

Proof. Combine Lemma 2.6.12 with Lemma 2.6.10. □

Fix a ring R and write $R \oplus [\epsilon_{d_1}, \ldots, \epsilon_{d_n}]$ for the trivial square zero extension of R by the free R-module with generators in degree $d_1, \ldots, d_n \geq 0$. We can "tensor up" the above splitting of $C_{\mathrm{coLie}}(S^{d_1} \vee \cdots \vee S^{d_k})$ (proved using Theorem 2.3.11) "from $\mathbb{F}_1$ to R" and deduce the following result from Theorem 2.6.13:

Corollary 2.6.14. *There is a splitting*

$$AQ_*^R(R \oplus [\epsilon_{d_1}, \ldots, \epsilon_{d_k}]) \cong \bigoplus_{\substack{\ell_1, \ldots, \ell_k \\ w \in B(\ell_1, \ldots, \ell_k)}} AQ_*^R(R \oplus [\epsilon_{(1+d_1)\ell_1 + \ldots + (1+d_k)\ell_k - 1}])$$

Over $\mathbb{F}_2$, this has been proven by Goerss [Goe90] with algebraic means. We find it remarkable that our purely combinatorial techniques have such a nontrivial consequence in derived algebraic geometry.

Chapter 3

Intertwining Topological and Algebraic Koszul Duality

In this chapter, we fix an $\mathbb{E}_2$-ring R for which $\pi_*(R)$ is Noetherian and an ideal $I \subset R_0$. We write $\mathrm{Mod}_R^{Cpl(I)}$ for the ∞-category of I-complete R-module spectra as introduced in Appendix A. Given an augmented monad $T \in \mathrm{Alg}^{aug}(\mathrm{End}(\mathrm{Mod}_R^{Cpl(I)}))$ with Koszul dual comonad $\mathrm{KD}(T) = |\operatorname{Bar}_\bullet(1, T, 1)|$ (as defined in [Lur11b]), we can ask:

Question.

1. *Can we produce cohomotopy operations on* $\mathrm{KD}(T)$*-coalgebras from homotopy operations on* T*-algebras?*
2. *Can we compute the composition of cohomotopy operations on* $\mathrm{KD}(T)$*-coalgebras from the composition of homotopy operations on* T*-algebras?*

By Corollary 5.1.21 in Appendix A, the homotopy category $h\,\mathrm{Mod}_{R,f}^{Cpl(I)}$ of completed-free R-module spectra is equivalent to the "*algebraic*" category $\mathrm{Mod}_{R_*,f}^{Cpl(I)}$ of completed-free graded modules over the graded ring R_*. This fact implies that if T preserves completed-free module spectra, then operations on the homotopy of T-algebras are controlled by an augmented monad $\hat{\mathbb{T}}$ on the algebraic category $\mathrm{Mod}_{R_*,f}^{Cpl(I)}$.

A natural first guess for an answer to the question raised above is that operations on $\mathrm{KD}(T)$-algebras are controlled by the Koszul dual $\mathrm{KD}(\hat{\mathbb{T}}) = |\operatorname{Bar}_\bullet(\hat{\mathbb{T}})|$ of this algebraic comonad.

In this chapter, we give a precise formulation of this statement and introduce conditions under which it is true.

In the fourth chapter of this thesis, we will then apply our machinery to the monad $T = \bigoplus_{n\geq 1} X_{h\Sigma_n}^{\otimes n}$ and use this to study $K(h)$-local Lie algebras in module spectra over Morava E-theory.

3.1 Operations on the (co)Homotopy of (co)Algebras

The ∞-category $\mathrm{End}(\mathrm{Mod}_R^{Cpl(I)})$ is naturally endowed with a monoidal structure given by composition, and $\mathrm{Mod}_R^{Cpl(I)}$ is left-tensored over $\mathrm{End}(\mathrm{Mod}_R^{Cpl(I)})$ (see Section 5.2.3 in Appendix B).

Given a monad T on $\mathrm{Mod}_R^{Cpl(I)}$, i.e. an algebra object in $\mathrm{End}(\mathrm{Mod}_R^{Cpl(I)})$, we can consider left T-modules in $\mathrm{Mod}_R^{Cpl(I)}$. In order to conform with classical nomenclature, we shall call such modules *T-algebras* and write $\mathrm{Alg}_T(\mathrm{Mod}_R^{Cpl(I)})$ for the resulting ∞-category. Note that this diverges from the notation used in [Lur14]. There is a forgetful-free adjunction[1] $T : \mathrm{Mod}_R^{Cpl(I)} \leftrightarrows \mathrm{Alg}_T(\mathrm{Mod}_R^{Cpl(I)}) : U$.

In this section, we will set up the language which we will later use to study the operations on the homotopy groups of T-algebras. The monad T induces a monad hT on the homotopy category $h\,\mathrm{Mod}_R^{Cpl(I)}$.

Write $h\,\mathrm{Alg}_T^{ff}$ for the homotopy category of the full subcategory of Alg_T spanned by all T-algebras of the form $T(\Sigma^{i_1}R \oplus \cdots \oplus \Sigma^{i_k}R)$.

Remark 3.1.1. A simple argument shows that $h\,\mathrm{Alg}_T^{ff}$ is equivalent to Alg_{hT}^{ff}, where Alg_{hT}^{ff} denotes the full subcategory of Alg_{hT} spanned by all hT-algebras of the form $T(\Sigma^{i_1}R \oplus \cdots \oplus \Sigma^{i_k}R)$.

The category $\mathcal{P}_T = (h\,\mathrm{Alg}_T^{ff})^{op}$ then forms a $\mathbb{Z}$-graded algebraic theory in the sense of Definition 5.3.4 in Appendix C. Define the required functor $F : \mathbb{Z}^* \to \mathcal{P}_T$ by sending a word $(t_1, \ldots, t_k)$ to $T(\Sigma^{t_1}R \oplus \cdots \oplus \Sigma^{t_k}R)$. A morphism $s_1 \ldots s_n \to t_1 \ldots t_k$ lying over the function $f : \{1, \ldots, k\} \to \{1, \ldots n\}$ with $s_{f(i)} = t_i$ for all i is sent to the arrow $T(\Sigma^{t_1}R \oplus \cdots \oplus \Sigma^{t_k}R) \to T(\Sigma^{s_1}R \oplus \cdots \oplus \Sigma^{s_n}R)$ fitting into the commutative square

$$\begin{array}{ccc} \Sigma^{t_1}R \oplus \cdots \oplus \Sigma^{t_k}R & \longrightarrow & \Sigma^{s_1}R \oplus \cdots \oplus \Sigma^{s_n}R \\ \downarrow & & \downarrow \\ T(\Sigma^{t_1}R \oplus \cdots \oplus \Sigma^{t_k}R) & \longrightarrow & T(\Sigma^{s_1}R \oplus \cdots \oplus \Sigma^{s_n}R) \end{array}$$

Here the top arrow is a map of spectra obtained by using codiagonals according to the structure of f and the lower arrow is a map of hT-algebras.

Given any T-algebra M, the graded set $\pi_i(M) = \pi_0(\mathrm{Map}_{\mathrm{Mod}_R^{Cpl(I)}}(\Sigma^i R, M))$ forms an algebra over $\mathcal{P}_T$ (in the sense of Definition 5.3.4). We shall write

$$P^j_{i_1,\ldots,i_k}(T) := \pi_0\,\mathrm{Map}_{\mathrm{Alg}_T}(T(\Sigma^j R), T(\Sigma^{i_1}R \oplus \cdots \oplus \Sigma^{i_1}R)) = \mathrm{Map}_{\mathcal{P}}(T(\Sigma^{i_1}R \oplus \cdots \oplus \Sigma^{i_1}R), T(\Sigma^j R))$$

for the group of operations with k inputs in degrees $i_1, \ldots i_k$ and one output in degree j.

[1] These notions are defined carefully in section 4.7 of [Lur14].

Applying T to the codiagonal gives rise to a morphism $T(\Sigma^i R \oplus \ldots \oplus \Sigma^i R) \to T(\Sigma^i R)$ which induces a map $P^j_{i,\ldots,i}(T) \to P^j_i(T)$ corresponding to plugging in the same variable into all slots of a given operation.

Proposition 3.1.2. *Every map of monads $T_1 \xrightarrow{\alpha} T_2$ naturally induces a morphism of associated $\mathbb{Z}$-graded algebraic theories $\mathcal{P}_\alpha : \mathcal{P}_{T_1} \to \mathcal{P}_{T_2}$.*

Proof. We shall construct a morphism $h\,\mathrm{Alg}^{ff}_{T_1} \to h\,\mathrm{Alg}^{ff}_{T_2}$ and then apply $(-)^{op}$ in the end.

On the level of objects, we send $T_1(M)$ to $\mathcal{P}_\alpha(T_1(M)) := T_2(M)$.

Given a morphism $f : T_1(A) \to T_1(B)$, we first produce the arrow $A \to T_1(A) \xrightarrow{f} T_1(B) \xrightarrow{\alpha_B} T_2(B)$ and then induce up to obtain a map of T_2-algebras $P_\alpha(f) : T_2(A) \to T_2(B)$.

Given $i = 1, 2$ and two morphisms represented by $A \xrightarrow{\overline{f}} T_i(B)$ and $B \xrightarrow{\overline{g}} T_i(C)$, the axioms of a monad imply that the composite $g \circ f$ is represented by the arrow $A \xrightarrow{\overline{f}} T_i(B) \xrightarrow{T_i(\overline{g})} T_i(T_i(C)) \xrightarrow{\mu_C} T_i(C)$.

The functoriality of $\mathcal{P}_\alpha$ is now proven by observing the following commutative diagram:

$$\begin{array}{ccccccc}
A & \xrightarrow{\overline{f}} & T_1(B) & \xrightarrow{T_1(\overline{g})} & T_1(T_1(C)) & \xrightarrow{\mu_1} & T_1(C) \\
& \searrow^{\overline{P_\alpha(f)}} & \downarrow{\scriptstyle \alpha_B} & & \downarrow{\scriptstyle \alpha_{T_1(C)}} & & \downarrow \\
& & T_2(B) & \xrightarrow{T_2(\overline{g})} & T_2(T_1(C)) & & \downarrow \\
& & & \searrow^{T_2(\overline{P_\alpha(g)})} & \downarrow & & \downarrow \\
& & & & T_2(T_2(C)) & \xrightarrow{\mu_2} & T_2(C)
\end{array}$$

□

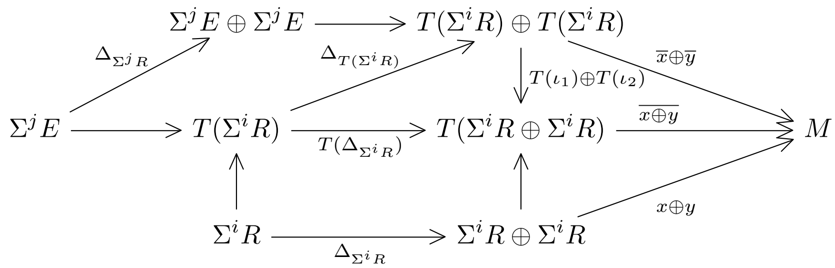
Some operations preserve additive structure:

Lemma 3.1.3. *An operation $\alpha \in P^j_i(T) = \pi_j T(\Sigma^i R)$ acts additively on the homotopy of T-algebras if and only if it lies in the equaliser of the two following maps:*

$$\begin{array}{ccc}
\pi_j T(\Sigma^i R) & \xrightarrow{\pi_j T(\Delta)} & \pi_j T(\Sigma^i R \oplus \Sigma^i R) \\
& \searrow^{\pi_j \Delta_{T(\Sigma^i R)}} & \uparrow{\scriptstyle \pi_j T(\iota_1) \oplus \pi_j T(\iota_2)} \\
& & \pi_j T(\Sigma^i R) \oplus \pi_j T(\Sigma^i R)
\end{array}$$

Here $\Delta_{(-)}$ denotes diagonal maps and $\iota_{(-)}$ stand for the inclusions of summands.

Proof. Let $\alpha \in P^j_i(T)$ be in the required equaliser and fix $x, y \in \pi_i(M)$ for M a T-algebra. Then we consider the following diagram:



where all squares and triangles except for the middle triangle commute and the various maps are defined in an evident manner. We notice that the top composite is given by $\alpha(x) + \alpha(y)$ whereas the horizontal composite gives $\alpha(x+y)$. If the middle triangle commutes, then these composites must agree. The converse direction follows by taking the universal case $M = T(\Sigma^i R \oplus \Sigma^i R)$. $\square$

A similar family of definitions can be introduced for operations on the cohomotopy groups of coalgebras over a comonad C. The i^{th} cohomotopy group of an R-module M is defined by $\pi^i(M) := \pi_0 \operatorname{Map}_{\operatorname{Mod}_R^{Cpl(I)}}(M, \Sigma^i R)$. We have a forgetful-free adjunction $U : \operatorname{coAlg}_C(\operatorname{Mod}_R^{Cpl(I)}) \leftrightarrows \operatorname{Mod}_R^{Cpl(I)} : C$.

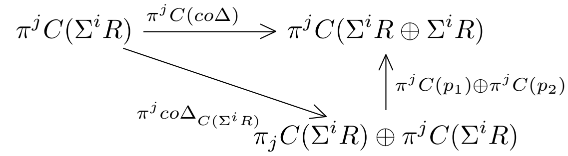
Definition 3.1.4. The $\mathbb{Z}$-graded algebraic theory $\mathcal{P}_C$ attached to a comonad $C \in \operatorname{coAlg}(\operatorname{End}(\operatorname{Mod}_R^{Cpl(I)}))$ is defined as the opposite of the full subcategory of $\operatorname{coAlg}(hC)$ spanned by all coalgebras of the form $(hC)(\Sigma^{i_1} R \oplus \ldots \oplus \Sigma^{i_k} R)$.

The cohomotopy groups of a C-coalgebra A in $\operatorname{Mod}_R^{Cpl(I)}$assemble into a $\mathcal{P}_C$-module $M(A)_i = \pi^i(A)$. Write

$$P^j_{i_1,\ldots,i_k}(C) := \pi_0 \operatorname{Map}_{\operatorname{coAlg}_C}(C(\Sigma^{i_1} R \oplus \cdots \oplus \Sigma^{i_1} R), C(\Sigma^j R)) = \operatorname{Map}_{\mathcal{P}_C}(C(\Sigma^j R), C(\Sigma^{i_1} R \oplus \cdots \oplus \Sigma^{i_1} R))$$

Dual to our reasoning above, every map $C_1 \to C_2$ of comonads induces a morphism of associated graded algebraic theories $\mathcal{P}_{C_2} \to \mathcal{P}_{C_1}$. Dually to 3.1.3, one proves the following criterion for additivity of operations:

Lemma 3.1.5. *An operation $\alpha \in P^j_i(C) = \pi^j(C(\Sigma^i R))$ acts additively on the homotopy of C-coalgebras if and only if it lies in the equaliser of the two following maps:*



We will now introduce a certain new structure which will later help us to formulate additive operations in the case where C is defined by an operad:

Definition 3.1.6. The *weighted power category* $\mathcal{W}$ has objects $\mathrm{ob}(\mathcal{W}) = \{i \in \mathbb{Z}\}$ and morphisms $\mathrm{Hom}_{\mathcal{W}}(i,j) = \{w \in \mathbb{N}\}$. Composition is defined in an evident way from multiplication of natural numbers: if $i \xrightarrow{v} j$ and $j \xrightarrow{w} k$, then $w \circ v =: v \cdot w : i \to k$.

Definition 3.1.7. A *power ring* P is a lax functor $P : \mathcal{W} \to B\,\mathrm{Mod}_{\mathbb{Z}}$ to the symmetric monoidal category with a single object, with morphisms indexed by $\mathrm{Mod}_{\mathbb{Z}}$, and with product defined using the tensor product.

A *morphism of power algebra objects* $f : P \to Q$ consists of an oplax natural transformation $\alpha : P \Rightarrow Q$ for which $\alpha_i = \mathbb{Z} \in \mathrm{End}_{B\,\mathrm{Mod}_{\mathbb{Z}}}(*) = \mathrm{Mod}_{\mathbb{Z}}$ for all i. We write $Pow_{\mathbb{Z}}^{\mathcal{W}}$ for the resulting 1-category.

More concretely, a *power ring* is a collection of abelian groups $P_i^j[w]$ for each $(i,j,w) \in \mathbb{Z}^2 \times \mathbb{N}$ together with composition maps $P_i^j[v] \otimes P_j^k[w] \to P_i^k[vw]$ satisfying evident associativity conditions. A module over a power ring is a graded abelian group M_* together with multiplication maps $P_i^j[w] \otimes M_i \to M_j$ which satisfy the natural associativity conditions.

3.2 Bridged Koszul Duality

In this section, we introduce a technical tool which we will later use to relate monadic Koszul duality in algebra and topology.

Bridged Endofunctors. We begin by recalling the following definition of Lurie:

Definition 3.2.1. An ∞-category $\mathcal{C}$ is called a *socle* if it is locally small, admits small coproducts, every object is a cogroup, and there is an essentially small full subcategory $\mathcal{C}_0$ such that any object is a retract of a coproduct of objects in $\mathcal{C}_0$.

Definition 3.2.2. Given such a socle $\mathcal{C}$, we write $\mathcal{P}_\sigma(\mathcal{C}) \subset \mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ for the ∞-category of contravariant functors to spaces which preserve small products.

By Proposition 4.2.1.(6) in [Lur11a], this construction freely adds geometric realisations to $\mathcal{C}$:

Proposition 3.2.3. *Given any ∞-category $\mathcal{D}$ with geometric realisations, precomposition with the Yoneda embedding determines an equivalence* $\mathrm{Fun}_\sigma(\mathcal{P}_\sigma(\mathcal{C}), \mathcal{D}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$.
Here $\mathrm{Fun}_\sigma(-,-)$ *denotes the full subcategory spanned by all functors which preserve geometric realisations.*

Notation 3.2.4. In the above situation, we denote the realisation-preserving functor corresponding to $F : \mathcal{C} \to \mathcal{D}$ by $LF : P_\sigma(\mathcal{C}) \to \mathcal{D}$ and think of it as the left derived functor of F.

Coming back to the situation of interest to us, we observe that the ∞-categories $\mathrm{Mod}_{R,f}^{Cpl(I)}$ and $h\,\mathrm{Mod}_{R,f}^{Cpl(I)} \cong \mathrm{Mod}_{R_*,f}^{Cpl(I)}$ introduced in Appendix A are evidently socles. Proposition 3.2.3 gives rise to a canonical diagram of ∞-categories

$$\begin{array}{ccccc} & & \mathrm{Mod}_{R,f}^{Cpl(I)} & \xrightarrow{h} & \mathrm{Mod}_{R_*,f}^{Cpl(I)} \\ & \swarrow^{\iota} & \downarrow & & \downarrow \\ \mathrm{Mod}_R^{Cpl(I)} & \xleftarrow{L\iota} & P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)}) & \xrightarrow{Lh} & P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)}) \end{array}$$

such that both $L\iota$ and Lh preserve geometric realisations.

Let S be the category $(\bullet \leftarrow \bullet \rightarrow \bullet)$ and consider the diagram

$$\mathrm{Mod}_R^{Cpl(I)} \xleftarrow{L\iota} P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)}) \xrightarrow{Lh} P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)})$$

as an element in the ∞-category $\mathrm{Fun}(\mathrm{N}(S), \widehat{Cat}_\infty)$.

Definition 3.2.5. The coCartesian fibration $\mathcal{B} \xrightarrow{p} \mathrm{N}(S)$ obtained from this functor by unstraightening is called the *resolution bridge*.

The three categories appearing as fibres of p admit small colimits. The functors $L\iota$ and Lh preserve geometric realisations and ι and h preserve small coproducts. By Proposition 4.2.12 and Proposition 4.2.11.(3) in [Lur11a], we conclude that both $L\iota$ and Lh preserve all small colimits. Corollary 4.3.1.11 in [Lur09] implies that the category $\mathcal{B}$ admits all small p-colimits.

Definition 3.2.6. We write $\mathrm{End}_S^c(\mathcal{B})$ for the full monoidal subcategory of $\mathrm{End}_S(\mathcal{B})$ spanned by all functors which preserve coCartesian edges.

We will make use of the following result of Shah [Sha17]:

Lemma 3.2.7. *Let $p : C \to S$ be a coCartesian fibration of ∞-categories. Suppose we are given a diagram $\phi : K \to \mathrm{Fun}_S(C, D)$, and suppose that for every $x \in C$, the evaluation functor $\mathrm{ev}_x \phi : K \to D_{p(x)}$ has a colimit, and for every $f : x \to y$, the canonical map $\mathrm{colim}_K(\mathrm{ev}_{f_!x}\,\phi) \to f_! \,\mathrm{colim}_K\,(ev_x\phi)$ is an equivalence. Then $\mathrm{colim}_K\phi : C \to D$ exists and is computed on objects $x \in C$ by $\mathrm{colim}_K(ev_x\phi)$.*

Corollary 3.2.8. *The ∞-category $\mathrm{End}_S^c(\mathcal{B})$ admits geometric realisations. Given some $c \in \mathcal{B}$, the evaluation functor ev_c preserves them.*

Definition 3.2.9. The monoidal ∞-category $\mathrm{End}_S^{c,\sigma}(\mathcal{B})$ of *bridged endofunctors* of $\mathrm{Mod}_R^{Cpl(I)}$ is given by the full subcategory of $\mathrm{End}_S^c(\mathcal{B})$ spanned by all functors which preserve p-geometric realisations.

The functor category $\mathrm{End}_S^{c,\sigma}(\mathcal{B})$ is closed under geometric realisations.

Proposition 3.2.10. *Restriction gives rise to a diagram of strictly monoidal realisation preserving functors*

$$\begin{array}{ccccc} & & \mathrm{End}^{c,\sigma}_S(\mathcal{B}) & & \\ & \swarrow^{p_1} & \downarrow^{p_2} & \searrow^{p_3} & \\ \mathrm{End}^\sigma(\mathrm{Mod}_R^{Cpl(I)}) & & \mathrm{End}^\sigma(P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)})) & & \mathrm{End}^\sigma(P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)})) \end{array}$$

Proof. Since the induced functors $L\iota$ and Lh between the fibres of p preserve geometric realisations, Proposition 4.3.1.10 in [Lur09] implies that restricting an endofunctor which preserves p-geometric realisations to one of the three fibres of $p : \mathcal{B} \to S$ yields a functor which preserves realisations. The restriction functors are evidently strictly monoidal. □

Remark 3.2.11. Informally, a bridged endofunctor is therefore given by a diagram

$$\begin{array}{ccccc} \mathrm{Mod}_R^{Cpl(I)} & \xleftarrow{L\iota} & P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)}) & \xrightarrow{Lh} & P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)}) \\ {\scriptstyle A}\downarrow & & {\scriptstyle B}\downarrow & & \downarrow{\scriptstyle C} \\ \mathrm{Mod}_R^{Cpl(I)} & \xleftarrow{L\iota} & P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)}) & \xrightarrow{Lh} & P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)}) \end{array}$$

where A,B, and C preserve geometric realisations.

We see that every algebra object $T \in \mathrm{Alg}(\mathrm{End}^{c,\sigma}_S(\mathcal{B}))$ gives rise to three monads

$$p_1(T) \in \mathrm{Alg}(\mathrm{End}^\sigma(\mathrm{Mod}_R^{Cpl(I)})), \quad p_2(T) \in \mathrm{Alg}(\mathrm{End}^\sigma(P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)}))), \quad p_3(T) \in \mathrm{Alg}(\mathrm{End}^\sigma(P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)})))$$

We will now construct functors

$$\mathrm{Alg}_{p_1(T)}\left(\mathrm{Mod}_R^{Cpl(I)}\right) \longleftarrow \mathrm{Alg}_{p_2(T)}\left(P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)})\right) \longrightarrow \mathrm{Alg}_{p_3(T)}\left(P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)})\right)$$

For this, we introduce a technical gadget:

Definition 3.2.12. The ∞-category $\mathcal{E}$ of *bridged I-complete R-module spectra* consists of the ∞-category of coCartesian sections of $\mathcal{B} \to S$.

We have a natural diagram

$$\begin{array}{ccccc} & & \mathcal{E} & & \\ & \swarrow & \downarrow & \searrow & \\ \mathrm{Mod}_R^{Cpl(I)} & & P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)}) & & P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)}) \end{array} \tag{3.1}$$

Remark 5.4.7.16. in [Lur09] implies the following fact:

Proposition 3.2.13. *Evaluating a section on the middle object of S defines an equivalence $\mathcal{E} \xrightarrow{\simeq} P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)})$.*

Remark 3.2.14. Thinking of an object in $P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)})$ as a formal geometric realisation of a simplicial diagram $X_\bullet : \mathrm{N}(\Delta^{op}) \to \mathrm{Mod}_{R,f}^{Cpl(I)}$, we can think of bridged module as a triple $(|X_\bullet|, X_\bullet, \pi_*(X_\bullet))$.

In order to construct the aforementioned functors, we observe the following diagram in the ordinary category $\widehat{\mathbf{CatMod}}^{ord}$ of strictly tensored ∞-categories introduced in Section 5.2.3 in Appendix B:

$$
\begin{array}{ccccc}
 & & (\mathrm{End}_S^{c,\sigma}(\mathcal{B}), \mathcal{E}) & & \\
 & \swarrow & \downarrow \cong & \searrow & \\
(\mathrm{End}_S^{c,\sigma}(\mathcal{B}), \mathrm{Mod}_R^{Cpl(I)}) & & (\mathrm{End}_S^{c,\sigma}(\mathcal{B}), P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)})) & & (\mathrm{End}_S^{c,\sigma}(\mathcal{B}), P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)})) \\
\downarrow & & \downarrow & & \downarrow \\
 & & (\mathrm{End}^\sigma(P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)})), P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)})) & & \\
(\mathrm{End}^\sigma(\mathrm{Mod}_R^{Cpl(I)}), \mathrm{Mod}_R^{Cpl(I)}) & & & & (\mathrm{End}^\sigma(P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)})), P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)}))
\end{array}
$$

Applying the construction $\Theta : N(\widehat{\mathbf{CatMod}}^{ord}) \to \widehat{\mathrm{CatMod}}$ defined in Section 5.2.3 in Appendix B, we obtain a diagram of tensored ∞-categories in the sense of Definition 5.2.4 (in the interest of readability, we will drop the arrows and just use commata instead):

$$
\begin{array}{ccccc}
 & & (\mathcal{E}^\circledast, \mathrm{End}_S^{c,\sigma}(\mathcal{B})^\circledast) & & \\
 & \swarrow & \downarrow \cong & \searrow & \\
(\mathrm{Mod}_R^{Cpl(I)^\circledast}, \mathrm{End}_S^{c,\sigma}(\mathcal{B})^\circledast) & & (P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)})^\circledast, \mathrm{End}_S^{c,\sigma}(\mathcal{B})^\circledast) & & (P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)})^\circledast, \mathrm{End}_S^{c,\sigma}(\mathcal{B})^\circledast) \\
\downarrow{\scriptstyle restrictive} & & {\scriptstyle restrictive}\downarrow & & {\scriptstyle restrictive}\downarrow \\
 & & (P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)})^\circledast, \mathrm{End}^\sigma(P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)}))^\circledast) & & \\
(\mathrm{Mod}_R^{Cpl(I)\circledast}, \mathrm{End}^\sigma(\mathrm{Mod}_R^{Cpl(I)})^\circledast) & & & & (P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)})^\circledast, \mathrm{End}^\sigma(P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)}))^\circledast)
\end{array}
$$

The lower three vertical arrows exhibit the upper tensored ∞-categories as obtained by restriction from the lower ones by Lemma 5.2.15 in Appendix B.

Proposition 3.2.15. *The top middle functor $\mathcal{E}^\circledast \to (P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)}))^\circledast$ in the above diagram is an equivalence of $\mathrm{End}_S^{c,\sigma}(\mathcal{B})$-tensored ∞-categories.*

Proof. By Corollay 4.2.3.2 in [Lur14], it suffices to check that the induced map of underlying ∞-categories is an equivalence, which holds true by Proposition 3.2.13 above □

Lemma 3.2.16. *Given an algebra object $T \in \mathrm{Alg}(\mathrm{End}_S^{c,\sigma}(\mathcal{B}))$, we obtain a diagram*

$$
\begin{array}{ccccc}
 & & \mathrm{Alg}_T(\mathcal{E}) & & \\
 & \swarrow & \cong\downarrow & \searrow & \\
\mathrm{Alg}_T\left(\mathrm{Mod}_R^{Cpl(I)}\right) & & \mathrm{Alg}_T\left(P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)})\right) & & \mathrm{Alg}_T\left(P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)})\right) \\
\cong\downarrow & & \cong\downarrow & & \cong\downarrow \\
\mathrm{Alg}_{p_1(T)}\left(\mathrm{Mod}_R^{Cpl(I)}\right) & & \mathrm{Alg}_{p_2(T)}\left(P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)})\right) & & \mathrm{Alg}_{p_3(T)}\left(P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)})\right)
\end{array}
$$

By construction, this diagram lies over diagram (3.1) on p.61.

Proof. The top middle arrow is an equivalence by Proposition 3.2.15. The lower three vertical arrows are equivalences by Lemma 5.2.16 in Appendix B. □

We have therefore defined functors

$$
\mathrm{Alg}_{p_1(T)}\left(\mathrm{Mod}_R^{Cpl(I)}\right) \longleftarrow \mathrm{Alg}_{p_2(T)}\left(P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)})\right) \longrightarrow \mathrm{Alg}_{p_3(T)}\left(P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)})\right)
$$

We can set up similar functors for comonads. Indeed, we apply $(-)^{op} : \widehat{\mathrm{CatMod}} \to \widehat{\mathrm{CatMod}}$ to the second diagram of the last paragraph to obtain a diagram of tensored ∞-categories (again, we indicate arrows by commata):

$$
\begin{array}{ccccc}
 & & (\mathcal{E}^{op\circledast}, \mathrm{End}_S^{c,\sigma}(\mathcal{B})^{op\circledast}) & & \\
 & \swarrow & \cong\downarrow & \searrow & \\
(\mathrm{Mod}_R^{Cpl(I)op\circledast}, \mathrm{End}_S^{c,\sigma}(\mathcal{B})^{op\circledast}) & & (P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)})^{op\circledast}, \mathrm{End}_S^{c,\sigma}(\mathcal{B})^{op\circledast}) & & (P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)})^{op\circledast}, \mathrm{End}_S^{c,\sigma}(\mathcal{B})^{op\circledast}) \\
\downarrow{\scriptstyle restrictive} & & {\scriptstyle restrictive}\downarrow & & {\scriptstyle restrictive}\downarrow \\
(\mathrm{Mod}_R^{Cpl(I)op\circledast}, \mathrm{End}^\sigma(\mathrm{Mod}_R^{Cpl(I)})^{op\circledast}) & & (P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)})^{op\circledast}, \mathrm{End}^\sigma(P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)}))^{op\circledast}) & & (P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)})^{op\circledast}, \mathrm{End}^\sigma(P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)}))^{op\circledast})
\end{array}
$$

The lower three vertical arrows are again restrictive by Proposition 5.2.14 in Appendix B.

Proposition 3.2.17. *The top middle functor $(\mathcal{E}^{op})^\circledast \to (P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)})^{op})^\circledast$ in the above diagram is an equivalence of $(\mathrm{End}_S^{c,\sigma,f}(\mathcal{B}))^{op}$-tensored ∞-categories.*

Proof. Identical to proof of Proposition 3.2.15. □

Lemma 3.2.18. *Given $C \in \mathrm{coAlg}(\mathrm{End}_S^{c,\sigma}(\mathcal{B})) = (\mathrm{Alg}(\mathrm{End}_S^{c,\sigma}(\mathcal{B})^{op}))^{op}$, we obtain a natural diagram*

$$\begin{array}{ccccc}
 & & \mathrm{coAlg}_C(\mathcal{E}) & & \\
 & \swarrow & \cong\downarrow & \searrow & \\
\mathrm{coAlg}_C\left(\mathrm{Mod}_R^{Cpl(I)}\right) & & \mathrm{coAlg}_C\left(P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)})\right) & & \mathrm{coAlg}_C\left(P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)})\right) \\
\cong\downarrow & & \cong\downarrow & & \cong\downarrow \\
\mathrm{coAlg}_{p_1(C)}\left(\mathrm{Mod}_R^{Cpl(I)}\right) & & \mathrm{coAlg}_{p_2(C)}\left(P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)})\right) & & \mathrm{coAlg}_{p_3(C)}\left(P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)})\right)
\end{array}$$

This diagram lies over diagram (3.1) from p.61.

We have therefore defined the following useful functors:

$$\mathrm{coAlg}_{p_1(C)}\left(\mathrm{Mod}_R^{Cpl(I)}\right) \longleftarrow \mathrm{coAlg}_{p_2(C)}\left(P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)})\right) \longrightarrow \mathrm{coAlg}_{p_3(C)}\left(P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)})\right)$$

Completed-Free (co)Monads. We shall now focus on a particularly convenient class of endofunctors:

Definition 3.2.19. An endofunctor $T \in \mathrm{End}(\mathrm{Mod}_R^{Cpl(I)})$ is said to be *completed-free* if it preserves the full subcategory $\mathrm{Mod}_{R,f}^{Cpl(I)}$ of completed-free I-complete R-module spectra in the sense of Definition 5.1.13. We write $\mathrm{End}^f(\mathrm{Mod}_{R,f}^{Cpl(I)})$ for the full monoidal subcategory spanned by all such functors.

Definition 3.2.20. The ∞-category $\mathrm{End}_S^{c,\sigma,f}(\mathcal{B})$ of *completed-free* bridged endofunctors is given by the full subcategory of $\mathrm{End}_S^{c,\sigma}(\mathcal{B})$ spanned by all functors whose restriction to $P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)})$ preserves $\mathrm{Mod}_{R,f}^{Cpl(I)}$.

Every realisation-preserving completed-free $T \in \mathrm{End}^{\sigma,f}(\mathrm{Mod}_R^{Cpl(I)})$ determines a $T_{\mathcal{B}} \in \mathrm{End}^{\sigma,c,f}(\mathfrak{B})$ given by

$$\begin{array}{ccccc}
\mathrm{Mod}_R^{Cpl(I)} & \xleftarrow{L\iota} & P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)}) & \xrightarrow{Lh} & P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)}) \\
T\downarrow & & \mathfrak{T}\downarrow{:=}LT|_{\mathrm{Mod}_{R,f}^{Cpl(I)}} & & \hat{\mathbb{T}}\downarrow{:=}LhT|_{\mathrm{Mod}_{R,f}^{Cpl(I)}} \\
\mathrm{Mod}_R^{Cpl(I)} & \xleftarrow{L\iota} & P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)}) & \xrightarrow{Lh} & P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)})
\end{array}$$

This observation implies that the restriction functor $\mathrm{End}_S^{c,\sigma,f}(\mathcal{B}) \to \mathrm{End}^{\sigma,f}(\mathrm{Mod}_R^{Cpl(I)})$ is an equivalence of monoidal ∞-categories. We write $(-)_{\mathfrak{B}}$ for the inverse functor.

Definition 3.2.21. The *analytic approximation* $\hat{\mathbb{T}}$ to a completed-free monad $T \in \mathrm{Alg}(\mathrm{End}^{\sigma,f}(\mathrm{Mod}_R^{Cpl(I)}))$ is given by $\hat{\mathbb{T}} = LhT|_{\mathrm{Mod}_{R,f}^{Cpl(I)}}$, i.e. by the image of T under the composition of the strictly monoidal functors:

$$\mathrm{End}^{\sigma,f}(\mathrm{Mod}_R^{Cpl(I)}) \xrightarrow{\simeq} \mathrm{End}_S^{c,\sigma,f}(\mathcal{B}) \to \mathrm{End}(P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)}))$$

The ∞-category $P_\sigma(\mathrm{Mod}_{R_*,f})$ is in fact prestable, and we therefore have an embedding into the stabilisation $Sp(P_\sigma(\mathrm{Mod}_{R_*,f}))$. The heart of the natural t-structure turns out to be $P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)})^\heartsuit \cong \mathrm{Mod}_{R_*}^{Cpl(I)}$ (see Proposition 4.2.4), but we will not need this for our considerations.

Operations on algebras over completed-free monads can be understood well in terms of graded R_*-modules since there is an evident equivalence of $\mathbb{Z}$-graded algebraic theories $\mathcal{P}_T \cong \mathcal{P}_{h\hat{\mathbb{T}}}$.

Dually, we can define the analytic approximation $\hat{\mathbb{C}}$ to a completed-free comonad $C \in \mathrm{coAlg}(\mathrm{End}^{\sigma,f}(\mathrm{Mod}_R^{Cpl(I)}))$, and the $\mathbb{Z}$-graded algebraic theory associated with C agrees with the algebraic theory for $\hat{\mathbb{C}}$.

Bridged Koszul Duality. Given an ∞-category $\mathcal{C}$ containing geometric realisations, we can apply Theorem 4.3.1 of [Lur11b] to the monoidal ∞-category $\mathrm{End}^{aug}(\mathcal{C})_{\mathrm{id}\,/-/\,\mathrm{id}}$ and obtain the Koszul duality functor[2] $\mathrm{Alg}^{aug}(\mathrm{End}(\mathcal{C})) \xrightarrow{\mathrm{KD}} \mathrm{coAlg}^{aug}(\mathrm{End}(\mathcal{C}))$ from monads to comonads. The underlying functor of the Koszul dual of T is given by $|\,\mathrm{Bar}_\bullet(T)|$. By Example 4.4.19 in [Lur11b], the restrictions define commutative squares

$$\begin{array}{ccc} \mathrm{Alg}^{aug}(\mathrm{End}_S^{c,\sigma}(\mathfrak{B})) & \xrightarrow{\mathrm{KD}} & \mathrm{coAlg}^{aug}(\mathrm{End}_S^{c,\sigma}(\mathfrak{B})) \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_1} \\ \mathrm{Alg}^{aug}(\mathrm{End}^\sigma(\mathrm{Mod}_R^{Cpl(I)})) & \xrightarrow{\mathrm{KD}} & \mathrm{coAlg}^{aug}(\mathrm{End}^\sigma(\mathrm{Mod}_R^{Cpl(I)})) \end{array}$$

$$\begin{array}{ccc} \mathrm{Alg}^{aug}(\mathrm{End}_S^{c,\sigma}(\mathfrak{B})) & \xrightarrow{\mathrm{KD}} & \mathrm{coAlg}^{aug}(\mathrm{End}_S^{c,\sigma}(\mathfrak{B})) \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_2} \\ \mathrm{Alg}^{aug}(\mathrm{End}^\sigma(P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)}))) & \xrightarrow{\mathrm{KD}} & \mathrm{coAlg}^{aug}(\mathrm{End}^\sigma(P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)}))) \end{array}$$

$$\begin{array}{ccc} \mathrm{Alg}^{aug}(\mathrm{End}_S^{c,\sigma}(\mathfrak{B})) & \xrightarrow{\mathrm{KD}} & \mathrm{coAlg}^{aug}(\mathrm{End}_S^{c,\sigma}(\mathfrak{B})) \\ {\scriptstyle p_3}\downarrow & & \downarrow{\scriptstyle p_3} \\ \mathrm{Alg}^{aug}(\mathrm{End}^\sigma(P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)}))) & \xrightarrow{\mathrm{KD}} & \mathrm{coAlg}^{aug}(\mathrm{End}^\sigma(P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)}))) \end{array}$$

We fix an augmented realisation-preserving completed-free monad $T \in \mathrm{Alg}^{aug}(\mathrm{End}^{\sigma,f}(\mathrm{Mod}_R^{Cpl(I)}))$ with associated bridged monad $T_{\mathfrak{B}} \in \mathrm{Alg}^{aug}(\mathrm{End}^{c,\sigma,f}(\mathrm{Mod}_R^{Cpl(I)}))$ and write $\mathfrak{T} = LT|_{\mathrm{Mod}_{R,f}^{Cpl(I)}}$ and $\hat{\mathbb{T}} = LhT|_{\mathrm{Mod}_{R_*,f}^{Cpl(I)}}$. We introduce notation for the comonads obtained by restricting the Koszul dual of $T_{\mathfrak{B}}$:

$$C := p_1(\mathrm{KD}(T_{\mathfrak{B}})) = \mathrm{KD}(T) \in \mathrm{coAlg}^{aug}(\mathrm{End}(\mathrm{Mod}_R^{Cpl(I)}))$$

$$\mathfrak{C} := p_2(\mathrm{KD}(T_{\mathfrak{B}})) = \mathrm{KD}(\mathfrak{T}) \in \mathrm{coAlg}^{aug}(\mathrm{End}(P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)})))$$

$$\hat{\mathbb{C}} := p_3(\mathrm{KD}(T_{\mathfrak{B}})) = \mathrm{KD}(\hat{\mathbb{T}}) \in \mathrm{coAlg}^{aug}(\mathrm{End}(P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)})))$$

[2]We will differ in notation and write KD instead of $\mathfrak{D}_\lambda$.

Observe that our definitions are set up to make the following diagram commute:

$$\begin{array}{ccccc}
\mathrm{Mod}_R^{Cpl(I)} & \xleftarrow{Li} & P_\sigma(\mathrm{Mod}_R^{Cpl(I)}) & \xrightarrow{Lh} & P_\sigma(\mathrm{Mod}_{R_*}^{Cpl(I)}) \\
{\scriptstyle C}\downarrow & & {\scriptstyle \mathfrak{C}}\downarrow & & {\scriptstyle \hat{\mathbb{C}}}\downarrow \\
\mathrm{Mod}_R^{Cpl(I)} & \xleftarrow{Li} & P_\sigma(\mathrm{Mod}_R^{Cpl(I)}) & \xrightarrow{Lh} & P_\sigma(\mathrm{Mod}_{R_*}^{Cpl(I)})
\end{array}$$

Lemma 3.2.18 from the last section gives natural functors

$$\mathrm{coAlg}_C(\mathrm{Mod}_R^{Cpl(I)}) \longleftarrow \mathrm{coAlg}_{\mathfrak{C}}(P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)})) \longrightarrow \mathrm{coAlg}_{\hat{\mathbb{C}}}(P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)}))$$

Our principal aim is to study unary operations on the cohomotopy groups of C-coalgebras, i.e. to understand the full subcategory of $\mathrm{coAlg}_C(\mathrm{Mod}_R^{Cpl(I)})$ spanned by free coalgebras $C(\Sigma^i R)$ on some suspension of the unit R. For this, it is convenient to have a large family of preimages of $\Sigma^i R$ under the geometric realisation functor $P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)}) \to \mathrm{Mod}_R$ at our disposal.

Notation 3.2.22. Given a socle $\mathcal{C}$, the a^{th} suspension of an object X computed in the pointed ∞-category $P_\sigma(\mathcal{C})$ is denoted by $S^a \wedge X$ and called the a^{th} *simplicial suspension* of X.

Warning. *If X is an object of the full subcategory $\mathrm{Mod}_{R,f}^{Cpl(I)} \hookrightarrow P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)})$, then the a^{th} simplicial suspension $S^a \wedge X$ is different from $\Sigma^a X$, which denotes the ∞-categorical suspension of X computed in $\mathrm{Mod}_R^{Cpl(I)}$.*

However, the natural functors $\mathrm{Mod}_R^{Cpl(I)} \leftarrow P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)}) \to P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)})$ preserve small colimits and we therefore see that simplicial suspension in the middle goes to ordinary suspension on the left and simplicial suspension on the right. Given a completed-free R-module spectrum $X \in \mathrm{Mod}_{R,f}^{Cpl(I)}$ and some nonnegative integer a, we can define a bridged module which we can informally write as $(\Sigma^a X, S^a \wedge X, S^a \wedge \pi_*(X))$.

We introduce the following notation for certain groups of operations:

Definition 3.2.23. Given integers i, j and nonnegative integers a, b, we write

$$Q_i^j = Q_i^j(T) := \pi_0 \mathrm{Map}_{\mathrm{coAlg}(C)}(C(\Sigma^i R), C(\Sigma^j R))$$

$$\mathfrak{Q}_{S^a(i)}^{S^b(j)} = \mathfrak{Q}_{S^a(i)}^{S^b(j)}(T) := \pi_0 \mathrm{Map}_{\mathrm{coAlg}(\mathfrak{C})}(\mathfrak{C}(S^a \wedge \Sigma^i R), \mathfrak{C}(S^b \wedge \Sigma^j R))$$

$$\mathbb{Q}_{S^a(i)}^{S^b(j)} = \mathbb{Q}_{S^a(i)}^{S^b(j)}(T) := \pi_0 \mathrm{Map}_{\mathrm{coAlg}(\hat{\mathbb{C}})}(\hat{\mathbb{C}}(S^a \wedge \Sigma^i R_*), \hat{\mathbb{C}}(S^b \wedge \Sigma^j R_*))$$

Using functoriality of the construction in Lemma 3.2.18, we obtain a diagram for all $i, j, k \in \mathbb{Z}$ and $a, b \in \mathbb{Z}_{\geq 0}$:

$$\begin{array}{ccccc}
Q_i^{j+a} & \times & Q_{j+a}^{k+a+b} & \longrightarrow & Q_i^{k+a+b} \\
\uparrow & & \uparrow & & \uparrow \\
\mathfrak{Q}_{S^0(i)}^{S^a(j)} & \times & \mathfrak{Q}_{S^a(j)}^{S^{a+b}(k)} & \longrightarrow & \mathfrak{Q}_{S^0(i)}^{S^{a+b}(k)} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Q}_{S^0(i)}^{S^a(j)} & \times & \mathbb{Q}_{S^a(j)}^{S^{a+b}(k)} & \longrightarrow & \mathbb{Q}_{S^0(i)}^{S^{a+b}(k)}
\end{array}$$

The bottom row is determined algebraically in terms of the functor $\hat{\mathbb{T}}$ and therefore should be thought of as the *computable* part of this diagram.

Our rough strategy in the next sections is to first lift elements along the lower vertical maps and then compose them. Unfortunately, none of the bottom arrows is an isomorphism in examples of interest and we therefore have many possible lifts. In the next sections, we shall address this difficulty.

3.3 Weighted Structures

Fix a completed-free realisation-preserving augmented monad $T \in \mathrm{Alg}^{aug}(\mathrm{End}^{\sigma,f}(\mathrm{Mod}_R^{Cpl(I)}))$ and write $C = \mathrm{KD}(T) = |\,\mathrm{Bar}_\bullet(T)|$ for its Koszul dual comonad.

Definition 3.3.1. A *weighted structure* on T consists of a collection $\{B_\bullet^{[w]} \to \mathrm{Bar}_\bullet(T)\}_{w\in\mathbb{Z}_{\geq 1}}$ of simplicial completed-free realisation-preserving endofunctors over $\mathrm{Bar}_\bullet(T)$ such that

1. The induced map $\bigoplus_w B_\bullet^{[w]} \to \mathrm{Bar}_\bullet(T)$ is an equivalence.
2. For all r, s, and t in $\mathbb{Z}_{\geq 1}$ with $t \neq r \cdot s$, the maps

$$C[t] \to C \to C \circ C \to C[r] \circ C[s], \qquad T[r] \circ T[s] \to T \circ T \to T \to T[t]$$

   are null for $C[w] = |B_\bullet^{[w]}|$ and $T[w] = B_1^{[w]}$.

Given a weighted structure on T, we obtain a decomposition of $\mathrm{KD}(T_{\mathfrak{B}}) = |\,\mathrm{Bar}_\bullet(T_{\mathfrak{B}})| = |\,\mathrm{Bar}_\bullet(T)_{\mathfrak{B}}|$. Applying p_1, p_2, and p_3, this gives rise to corresponding decompositions

$$C = \mathrm{KD}(T) \cong \bigoplus_w C[w], \qquad \mathfrak{C} = \mathrm{KD}(\mathfrak{T}) \cong \bigoplus_w \mathfrak{C}[w], \qquad \hat{\mathbb{C}} = \mathrm{KD}(\hat{\mathbb{T}}) \cong \bigoplus_w \hat{\mathbb{C}}[w]$$

The weighted structure on T hence allows us to define *weighted* operations:

Definition 3.3.2. Given integers i, j, nonnegative integers a, b, and w, we define

$$Q_i^j[w] := \pi_0 \,\mathrm{Map}_{\mathrm{Mod}_R^{Cpl(I)}}(C[w](\Sigma^i R), \Sigma^j R)$$

$$\mathfrak{Q}_{S^a(i)}^{S^b(j)}[w] := \pi_0 \,\mathrm{Map}_{P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)})}(\mathfrak{C}[w](S^a \wedge \Sigma^i R), (S^b \wedge \Sigma^j R))$$

$$\mathbb{Q}_{S^a(i)}^{S^b(j)}[w] := \pi_0 \,\mathrm{Map}_{P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)})}(\hat{\mathbb{C}}[w](S^a \wedge \Sigma^i R_*), S^b \wedge \Sigma^j R_*)$$

The direct sum decompositions of $C, \mathfrak{C}$, and $\hat{\mathbb{C}}$ give natural product decompositions

$$Q_i^j \cong \prod_{w\geq 1} Q_i^j[w], \quad \mathfrak{Q}_{S^a(i)}^{S^b(j)} \cong \prod_{w\geq 1} \mathfrak{Q}_{S^a(i)}^{S^b(j)}[w], \quad \mathbb{Q}_{S^a(i)}^{S^b(j)} \cong \prod_{w\geq 1} \mathbb{Q}_{S^a(i)}^{S^b(j)}[w]$$

Zero elements give inclusions from the individual factors into these products.

The maps $Q_{i+a}^{j+b} \leftarrow \mathfrak{Q}_{S^a(i)}^{S^b(j)} \to \mathbb{Q}_{S^a(i)}^{S^b(j)}$ are induced by products of evident maps on the factors.

The Canonical Lifting. We fix an augmented completed-free monad $T \in \mathrm{Alg}^{aug}(\mathrm{End}^{\sigma,f}(\mathrm{Mod}_R^{Cpl(I)}))$ endowed with a weighted structure $\mathrm{Bar}_\bullet(T) = \bigoplus_w B_\bullet^{[w]}$. On p.64, we have constructed functors

$$\mathrm{coAlg}_C\left(\mathrm{Mod}_R^{Cpl(I)}\right) \longleftarrow \mathrm{coAlg}_{\mathfrak{C}}\left(P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)})\right) \longrightarrow \mathrm{coAlg}_{\hat{\mathbb{C}}}\left(P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)})\right)$$

In order to study lifting properties of maps under the right functor, we introduce a technical definition. Given a completed-free R_*-module M and a weight w, we can consider the chain complex of completed-free R_*-modules $hB_*^{[w]}(M)$ corresponding to the simplicial object $hB_\bullet^{[w]}(M) \in s\,\mathrm{Mod}_{R_*,f}^{Cpl(I)}$.

Definition 3.3.3. The completed-free R_*-module M is said to be *p-Koszul* for T if the cohomology of the complex $H_s(hB_*^{[w]}(M))$ vanishes whenever $s \neq \log_p(w)$ and is completed-free if $s = \log_p(w)$.

Remark 3.3.4. Observe that if M is p-Koszul, then $H_s(hB_*^{[w]}(M)) = 0$ for all s whenever w is not a power of p.

Lemma 3.3.5. *Assume $K \in \mathrm{Mod}_{R,f}^{Cpl(I)}$ is such that $\pi_*(K)$ is p-Koszul. Let $j \in \mathbb{Z}$. Then:*

1. $\pi_0 \mathrm{Map}_{\mathcal{P}_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)})}(\mathfrak{C}[w](K), S^a \wedge \Sigma^j R) \to \pi_0 \mathrm{Map}_{\mathcal{P}_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)})}(\hat{\mathbb{C}}[w](\pi_* K), S^a \wedge \Sigma^j R_*)$ *is an isomorphism if $a = \log_p(w)$. If $a \neq \log_p(w)$, the right hand side vanishes.*
2. $\pi_0 \mathrm{Map}_{\mathcal{P}_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)})}(\mathfrak{C}[w](K), S^a \wedge \Sigma^j R) \to \pi_0 \mathrm{Map}_{\mathrm{Mod}_R^{Cpl(I)}}(C[w](K), \Sigma^{j+a} R)$ *is an isomorphism whenever $a \leq \log_p(w)$. If $a > \log_p(w)$, the left hand side vanishes.*
3. *If w is not a power of p, then all of the above groups vanish. If $w = p^a$, all three groups are isomorphic to* $\mathrm{Map}_{\mathrm{Mod}_{R_*,f}^{Cpl(I)}}(H_a(hB_\bullet^{[w]}(\pi_* K)), \Sigma^j R_*)$.

We spell out Lemma 3.3.5 in the specific case of interest to us (using Definition 3.3.2):

Corollary 3.3.6. *If $\Sigma^i R_*$ is p-Koszul for T at weight w, then:*

1. $\mathfrak{Q}_{S^0(i)}^{S^a(j)}[w] \to \mathbb{Q}_{S^0(i)}^{S^a(j)}[w]$ *is an isomorphism if $a = \log_p(w)$. If $a \neq \log_p(w)$, the right hand side vanishes.*
2. $\mathfrak{Q}_{S^0(i)}^{S^a(j)}[w] \to Q_i^{j+a}[w]$ *is an isomorphism if $a \leq \log_p(w)$. If $a > \log_p(w)$, the left hand side vanishes.*
3. *If w is not a power of p, then $\mathfrak{Q}_{S^0(i)}^{S^a(j)}[w]$, $\mathbb{Q}_{S^0(i)}^{S^a(j)}[w]$, $Q_i^{a+j}[w]$ vanish. If $w = p^a$, then all three groups are isomorphic.*

We first deduce a useful statement:

Corollary 3.3.7. *If $\Sigma^i R_*$ is p-Koszul for T, then there are natural factorisations*

$$\begin{array}{ccc}
\mathfrak{Q}_{S^0(i)}^{S^a(j)}[v] \times \mathfrak{Q}_{S^a(j)}^{S^b(k)}[w] & \dashrightarrow & \mathfrak{Q}_{S^0(i)}^{S^b(k)}[vw] \\
\downarrow & & \downarrow \\
\mathfrak{Q}_{S^0(i)}^{S^a(j)} \times \mathfrak{Q}_{S^a(j)}^{S^b(k)} & \longrightarrow & \mathfrak{Q}_{S^0(i)}^{S^b(k)}
\end{array}$$

Proof of 3.3.7. Given some weight $t \neq v \cdot w$, we consider the diagram

$$\begin{array}{ccc}
\mathfrak{Q}^{S^a(j)}_{S^0(i)}[v] \times \mathfrak{Q}^{S^b(k)}_{S^a(j)}[w] & \longrightarrow & Q^{a+j}_i[v] \times Q^{b+k}_{a+j}[w] \\
\downarrow & & \downarrow \\
\mathfrak{Q}^{S^b(k)}_{S^0(i)} & \longrightarrow & Q^{b+k}_i \\
\downarrow & & \downarrow \\
\mathfrak{Q}^{S^b(k)}_{S^0(i)}[t] & \longrightarrow & Q^{b+k}_i[t]
\end{array}$$

The long arrow on the right is null by condition (2) of the definition of weighted structures. The lowest horizontal map is an injection by Corollary 3.3.6 (2). □

Proof of 3.3.5. The map in part (1) can be obtained by applying π_0 to the totalisation of the following map $X^\bullet \to Y^\bullet$ of (pointed) cosimplicial spaces:

$$\mathrm{Map}_{\mathcal{P}_\sigma(\mathrm{Mod}^{Cpl(I)}_{R,f})}(B^{[w]}_\bullet(K), S^a \wedge \Sigma^j R) \to \mathrm{Map}_{\mathcal{P}_\sigma(\mathrm{Mod}^{Cpl(I)}_{R_*,f})}(hB^{[w]}_\bullet(K), S^a \wedge \Sigma^j R_*)$$

Let B^a denote the a-fold (connected) delooping of topological monoids (this is of course not related to $B^{[w]}_\bullet$) and write $N^*G^\bullet$ for the normalised cochain complex of a cosimplicial abelian group $G^\bullet$.

The E_1-page of the Bousfield-Kan spectral sequence for $X^\bullet$ has for $t \geq s \geq 0$:

$$E^{s,t}_1 = N^s\pi_t(\mathrm{Map}_{\mathcal{P}_\sigma(\mathrm{Mod}^{Cpl(I)}_{R,f})}(B^{[w]}_\bullet(K), S^a \wedge \Sigma^j R)) = N^s\pi_t(B^a \,\mathrm{Map}_{\mathrm{Mod}^{Cpl(I)}_{R,f}}(B^{[w]}_\bullet(K), \Sigma^j R))$$

$$= \begin{cases} N^s\pi_{t-a}(\mathrm{Map}_{\mathrm{Mod}^{Cpl(I)}_{R,f}}(B^{[w]}_\bullet(K), \Sigma^j R)) & \text{if } t \geq a \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} N^s\pi_0(\mathrm{Map}_{\mathrm{Mod}^{Cpl(I)}_{R,f}}(B^{[w]}_\bullet(K), \Sigma^{j+a-t} R)) & \text{if } t \geq a \\ 0 & \text{else} \end{cases}$$

$$\cong \begin{cases} N^s \,\mathrm{Map}_{\mathrm{Mod}^{Cpl(I)}_{R_*,f}}(hB^{[w]}_\bullet(\pi_* K), \Sigma^{j+a-t} R_*)) & \text{if } t \geq a \\ 0 & \text{else} \end{cases}$$

Hence for $t \geq s \geq 0$, we have

$$E^{s,t}_2 \cong \begin{cases} \pi^s \,\mathrm{Map}_{\mathrm{Mod}^{Cpl(I)}_{R_*,f}}(hB^{[w]}_\bullet(\pi_* K), \Sigma^{j+a-t} R_*)) & \text{if } t \geq a \\ 0 & \text{else} \end{cases}$$

Since the homology of the complex $hB_\bullet^{[w]}(\pi_* K)$ is projective in the abelian category $\mathrm{Mod}_{R_*}^{Cpl(I)}$, the universal coefficient spectral sequence implies that

$$E_2^{s,t} \cong \begin{cases} \mathrm{Map}_{\mathrm{Mod}_{R_*,f}^{Cpl(I)}}(H_s(hB_\bullet^{[w]}(\pi_* K)), \Sigma^{j+a-t} R_*) & \text{if } s = \log_p(w), t \geq a \\ 0 & \text{else} \end{cases}$$

The spectral sequence evidently degenerates at E_2.

The Bousfield-Kan spectral sequence for $Y^\bullet$ has for $t \geq s \geq 0$:

$$\tilde{E}_1^{s,t} = N^s \pi_t(\mathrm{Map}_{\mathcal{P}_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)})}(hB_\bullet^{[w]}(\pi_* K), S^a \wedge \Sigma^j R_*)) = N^s \pi_t(B^a \,\mathrm{Map}_{\mathrm{Mod}_{R_*,f}^{Cpl(I)}}(hB_\bullet^{[w]}(\pi_* K), \Sigma^j R_*))$$

$$= \begin{cases} N^s \pi_{t-a}(\mathrm{Map}_{\mathrm{Mod}_{R_*,f}^{Cpl(I)}}(hB_\bullet^{[w]}(\pi_* K), \Sigma^j R_*) & \text{if } t \geq a \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} N^s \,\mathrm{Map}_{\mathrm{Mod}_{R_*,f}^{Cpl(I)}}(hB_\bullet^{[w]}(\pi_* K), \Sigma^j R_*) & \text{if } t = a \\ 0 & \text{else} \end{cases}$$

The last step follows since the mapping spaces in the category $\mathrm{Mod}_{R_*,f}^{Cpl(I)}$ are discrete.

As before, we conclude $\tilde{E}_2^{s,t} = \begin{cases} \mathrm{Map}_{\mathrm{Mod}_{R_*,f}^{Cpl(I)}}(H_s(hB_\bullet^{[w]}(\pi_* K)), \Sigma^j R_*) & \text{if } s = \log_p(w), t = a \\ 0 & \text{else} \end{cases}$.

This spectral sequence therefore also degenerates for obvious reasons.

We can depict the map of spectral sequences induced by $X^\bullet \to Y^\bullet$ as follows:

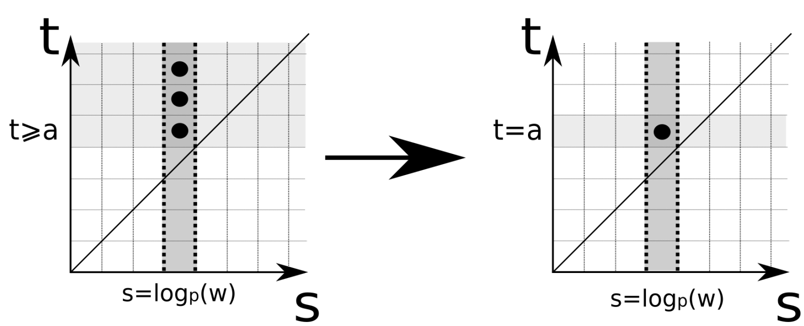


Figure 9: The map of spectral sequences for $X^\bullet$ and $Y^\bullet$ on the respective E_2-pages.

The natural map $E_{s,t}^\infty \to \tilde{E}_{s,t}^\infty$ is therefore the identity if $t = a$ and $s = \log_p(w)$ and projection to zero otherwise. Both spectral sequences vanish outside the vertical line $s = \log_p(w)$ and converge.

If w is not a power of p, this means that both groups are zero.

If $w = p^s$, then there are no extension problems and we obtain a commutative square:

$$\begin{array}{ccc} \pi_0 \operatorname{Map}_{\mathcal{P}_\sigma(\operatorname{Mod}_{R,f}^{Cpl(I)})}(\mathfrak{C}[w](K), S^a \wedge \Sigma^j R) & \longrightarrow & \pi_0 \operatorname{Map}_{\mathcal{P}_\sigma(\operatorname{Mod}_{R_*,f}^{Cpl(I)})}(\hat{\mathbb{C}}[w](\pi_* K), S^a \wedge \Sigma^j R_*) \\ \cong \downarrow & & \cong \downarrow \\ E^\infty_{s,s} & \longrightarrow & \tilde{E}^\infty_{s,s} \end{array}$$

If $a \neq s$, the right hand side is zero. If $a = s$, then the lower map is an isomorphism.

Let us now proceed to statement (2). We can obtain the map in question by applying π_0 to the totalisation of the following map $X^\bullet \to Z^\bullet$ of (pointed) cosimplicial spaces:

$$\operatorname{Map}_{\mathcal{P}_\sigma(\operatorname{Mod}_{R,f}^{Cpl(I)})}(B^{[w]}_\bullet(K), S^a \wedge \Sigma^j R) \to \operatorname{Map}_{\operatorname{Mod}_R^{Cpl(I)}}(B^{[w]}_\bullet(K), \Sigma^{j+a} R)$$

The Bousfield-Kan spectral sequence for $Z^\bullet$ has for $t \geq s \geq 0$:

$$\tilde{\tilde{E}}_1^{s,t} = N^s \pi_t(\operatorname{Map}_{\operatorname{Mod}_R^{Cpl(I)}}(B^{[w]}_\bullet(K), \Sigma^{j+a} R)) = N^s \pi_0(\operatorname{Map}_{\operatorname{Mod}_R^{Cpl(I)}}(B^{[w]}_\bullet(K), \Sigma^{j+a-t} R))$$

$$= N^s \pi_0(\operatorname{Map}_{\operatorname{Mod}_{R_*}^{Cpl(I)}}(hB^{[w]}_\bullet(\pi_* K), \Sigma^{j+a-t} R_*)$$

Once again, we conclude $\tilde{\tilde{E}}_2^{s,t} = \begin{cases} \operatorname{Map}_{\operatorname{Mod}_{R_*,f}^{Cpl(I)}}(H_s(hB^{[w]}_\bullet(\pi_* K)), \Sigma^{j+a-t} R_*) & \text{if } s = \log_p(w) \\ 0 & \text{else} \end{cases}$.

This spectral sequence degenerates along a line and we obtain a square

$$\begin{array}{ccc} \pi_0 \operatorname{Map}_{\mathcal{P}_\sigma(\operatorname{Mod}_{R,f}^{Cpl(I)})}(\mathfrak{C}[w](K), S^a \wedge \Sigma^j R) & \longrightarrow & \pi_0 \operatorname{Map}_{\operatorname{Mod}_{R,f}^{Cpl(I)}}(C[w](K), \Sigma^{j+a} R) \\ \cong \downarrow & & \cong \downarrow \\ E^\infty_{s,s} & \longrightarrow & \tilde{\tilde{E}}^\infty_{s,s} \end{array}$$

if $s = \log_p(w)$. We can read off the second claim. □

3.4 Shearing and Suspending

In the situation of interest to us, the module $\Sigma^i R_*$ will be p-Koszul whenever i is odd. This means that we can lift along the map $\mathfrak{Q}^{S^a(j)}_{S^0(i)}[p^a] \xrightarrow{\cong} \mathbb{Q}^{S^a(j)}_{S^0(i)}[p^a]$ and thereby produce certain operations in $Q_i^{j+a}[p^a]$ "from algebra".

Our aim is to understand the composition of these lifted operations in topology in terms of algebra. At first, this seems problematic as we cannot directly postcompose in algebra since our representing classes all lie in groups $\mathbb{Q}^{S^a(j)}_{S^0(i)}[p^a]$.

Shearing Operations. Fix $i, j, k \in \mathbb{Z}$ and $a, b \in \mathbb{Z}_{\geq 0}$. We shall first produce a commutative diagram

$$\begin{array}{ccccc} Q^{k+a+b}_{j+a}[p^b] & \longleftarrow & \mathfrak{Q}^{S^b(k+a)}_{S^0(j+a)}[p^b] & \longrightarrow & \mathbb{Q}^{S^b(k+a)}_{S^0(j+a)}[p^b] \\ \| & & \downarrow & & \downarrow \\ Q^{k+a+b}_{j+a}[p^b] & \longleftarrow & \mathfrak{Q}^{S^{a+b}(k)}_{S^a(j)}[p^b] & \longrightarrow & \mathbb{Q}^{S^{a+b}(k)}_{S^a(j)}[p^b] \end{array}$$

This will allow us to represent operations in topology by classes in algebra by which we can postcompose. We shall call our technique *shearing* and denote all corresponding maps by Sh^a.

Fix $X \in \mathrm{Mod}^{Cpl(I)}_{R,f}$. Let $S^a_\bullet = \Delta^a/\partial\Delta^a$ be the standard simplicial a-sphere, pointed by $*_\bullet \to S^a_\bullet$. We obtain a simplicial object $S^a_\bullet \wedge X$ whose n-simplices are given by $(S^a_\bullet \wedge X)_n = \mathrm{cofib}\left(\bigoplus_{*_n} X \to \bigoplus_{S^a_n} X\right)$. The colimit of this simplicial diagram in $\mathrm{Mod}^{Cpl(I)}_R$ is given by $\Sigma^a X$, and we therefore obtain an *augmented* simplicial object $\Delta^{op}_+ \to \mathrm{Mod}^{Cpl(I)}_R$ which we shall denote by $(S^a_\bullet \wedge X \to \Sigma^a X)$.

Let $B_\bullet : \Delta^{op} \to \mathrm{End}^{f,\sigma}(\mathrm{Mod}^{Cpl(I)}_R)$ be a simplicial object in completed-free functors which preserve realisations. Applying $B_\bullet$ to the above augmented simplicial object gives rise to a functor $\Delta^{op} \times \Delta^{op}_+ \to \mathrm{Mod}^{Cpl(I)}_R$ which we write as $(B_\bullet(S^a_\bullet \wedge X) \to B_\bullet(\Sigma^a X))$. This in turn can be thought of as an augmented simplicial object $F_\bullet : \Delta^{op}_+ \to \mathcal{C}$ in the stable ∞-category $\mathcal{C} := \mathrm{Fun}(\Delta^{op}, \mathrm{Mod}^{Cpl(I)}_R)$.

We observe that the diagram $G_\bullet : \Delta^{op}_+ \to \mathcal{C}$ given by $(S^a_\bullet \wedge \Sigma^{-a} B_\bullet(\Sigma^a X) \to B_\bullet(\Sigma^a X))$ is a left Kan extension of its restriction to $\Delta^{op}_{\leq a}$. By Lemma 1.2.4.19 in [Lur14], this implies that $G_\bullet$ is also the right Kan extension of its restriction to $\Delta^{op}_{+,\leq a-1}$. Since the restriction of $G_\bullet$ to $\Delta^{op}_{+,\leq a-1}$ agrees with $(F_\bullet)|_{\Delta^{op}_{+,\leq a-1}}$ (all non-degenerate simplices of S^a lie above dimension a), we deduce the identification $\mathrm{Ran}^{\Delta^{op}_+}_{\Delta^{op}_{+,\leq a-1}}(F_\bullet)|_{\Delta^{op}_{+,\leq a-1}} \cong G_\bullet$. Restricting the tautological map $F_\bullet \to \mathrm{Ran}^{\Delta^{op}_+}_{\Delta^{op}_{+,\leq a-1}}(F_\bullet)|_{\Delta^{op}_{+,\leq a-1}} \cong G_\bullet$ back to Δ^{op}, we obtain a map of *bi*simplicial objects $B_\bullet(S^a_\bullet \wedge X) \to S^a_\bullet \wedge \Sigma^{-a} B_\bullet(\Sigma^a X)$. Its value on objects is obtained by projecting onto summands and using the canonical map $B_\bullet \to \Sigma^{-a} B_\bullet \Sigma^a$.

The colimit of this diagram in $P_\sigma(\mathrm{Mod}^{Cpl(I)}_{R,f})$ gives rise to an arrow $|B_\bullet|(S^a \wedge X) \to S^a \wedge (L\Sigma^{-a})|B_\bullet|(\Sigma^a X)$. If $B_\bullet = B^{[w]}_\bullet$ for a weighted structure on a monad T, we obtain maps $\mathrm{Sh}^a : \mathfrak{Q}^{S^b(k+a)}_{S^0(j+a)}[w] \to \mathfrak{Q}^{S^{a+b}(k)}_{S^a(j)}[w]$ given by:

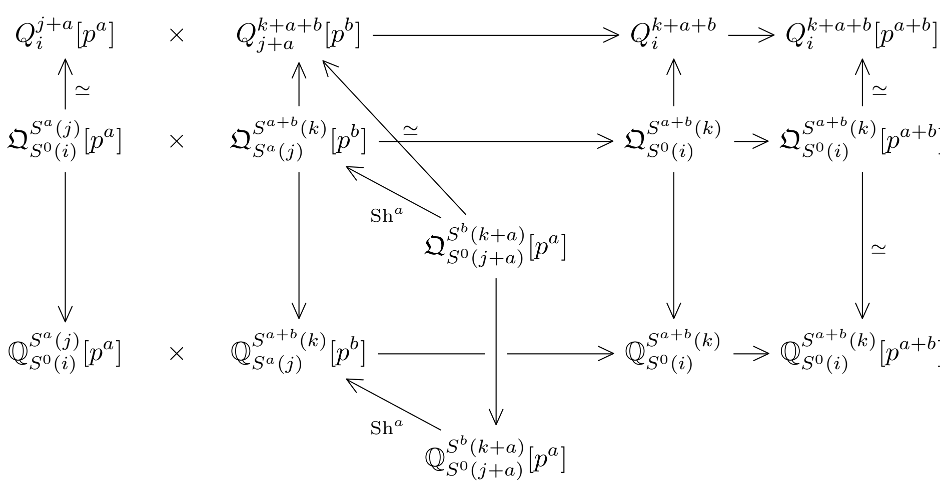
$$\mathrm{Map}_{P_\sigma(\mathrm{Mod}^{Cpl(I)}_{R,f})}(\mathfrak{C}[w](\Sigma^{j+a}R), S^b \wedge \Sigma^{k+a}R) \to \mathrm{Map}_{P_\sigma(\mathrm{Mod}^{Cpl(I)}_{R,f})}(S^a \wedge (L\Sigma^{-a})\mathfrak{C}[w](\Sigma^{j+a}R), S^{a+b} \wedge \Sigma^k R)$$

$$\to \mathrm{Map}_{P_\sigma(\mathrm{Mod}^{Cpl(I)}_{R,f})}(\mathfrak{C}[w](S^a \wedge \Sigma^j R), S^{a+b} \wedge \Sigma^k R)$$

Applying the functors $L\iota : P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)}) \to \mathrm{Mod}_R^{Cpl(I)}$, $Lh : P_\sigma(\mathrm{Mod}_{R,f}^{Cpl(I)}) \to P_\sigma(\mathrm{Mod}_{R_*,f}^{Cpl(I)})$ gives corresponding maps of mapping spaces which fit into the following diagram:

$$\begin{array}{ccccc}
Q_{j+a}^{k+a+b}[p^b] & \longleftarrow & \mathfrak{Q}_{S^0(j+a)}^{S^b(k+a)}[p^b] & \longrightarrow & \mathbb{Q}_{S^0(j+a)}^{S^b(k+a)}[p^b] \\
\Vert & & \downarrow{\scriptstyle \mathrm{Sh}^a} & & \downarrow{\scriptstyle \mathrm{Sh}^a} \\
Q_{j+a}^{k+a+b}[p^b] & \longleftarrow & \mathfrak{Q}_{S^a(j)}^{S^{a+b}(k)}[p^b] & \longrightarrow & \mathbb{Q}_{S^a(j)}^{S^{a+b}(k)}[p^b]
\end{array}$$

For each $i, j, k \in \mathbb{Z}$ and $a, b \in \mathbb{N}$, we therefore obtain a diagram



This diagram allows us to understand composition in topology in terms of (nonadditive) derived functors in algebra whenever $\Sigma^i R, \Sigma^{j+a} R$ are both p-Koszul. We will elaborate on this below.

Suspending Operations. We can also suspend operations. Given a map $S \to T$ of completed-free realisation-preserving monads, we obtain:

$$\begin{array}{ccccc}
Q_{i+a}^{j+b}(T) & \leftarrow & \mathfrak{Q}_{S^a(i)}^{S^b(j)}(T) & \rightarrow & \mathbb{Q}_{S^a(i)}^{S^b(j)}(T) \\
\downarrow & & \downarrow & & \downarrow \\
Q_{i+a}^{j+b}(S) & \leftarrow & \mathfrak{Q}_{S^a(i)}^{S^b(j)}(S) & \rightarrow & \mathbb{Q}_{S^a(i)}^{S^b(j)}(S)
\end{array}$$

Let T be a completed-free monad. The endofunctor $T^{\Sigma^s} = \Sigma^s T \Sigma^{-s}$ inherits the structure of a completed-free monad, and there is an evident morphism of monads $T^{\Sigma^s} \to T$. We observe equivalences

$$\begin{array}{ccccc}
Q_{i+a}^{j+b}(T^{\Sigma^s}) & \leftarrow & \mathfrak{Q}_{S^a(i)}^{S^b(j)}(T^{\Sigma^s}) & \rightarrow & \mathbb{Q}_{S^a(i)}^{S^b(j)}(T^{\Sigma^s}) \\
{\scriptstyle \simeq}\downarrow & & {\scriptstyle \simeq}\downarrow & & {\scriptstyle \simeq}\downarrow \\
Q_{i+a-s}^{j+b-s}(T) & \leftarrow & \mathfrak{Q}_{S^a(i-s)}^{S^b(j-s)}(T) & \rightarrow & \mathbb{Q}_{S^a(i-s)}^{S^b(j-s)}(T)
\end{array}$$

Combining this with the previous diagram (for $S = T^{\Sigma}$), we obtain the diagram

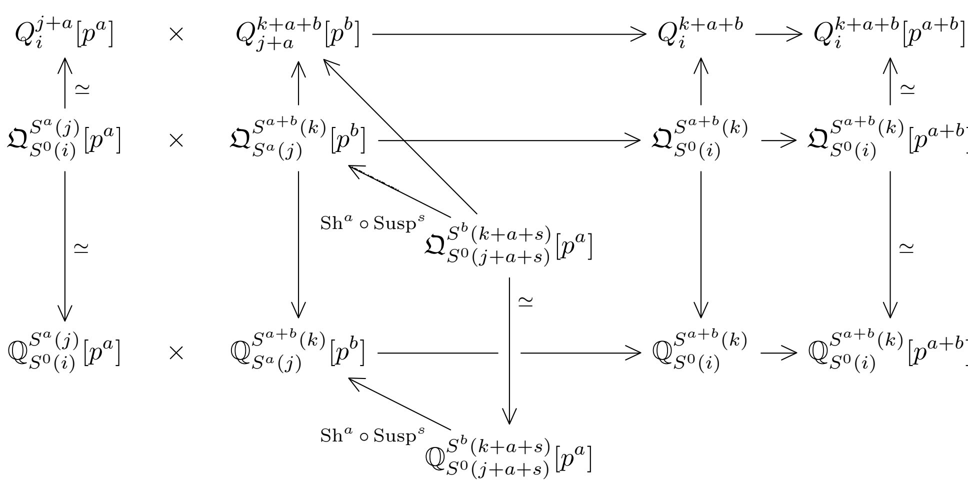
$$\begin{array}{ccccc}
Q^{j+b}_{i+a}(T) & \longleftarrow & \mathfrak{Q}^{S^b(j)}_{S^a(i)}(T) & \longrightarrow & \mathbb{Q}^{S^b(j)}_{S^a(i)}(T) \\
\big\downarrow{\scriptstyle \mathrm{Susp}^s} & & \big\downarrow{\scriptstyle \mathrm{Susp}^s} & & \big\downarrow{\scriptstyle \mathrm{Susp}^s} \\
Q^{j+b-s}_{i+a-s}(T) & \longleftarrow & \mathfrak{Q}^{S^b(j-s)}_{S^a(i-s)}(T) & \longrightarrow & \mathbb{Q}^{S^b(j-s)}_{S^a(i-s)}(T)
\end{array}$$

A corresponding diagram is induced on the weighted pieces.

3.5 Composing Operations in Algebra and Topology

We can now combine our results and constructions from the preceding sections, define certain operations in topology from operations in algebra (via the canonical lifts described in Section 3.3), and prove that their compositions are compatible:

Theorem 3.5.1. *Let $T \in \mathrm{Alg}^{aug}(\mathrm{End}^{\sigma,f}(\mathrm{Mod}_R^{Cpl(I)}))$ be an augmented completed-free realisation-preserving monad endowed with a weighted structure $\mathrm{Bar}_\bullet(T) = \bigoplus_w B^{[w]}_\bullet$. Fix integers i, j, k and nonnegative integers a, b, and s. Assume that $\Sigma^i R_*$ and $\Sigma^{j+a+s} R_*$ are p-Koszul. Then the following diagram commutes:*



We have therefore produced a commutative diagram

$$\begin{array}{ccccc}
\mathbb{Q}^{S^a(j)}_{S^0(i)}[p^a] \times \mathbb{Q}^{S^b(k+a+s)}_{S^0(j+a+s)}[p^b] & \xrightarrow{\mathrm{id} \times (\mathrm{Sh}^a \circ \mathrm{Susp}^s)} & \mathbb{Q}^{S^a(j)}_{S^0(i)}[p^a] \times \mathbb{Q}^{S^{a+b}(k)}_{S^a(j)}[p^b] & \longrightarrow & \mathbb{Q}^{S^{a+b}(k)}_{S^0(i)}[p^{a+b}] \\
\big\downarrow & & & & \big\downarrow{\scriptstyle \cong} \\
Q^{j+a}_i[p^a] \times Q^{k+a+b+s}_{j+a+s}[p^b] & \xrightarrow{\mathrm{id} \times \mathrm{Susp}^s} & Q^{j+a}_i[p^a] \times Q^{k+a+b}_{j+a}[p^b] & \longrightarrow & Q^{k+a+b}_i[p^{a+b}]
\end{array}$$

Chapter 4

Operations on the E-theory of Spectral Lie Algebras

In this chapter, we compute the operations which act naturally on the homotopy groups of Lie algebras in $K(h)$-local E-module spectra.

After introducing background and preliminary results in Section 4.1, we will construct three kinds of operations in Section 4.2: additive unary operations labelled by elements in the power ring $\mathcal{H}^{\mathrm{Lie}}$, the nonadditive unary operation θ, and the binary Lie bracket $[-,-]$. We then proceed to establish the various relations between these operations in Section 4.3. In Section 4.4, we axiomatise the resulting structure and define the notion of a *Hecke Lie algebra*. We then establish that, up to completion, we have indeed found all operations and all relations between them.

4.1 Preliminary Considerations

In this section, we will set the stage for our later computations. After briefly reviewing the basics of Morava E-theory in Section 4.1.1, we discuss symmetric sequences in Section 4.1.2 and use them to give a formal definition of spectral Lie algebras. We then collect several basic facts about Goodwillie's calculus of functors in Section 4.1.3 and discuss applications to the theory of spectral Lie algebras.

4.1.1 Lubin-Tate Theory

We fix a prime p, a natural number h, and a (1-dimensional, commutative) formal group G_0 of height h over the field $\overline{\mathbb{F}}_p$. By the theorem of Goerss–Hopkins–Miller, there is an essentially unique even periodic $\mathbb{E}_\infty$-ring spectrum E for which $E_0 = \pi_0(E)$ is complete local Noetherian with residue field $\overline{\mathbb{F}}_p$ and for which the formal group $G = \operatorname{Spf} \pi_0(E^{\mathbb{C}P^\infty})$ is a universal deformation of G_0. This spectrum E is usually called Morava E-theory. In order to make the nature of this object more readily accessible to a wider audience, we also use the name *Lubin-Tate theory* for E.

We also fix a coordinate for G, i.e. a formal group law $f \in \pi_0(E)[[x,y]]$ whose associated formal group is G. Such a complex orientation in particular determines a Thom isomorphism and an equivalence of naïve Σ_2-spectra $E \otimes (S^2 \otimes S^2) \cong E \otimes S^4$, where Σ_2 acts trivially on the right hand side and by swapping on the left hand side (cf. [MNN15]).

We recall that there is a (noncanonical) isomorphism $E_* = \pi_*(E) \cong W(\overline{\mathbb{F}_p})[[u_1, \ldots, u_{h-1}]][\beta^{\pm 1}]$ with β in degree 2 and write $I \subset E_0$ for the unique maximal ideal. Work by Hovey-Strickland [HS99] implies that the ∞-category $\mathrm{Mod}_E^{Cpl(I)}$ of I-complete E-module spectra (in the sense of Definition 5.1.2 in Appendix A) is naturally equivalent to the ∞-category of $K(h)$-local E-module spectra and that the completion functor $(-)^\wedge_I$ is given by $K(h)$-localisation.

The ∞-category $\mathrm{Mod}_E^{Cpl(I)}$ comes endowed with a symmetric monoidal structure $\otimes$ obtained by postcomposing the usual (relative) smash product of E-module spectra (over E) with $K(h)$-localisation. We shall write $\oplus$ for the coproduct in $\mathrm{Mod}_E^{Cpl(I)}$, which can be computed by first taking the coproduct in E-module spectra and then $K(h)$-localising.

The ∞-category $\mathrm{Mod}_E^{Cpl(I)}$ is naturally tensored over spaces and spectra, and we shall denote the product of a space or spectrum X with an object $M \in \mathrm{Mod}_E^{Cpl(I)}$ simply by $X \otimes M$.

There is a natural "forgetful-free"-adjunction $Sp_{K(h)} \leftrightarrows \mathrm{Mod}_E^{Cpl(I)}$. Write $\mathbb{G}$ for the Morava stabiliser group.

Definition 4.1.1. The ∞-category $\mathrm{Mod}_{E,\mathbb{G}}^{Cpl(I)}$ of $\mathbb{G}$-equivariant $K(h)$-local module spectra over Lubin-Tate space is the ∞-category of coalgebras for the comonad on $\mathrm{Mod}_E^{Cpl(I)}$ attached to the above adjunction.

One can combine Lurie's ∞-categorical Barr-Beck theorem (see [Lur14]) with the smash product theorem of Hopkins–Ravenel (see [Rav92]) to prove that the canonical map $Sp_{K(h)} \to \mathrm{Mod}_{E,\mathbb{G}}^{Cpl(I)}$ is an equivalence (see [Mat17]).

We will also consider the abelian category $\mathrm{Mod}_{E_*}^{Cpl(I)}$ of L-complete modules in the category Mod_{E_*} as formalised in Definition 5.1.1 in Appendix A. As before, we write $\mathrm{Mod}_{E_*,f}^{Cpl(I)}$ for the full subcategory spanned by all completed-free (also sometimes called pro-free) modules in the sense of Definition 5.1.14 in Appendix A.

Using the fact that E_* is a regular local ring, Hovey and Strickland prove in Theorem $A.9$ of [HS99] that an E_*-module M is completed-free if and only if it is projective in the abelian category $\mathrm{Mod}_{E_*}^{Cpl(I)}$.

The category $\mathrm{Mod}_{E_*}^{Cpl(I)}$ has enough projectives and we may therefore consider its nonnegative derived category $\mathcal{D}^-_{\geq 0}(\mathrm{Mod}_{E_*}^{Cpl(I)})$ (see section 1.3.2 in [Lur14] for a careful higher-categorical treatment). This classical object gives a concrete model for the ∞-category $P_\sigma(\mathrm{Mod}_{E_*}^{Cpl(I)})$ from Definition 3.2.2:

Proposition 4.1.2. *The derived functor ι of the canonical inclusion $\iota : \mathrm{Mod}_{E_*,f}^{Cpl(I)} \to \mathcal{D}^-_{\geq 0}(\mathrm{Mod}_{E_*}^{Cpl(I)})$ into the heart gives rise to an equivalence of ∞-categories $L\iota : P_\sigma(\mathrm{Mod}_{E_*,f}^{Cpl(I)}) \to \mathcal{D}^-_{\geq 0}(\mathrm{Mod}_{E_*}^{Cpl(I)})$.*

Proof. By Proposition 4.2.15 in [Lur11a], it suffices to check that ι is fully faithful and preserves small coproducts (which is evident), that every element in its image is projective (which is true by Proposition 7.2.2.6. in [Lur14]), and that we can detect equivalences $D \to D'$ in the derived category by considering mapping spaces out of elements in the image of ι (which holds because equivalences are detected in homology). □

4.1.2 ∞-Operads as Symmetric Sequences

We define a monoidal ∞-category of symmetric sequences and construct specific algebra objects of interest inside it, namely the $\mathbb{E}_\infty$-operad and the spectral Lie operad. The definition of the composition product on symmetric sequences is an ∞-categorical version of a 1-categorical construction due to Trimble [Tri]. We thank our advisor for a particularly helpful discussion related to the material of this section.

Let Pr^L denote the ∞-category of presentable ∞-categories and functors between them which preserve small colimits. This category can be endowed with the structure of a symmetric monoidal ∞-category: Given two ∞-categories $\mathcal{C}, \mathcal{D}$ in Pr^L, the ∞-category $\mathcal{C} \otimes \mathcal{D}$ is the universal presentable ∞-category which receives a functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ preserving small colimits in each variable. We refer to Section 4.8 in [Lur14] for a careful treatment.

Assume now that we are given a commutative algebra object $\mathcal{A} \in \mathrm{CAlg}(\mathrm{Pr}^L)$, i.e. a specific presentable symmetric monoidal ∞-category for which the symmetric monoidal product preserves small colimits in each variable. The forgetful functor $\mathrm{CAlg}(\mathrm{Pr}^L)_{\mathcal{A}/} \to \mathrm{CAlg}(\mathrm{Pr}^L)$ admits a left adjoint $\mathcal{A} \otimes (-)$ by Theorem 4.5.3.1. and Remark 4.8.1.23 in [Lur14].

Write $\mathrm{Fin}^{\cong}$ for the category of finite sets and bijections. Disjoint union endows this category with a symmetric monoidal structure, and the corresponding symmetric monoidal ∞-category $N(\mathrm{Fin}^{\cong})$ is in fact the free symmetric monoidal ∞-category on a point. The presheaf category $\mathcal{P}(N(\mathrm{Fin}^{\cong})) = \mathrm{Fun}(N(\mathrm{Fin}^{\cong})^{op}, \mathcal{S})$ inherits a canonical symmetric monoidal structure via Day convolution by 4.8.1.12 in [Lur14]. Since the Day convolution product commutes with small colimits separately in each variable, we can think of $\mathcal{P}(N(\mathrm{Fin}^{\cong}))$ as an object in $\mathrm{CAlg}_{\mathrm{Pr}^L}$. Given any other $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L)$, Remark 4.8.1.9. in [Lur14] implies that restrictions define an equivalence

$$\mathrm{Fun}_{\mathrm{CAlg}(\mathrm{Pr}^L)}(\mathcal{P}(N(\mathrm{Fin}^{\cong})), \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}_{\mathrm{CAlg}(\mathrm{Cat}_\infty)}(N(\mathrm{Fin}^{\cong}), \mathcal{C}) \xrightarrow{\simeq} \mathcal{C}$$

Definition 4.1.3. Given $\mathcal{A} \in \mathrm{CAlg}(\mathrm{Pr}^L)$ a presentable symmetric monoidal ∞-category for which the symmetric monoidal product distributes over small colimits, the ∞-category $\mathrm{SSeq}(\mathcal{A})$ of $\mathcal{A}$-valued symmetric sequences is defined as $\mathrm{SSeq}(\mathcal{A}) := \mathrm{Fun}(N(\mathrm{Fin}^{\cong}), \mathcal{A})$.

Using Proposition 4.8.1.16 of [Lur14], there are canonical equivalences

$$\mathrm{Fun}(N(\mathrm{Fin}^{\cong}), \mathcal{A}) \cong \mathrm{Fun}^L(\mathcal{P}(N(\mathrm{Fin}^{\cong})^{op}), \mathcal{A}) \cong \mathrm{Fun}^R(\mathcal{P}(N(\mathrm{Fin}^{\cong})), \mathcal{A}^{op})^{op} \cong \mathcal{P}(N(\mathrm{Fin}^{\cong})) \otimes \mathcal{A}$$

Given any $\mathcal{D} \in \mathrm{CAlg}(\mathrm{Pr}^L)_{A/}$, the following restrictions therefore define equivalences:

$$\mathrm{Fun}_{\mathrm{CAlg}(\mathrm{Pr}^L)_{A/}}(\mathrm{SSeq}(\mathcal{A}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}_{\mathrm{CAlg}(\mathrm{Pr}^L)}(\mathcal{P}(N(\mathrm{Fin}^{\cong})), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}_{\mathrm{CAlg}(\mathrm{Cat}_\infty)}(N(\mathrm{Fin}^{\cong}), \mathcal{D}) \xrightarrow{\simeq} \mathcal{D}$$

Taking $\mathcal{D} = \mathrm{SSeq}(\mathcal{A})$, the *reverse* of the evident *composition product* on the left endows $\mathrm{SSeq}(\mathcal{A})$ with the structure of a monoidal ∞-category. We call this product $\circ$ the *composition product of symmetric sequences.*

Definition 4.1.4. The ∞-category $\mathrm{Op}(\mathcal{A})$ of ∞-operads in $\mathcal{A}$ is given by $\mathrm{Alg}(\mathrm{SSeq}(\mathcal{A}))$.

We can identify $\mathcal{A}$ with the full subcategory of $\mathrm{SSeq}(\mathcal{A})$ spanned by all functors $N(\mathrm{Fin}^{\cong}) \to \mathcal{A}$ for which all values on nonempty sets are initial in $\mathcal{A}$. In $\mathrm{Fun}_{\mathrm{CAlg}(\mathrm{Pr}^L)_{A/}}(\mathrm{SSeq}(\mathcal{A}), \mathrm{SSeq}(\mathcal{A}))$, this corresponds to functors whose essential image is contained in $\mathcal{A}$. The subcategory of all such functors is evidently closed under precomposition, and this implies that the natural (left) $\mathrm{SSeq}(\mathcal{A})$-tensored structure on $\mathrm{SSeq}(\mathcal{A})$ restricts and makes $\mathcal{A}$ into a (left) $\mathrm{SSeq}(\mathcal{A})$-tensored ∞-category. We obtain a functor of monoidal ∞-categories $m : \mathrm{SSeq}(\mathcal{A}) \to \mathrm{End}(\mathcal{A})$. On algebra objects, this gives rise to a functor $m : \mathrm{Op}(\mathcal{A}) \to \mathrm{Alg}(\mathrm{End}(\mathcal{A}))$ assigning a monad to every operad. We abuse notation here by denoting these two functors by the same symbol.

Given a nonnegative integer n, we write $\mathrm{Fin}^{\cong}_n$ for the full subcategory of $\mathrm{Fin}^{\cong}$ spanned by all sets of cardinality exactly n. Assuming that $\mathcal{A}$ is pointed, there is a functor which is both left and right adjoint to the restriction functor $\mathrm{SSeq}(\mathcal{A}) \to \mathrm{Fun}(N(\mathrm{Fin}^{\cong}_n), \mathcal{A})$. We write p_n for the resulting endofunctor of $\mathrm{SSeq}(\mathcal{A})$ and obtain natural transformations $p_n \to \mathrm{id}$ and $\mathrm{id} \to p_n$. Informally speaking, p_n takes a symmetric sequence and changes all components other than the n^{th} one into the zero object.

If $L : \mathcal{A} \rightleftarrows \mathcal{B} : R$ is an adjunction between objects in $\mathrm{CAlg}(\mathrm{Pr}^L)$, one can consider the naturally induced adjunction $L : \mathrm{SSeq}(\mathcal{A}) \rightleftarrows \mathrm{SSeq}(\mathcal{B}) : R$. We observe that the functor L is monoidal for the composition product, which implies that its right adjoint R is lax monoidal.

Concrete Symmetric Sequences

We shall now construct specific elements of interest in $\mathrm{SSeq}(Sp)$. For this, we start with the cofibrantly generated model category **Sp** of S-modules in the sense of [EKMM97], Chapter VII.

We set $\mathbf{SSeq}(\mathbf{Sp}) := \mathrm{Fun}(\mathrm{Fin}^{\cong}, \mathbf{Sp})$. Ordinary Day convolution can be used to endow this functor category with the structure of a symmetric monoidal model category. Here weak equivalences and fibrations are defined pointwise. We write $\mathbf{SSeq}(\mathbf{Sp})^c$ for the category of cofibrant symmetric sequences in this model structure. We say that a symmetric sequence is *reduced* if $M(\underline{0}) = 0$ and $M(\underline{1}) = S^0$ is the (non-cofibrant) sphere spectrum. A symmetric sequence M is said to be Σ-*cofibrant*[1] if $M(n)$ is projectively cofibrant in the functor category $\mathrm{Fun}(\mathrm{Fin}^{\cong}_n, \mathbf{Sp})$ for every $n \neq 1$ and the spectrum $M(1)$ is either cofibrant or equal to S^0. Let $\mathbf{SSeq}(\mathbf{Sp})^{\mathbf{c}}$ denote the category of Σ-cofibrant symmetric sequences. As before, $\mathrm{Fin}^{\cong}_n$ denotes the category of sets of cardinality n.

We can define the *strict composition product* of two elements $M, N \in \mathbf{SSeq}(\mathbf{Sp})$ by the rule

$$(M \circ N)_J = \coprod_{r=0}^{\infty} \Big(\coprod_{J = J_1 \coprod \cdots \coprod J_r} M_r \otimes N_{J_1} \otimes \ldots \otimes N_{J_r} \Big)_{\Sigma_r}$$

We refer to Rezk's thesis [Rez96] for a nice treatment.

Warning. *The projective model structures on symmetric sequences is not left closed for the composition product and hence* $\mathbf{SSeq}(\mathbf{Sp})$ *does not satisfy the axioms of a monoidal model category for this product.*

The underlying ∞-category of $\mathbf{SSeq}(\mathbf{Sp})$ is given by the functor category $\mathrm{SSeq}(Sp) = \mathrm{Fun}(N(\mathrm{Fin}^{\cong}), Sp)$.

Definition 4.1.5. A *strict operad* in spectra is an algebra object in $\mathbf{SSeq}(\mathbf{Sp})$. Write $\mathbf{Op}(\mathbf{Sp})$ for the resulting category. Let $\mathbf{Op}^{\mathrm{red}}(\mathbf{Sp})$ be the subcategory of operads whose underlying symmetric sequence is reduced.

The following is essentially contained in Lemma 9.20 in [AC11] :

Lemma 4.1.6. *If M is a Σ-cofibrant symmetric sequence, then the two functors $F_M := (-) \circ M$ and $S_M := M \circ (-)$ both preserve cofibrant symmetric sequences and weak equivalences between them.*

Proof. Fix a finite set J and an integer $r \geq 0$. The proof of Lemma 9.20 in [AC11] implies that the two functors which attach to a symmetric sequence N the $\Sigma_J \times \Sigma_r$-spectra

$$\coprod_{J = J_1 \coprod \cdots \coprod J_r} M_r \otimes N_{J_1} \otimes \ldots \otimes N_{J_r}, \qquad \coprod_{J = J_1 \coprod \cdots \coprod J_r} N_r \otimes M_{J_1} \otimes \ldots \otimes M_{J_r}$$

[1] We deviate very slightly from the terminology used in [AC11].

both send (weak equivalences between) cofibrant symmetric sequences to (weak equivalences between) projectively cofibrant $\Sigma_J \times \Sigma_r$-spectra. The result follows from the definition of the composition product by observing that taking coinvariants for a group action is a left Quillen functor. □

We can identify **Sp** with the full subcategory of **SSeq**(**Sp**) spanned by all symmetric sequences which vanish on all nonempty finite sets. The functor $S_{(-)}$ from Lemma 4.1.6 restricts and thus gives rise to a monoidal functor $S_{(-)} : \mathbf{SSeq}(\mathbf{Sp}) \to \mathbf{End}(\mathbf{Sp})$ which sends M to $S_M(X) = \coprod_{n\geq 1}(M(n) \otimes X^{\otimes n})_{\Sigma_n}$.

Corollary 4.1.7. *If M is a Σ-cofibrant symmetric sequence, then the functor $S_M : \mathbf{Sp} \to \mathbf{Sp}$ preserves cofibrant S-modules and weak equivalences between them.*

Lemma 4.1.8. *If C is a Σ-cofibrant symmetric sequence, then then functor $(-)\circ C : \mathbf{SSeq}(\mathbf{Sp}) \to \mathbf{SSeq}(\mathbf{Sp})$ preserves Day convolution and small homotopy colimits.*

Proof. The first claim appears as Lemma 2.2.5 in Rezk's thesis [Rez96]. The second part is a standard. □

By the universal property of the "underlying ∞-category"-construction (see section 1.3.4. in [Lur14]), we obtain functor of monoidal ∞-categories:

$$\Phi : N(\mathbf{SSeq^c}(\mathbf{Sp})) \xrightarrow{C\mapsto(-)\circ C} \mathrm{Fun}_{\mathrm{CAlg}(\mathrm{Pr}^L)_{Sp/}}(\mathrm{SSeq}(Sp), \mathrm{SSeq}(Sp))^{rev} = \mathrm{SSeq}(Sp)$$

where the superscript $(-)^{rev}$ denotes the reverse monoidal structure $F \circ^{rev} G := G \circ F$.

By Theorem 9.8 in [AC11] (an elaboration on work by Basterra-Mandell [BM05]), the category $\mathbf{Op}^{\mathrm{red}}(\mathbf{Sp})$ of *reduced* operads in **Sp** carries a cofibrantly-generated simplicial model category structure with weak equivalences and fibrations defined termwise. By Proposition 9.14 in [AC11], the underlying symmetric sequence of any cofibrant reduced operad is Σ-cofibrant (as defined in the preceding section). Together with the above construction, we obtain a functor

$$\Phi : N(\mathbf{Op}^{\mathrm{red}}(\mathbf{Sp})) \to N(\mathrm{Alg}(\mathbf{SSeq^c}(\mathbf{Sp}))) \to \mathrm{Alg}(\mathrm{SSeq}(Sp))$$

We abuse notation and assign two meanings to the letter Φ – which one applies is clear from the context.

A slightly more general version of Theorem 9.8. in [AC11] shows that the category of left modules over a cofibrant reduced operad carries a cofibrantly generated simplicial model structure with termwise defined weak equivalences and fibrations. Cofibrant replacement in modules then gives rise to a functor

$$N(\mathrm{Mod}_{\mathbf{O}}(\mathbf{SSeq}(\mathbf{Sp}))) \to \mathrm{Mod}_{\Phi(\mathbf{O})}(\mathrm{SSeq}(Sp))$$

4.1.3 Goodwillie Derivatives

Write $\mathbf{sSet}_*$ for the model category of pointed simplicial sets and $[\mathbf{sSet}_*, \mathbf{sSet}_*]$ for the category of pointed simplicial homotopy functors $\mathbf{sSet}_* \to \mathbf{sSet}_*$. Attaching Goodwillie derivatives to symmetric sequences gives rise to a functor $\partial_* : [\mathbf{sSet}_*, \mathbf{sSet}_*] \to \mathbf{SSeq}(\mathbf{Sp})$ (see [AC11]).

Applying this to the identity functor $F = \mathrm{id}$, we obtain a symmetric sequence $\partial_*(\mathrm{id})$ of spectra. Its values have been computed in work by Johnson [Joh95] and Arone-Mahowald [AM99]: There is an equivalence of Σ_n-spectra $\partial_n(\mathrm{id}) \cong \mathbb{D}(\Sigma|\Pi_n|^\diamond)$ between the n^{th} Goodwillie derivative of the identity functor and the Spanier-Whitehead dual of the (suspended) partition complex. Ching's work [Chi05] on tree grafting endows the symmetric sequence $\mathbb{D}(\Sigma|\Pi_n|^\diamond)$ with the structure of a strict reduced operad in spectra (cf. also [Sal98]).

Definition 4.1.9. The (shifted) *spectral Lie operad* $\mathcal{O}_{\Sigma\,\mathrm{Lie}}$ is given by $\Phi(\mathbb{D}(\Sigma|\Pi_*|^\diamond)) \in \mathrm{Alg}(\mathrm{SSeq}(Sp))$.

We can also define a strict reduced operad $\mathbf{O}^{nu}_{\mathrm{Comm}} \in \mathbf{Op}^{red}(\mathbf{Sp})$ whose value is the sphere spectrum S^0 for every nonempty set, the zero spectrum on the empty set, and all of whose structure maps are the identity.

Definition 4.1.10. The *nonunital* $\mathbb{E}_\infty$-operad $\mathcal{O}^{nu}_{\mathrm{Comm}}$ is given by $\Phi(\mathbf{O}^{nu}_{\mathrm{Comm}}) \in \mathrm{Alg}(\mathrm{SSeq}(Sp))$.

Algebras over the monad associated with $\mathcal{O}^{nu}_{\mathrm{Comm}}$ are just nonunital commutative algebra objects in the symmetric monoidal ∞-category of spectra.

Assume that we are given $\mathcal{A} \in \mathrm{CAlg}(\mathrm{Pr}^L)_{Sp/}$. We write $\mathcal{O}^{nu\ \mathcal{A}}_{\mathrm{Comm}},\ \mathcal{O}^{\mathcal{A}}_{\Sigma\,\mathrm{Lie}} \in \mathrm{Op}(\mathcal{A})$ for the operads obtained by applying the monoidal functor $(\mathcal{A}\otimes -) : \mathrm{SSeq}(Sp) \to \mathrm{SSeq}(\mathcal{A})$ coming from the unique colimit-preserving symmetric monoidal functor $Sp \to \mathcal{A}$ sending the sphere spectrum to the unit (see Corollary 4.8.2.19 in [Lur14]). We let $T_{\mathcal{A}} = m(\mathcal{O}^{nu\ \mathcal{A}}_{\mathrm{Comm}})$ and $L_{\mathcal{A}} = m(\mathcal{O}^{\mathcal{A}}_{\Sigma\,\mathrm{Lie}})$ denote the corresponding monads on $\mathcal{A}$.

Definition 4.1.11. A *(shifted) Lie algebra or a (nonunital)* $\mathbb{E}_\infty$*-ring in* $\mathcal{A}$ is an algebra for the monad $L_{\mathcal{A}}$ or $T_{\mathcal{A}}$ respectively. We write $\mathrm{Alg}_{\Sigma\,\mathrm{Lie}}(\mathcal{A})$ or $\mathrm{Alg}_{\mathrm{Comm}^{nu}}(\mathcal{A})$ for the resulting ∞-categories. A *homotopy (shifted) Lie algebra or a nonunital* H_∞*-ring in* $h\mathcal{A}$ is an algebra for $hL_{\mathcal{A}} = \bigoplus_n \mathbb{D}(\Sigma|\Pi_n|^\diamond) \otimes_{h\Sigma_n} (-)^{\otimes n}$ or $hT_{\mathcal{A}} = \bigoplus_n (-)^{\otimes n}_{h\Sigma_n}$. Write $\mathrm{Alg}_{\Sigma\,\mathrm{Lie}}(h\mathcal{A})$ or $\mathrm{Alg}_{\mathrm{Comm}^{nu}}(h\mathcal{A})$ for the resulting categories.

We restrict attention to the case $\mathcal{A} = \mathrm{Mod}_E^{Cpl(I)}$ the ∞-category of $K(h)$-local E-module spectra. Theorem 4.3.1. in [Lur11b] gives a Koszul duality functor $\mathrm{KD} : \mathrm{Alg}^{aug}(\mathrm{SSeq}(\mathrm{Mod}_E^{Cpl(I)})) \to \mathrm{coAlg}^{aug}(\mathrm{SSeq}(\mathrm{Mod}_E^{Cpl(I)}))$. We can apply this functor to the nonunital $\mathbb{E}_\infty$-operad $\mathcal{O}^{nu\ \mathrm{Mod}_E^{Cpl(I)}}_{\mathrm{Comm}}$ to obtain the Koszul dual coalgebra in symmetric sequences $\mathrm{KD}(\mathcal{O}^{nu\ \mathrm{Mod}_E^{Cpl(I)}}_{\mathrm{Comm}}) \in \mathrm{coAlg}^{aug}(\mathrm{SSeq}(\mathrm{Mod}_E^{Cpl(I)}))$.

The following lemma holds in greater generality, but we will only present what we need in our computation:

Lemma 4.1.12. *Let X be a dualisable spectrum. There is a natural homotopy commutative diagram*

$$\begin{array}{ccc}
\mathcal{O}^{\mathrm{Mod}_E^{Cpl(I)}}_{\Sigma\,\mathrm{Lie}} \circ \mathcal{O}^{\mathrm{Mod}_E^{Cpl(I)}}_{\Sigma\,\mathrm{Lie}} \circ \mathbb{D}(E\otimes X) & \longrightarrow & \mathcal{O}^{\mathrm{Mod}_E^{Cpl(I)}}_{\Sigma\,\mathrm{Lie}} \circ \mathbb{D}(E\otimes X) \\
\downarrow{\scriptstyle \nu\circ\nu} & & \downarrow{\scriptstyle \nu} \\
\mathbb{D}\left(\mathrm{KD}(\mathcal{O}^{nu\ \mathrm{Mod}_E^{Cpl(I)}}_{\mathrm{Comm}}) \circ \mathrm{KD}(\mathcal{O}^{nu\ \mathrm{Mod}_E^{Cpl(I)}}_{\mathrm{Comm}}) \circ (E\otimes X)\right) & \longrightarrow & \mathbb{D}\left(\mathrm{KD}(\mathcal{O}^{nu\ \mathrm{Mod}_E^{Cpl(I)}}_{\mathrm{Comm}}) \circ (E\otimes X)\right)
\end{array}$$

Here $\mathbb{D}(X) = \underline{\mathrm{Map}}_{\mathrm{Mod}_E^{Cpl(I)}}(X, E)$ and $(E\otimes -) : Sp \to \mathrm{Mod}_E^{Cpl(I)}$ is the unique functor preserving small colimits and sending S to E.

Proof. Corollary 5.4.20 in Appendix D can be used to produce a homotopy commutative square:

$$\begin{array}{ccc}
\mathcal{O}^{Sp}_{\Sigma\,\mathrm{Lie}} \circ \mathcal{O}^{Sp}_{\Sigma\,\mathrm{Lie}} \circ \mathbb{D}X & \longrightarrow & \mathcal{O}^{Sp}_{\Sigma\,\mathrm{Lie}} \circ \mathbb{D}X \\
\downarrow & & \downarrow \\
\mathbb{D}(\mathrm{KD}(\mathcal{O}^{nu\ Sp}_{\mathrm{Comm}}) \circ \mathrm{KD}(\mathcal{O}^{nu\ Sp}_{\mathrm{Comm}}) \circ X) & \longrightarrow & \mathbb{D}(\mathrm{KD}(\mathcal{O}^{nu\ Sp}_{\mathrm{Comm}}) \circ X)
\end{array}$$

The vertical maps are given by norm maps and the canonical maps from coproducts to products. We apply the functor $(E\otimes -)$ and obtain:

$$\begin{array}{ccc}
\mathcal{O}^{\mathrm{Mod}_E^{Cpl(I)}}_{\Sigma\,\mathrm{Lie}} \circ \mathcal{O}^{\mathrm{Mod}_E^{Cpl(I)}}_{\Sigma\,\mathrm{Lie}} \circ (E\otimes \mathbb{D}X) & \longrightarrow & \mathcal{O}^{\mathrm{Mod}_E^{Cpl(I)}}_{\Sigma\,\mathrm{Lie}} \circ (E\otimes \mathbb{D}X) \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
E\otimes(\mathcal{O}^{Sp}_{\Sigma\,\mathrm{Lie}} \circ \mathcal{O}^{Sp}_{\Sigma\,\mathrm{Lie}} \circ \mathbb{D}X) & \longrightarrow & E\otimes(\mathcal{O}^{Sp}_{\Sigma\,\mathrm{Lie}} \circ \mathbb{D}X) \\
\downarrow & & \downarrow \\
E\otimes\mathbb{D}\left(\mathrm{KD}(\mathcal{O}^{nu\ Sp}_{\mathrm{Comm}}) \circ \mathrm{KD}(\mathcal{O}^{nu\ Sp}_{\mathrm{Comm}}) \circ X\right) & \longrightarrow & E\otimes(\mathbb{D}\left(\mathrm{KD}(\mathcal{O}^{nu\ Sp}_{\mathrm{Comm}}) \circ X\right)) \\
\downarrow & & \downarrow \\
\mathbb{D}\left(\mathrm{KD}(\mathcal{O}^{nu\ \mathrm{Mod}_E^{Cpl(I)}}_{\mathrm{Comm}}) \circ \mathrm{KD}(\mathcal{O}^{nu\ \mathrm{Mod}_E^{Cpl(I)}}_{\mathrm{Comm}}) \circ (E\otimes X)\right) & \longrightarrow & \mathbb{D}\left(\mathrm{KD}(\mathcal{O}^{nu\ \mathrm{Mod}_E^{Cpl(I)}}_{\mathrm{Comm}}) \circ (E\otimes X)\right)
\end{array}$$

Here we use that there is an equivalence $E\otimes \mathrm{KD}(\mathcal{O}^{nu\ Sp}_{\mathrm{Comm}}) \cong \mathrm{KD}(E\otimes\mathcal{O}^{nu\ Sp}_{\mathrm{Comm}})$. □

We write $\nu : \mathcal{O}^{\mathrm{Mod}_E^{Cpl(I)}}_{\Sigma\,\mathrm{Lie}} \circ \mathbb{D}(E\otimes X) \to \mathbb{D}(\mathrm{KD}(\mathcal{O}^{nu\ \mathrm{Mod}_E^{Cpl(I)}}_{\mathrm{Comm}}) \circ (E\otimes X))$ for the above transformation. We recall the functors p_j defined on p.79 and observe that the transformation ν has components given by $\nu_j : p_j\mathcal{O}^{\mathrm{Mod}_E^{Cpl(I)}}_{\Sigma\,\mathrm{Lie}} \circ \mathbb{D}(E\otimes X) \xrightarrow{\simeq} \mathbb{D}(p_j\,\mathrm{KD}(\mathcal{O}^{nu\ \mathrm{Mod}_E^{Cpl(I)}}_{\mathrm{Comm}}) \circ (E\otimes X))$ (which are equivalences since $K(h)$-local Tate spectra vanish) in the sense that the following two squares commute up to homotopy:

$$\begin{array}{ccc}
\mathcal{O}^{\mathrm{Mod}_E^{Cpl(I)}}_{\Sigma\,\mathrm{Lie}} \circ \mathbb{D}(E\otimes X) & \longrightarrow & \mathbb{D}(\mathrm{KD}(\mathcal{O}^{nu\ \mathrm{Mod}_E^{Cpl(I)}}_{\mathrm{Comm}}) \circ (E\otimes X)) \\
\downarrow\uparrow & & \downarrow\uparrow \\
p_j\mathcal{O}^{\mathrm{Mod}_E^{Cpl(I)}}_{\Sigma\,\mathrm{Lie}} \circ \mathbb{D}(E\otimes X) & \longrightarrow & \mathbb{D}(p_j\,\mathrm{KD}(\mathcal{O}^{nu\ \mathrm{Mod}_E^{Cpl(I)}}_{\mathrm{Comm}}) \circ (E\otimes X))
\end{array}$$

Arone-Ching [AC11] upgrade the above assignment to a functor $[\mathbf{sSet}_*, \mathbf{sSet}_*] \to \mathrm{Mod}_{\partial_*(\mathrm{id})}(\mathbf{SSeq}(\mathbf{Sp}))$ landing in left modules over the symmetric sequence $\partial_*(\mathrm{id})$ (they in fact produce *bi*modules, but we will not need this additional structure).

We compose with our previous assignments to obtain a functor $N([\mathbf{sSet}_*, \mathbf{sSet}_*]) \to \mathrm{Alg}_L(\mathrm{End}(\mathrm{Mod}_E^{Cpl(I)}))$. Given $X \in \mathrm{Mod}_E^{Cpl(I)}$, we furthermore have an evaluation functor $ev_X : \mathrm{End}(\mathrm{Mod}_E^{Cpl(I)}) \to \mathrm{Mod}_E^{Cpl(I)}$. This is a functor of $\mathrm{End}(\mathrm{Mod}_E^{Cpl(I)})$-tensored ∞-categories, and we deduce the existence of a natural map

$$N([\mathbf{sSet}_*, \mathbf{sSet}_*]) \to \mathrm{Mod}_{m(\partial_*(\mathrm{id}))}(\mathrm{End}(\mathrm{Mod}_E^{Cpl(I)})) \xrightarrow{ev_X} \mathrm{Alg}_{\Sigma\,\mathrm{Lie}}(\mathrm{Mod}_E^{Cpl(I)})$$

Write the Lie algebra corresponding to a module spectrum X and a functor F as $L_F(X)$. We observe that $L_{\mathrm{id}}(X) = L(X)$. Note that the underlying homotopy Lie algebra of $L_F(X)$ is simply given by $L_F(X) = \bigoplus_n \partial_n(F) \otimes_{h\Sigma_n} X^{\otimes n}$ where $\partial_n(F)$ denotes the usual Goodwillie derivatives.

Example 4.1.13 (Precomposing with Suspension). Given a functor F in $[\mathbf{sSet}_*, \mathbf{sSet}_*]$, we can *pre*compose with the suspension functor Σ^i to obtain a new functor $F\Sigma^i$. By Example 19.4 in [AC11] (cf. Section 2.2 of [Beh12]), the Goodwillie derivatives of $F\Sigma^i$ are given by $\partial_*(F\Sigma^i) = \partial_*(F) \otimes (S^i)^*$ where the left $\partial_*(\mathrm{id})$-module structure is obtained from the obvious structure maps

$$\partial_k(\mathrm{id}) \otimes (\partial_{n_1}(F) \otimes (S^i)^{n_1}) \otimes \ldots \otimes (\partial_{n_k}(F) \otimes (S^i)^{n_k}) \cong (\partial_k(\mathrm{id}) \otimes \partial_{n_1}(F) \otimes \ldots \otimes \partial_{n_k}(F)) \otimes (S^i)^{n_1} \otimes \ldots \otimes (S^i)^{n_k})$$

$$\longrightarrow \partial_{n_1+\ldots+n_k}(F) \otimes (S^i)^{n_1+\ldots+n_k}$$

We have an equivalence of Lie algebras $L_{F\circ\Sigma^i}(X) \cong L_F(\Sigma^i X)$.

Example 4.1.14 (Precomposing with a Power). We can also precompose a functor F with the "n^{th} power functor" $P^n(-) = (-)^{\wedge n}$. Writing n_* for the symmetric sequence $(*, \ldots, *, \Sigma_{n+}, *, *, \ldots)$, an evident generalisation of the argument of Behrens for $n = 2$ in Lemma 2.2.5 of [Beh12] establishes an equivalence of left $\partial_*(\mathrm{id})$-modules $\partial_*(FP^n) = \partial_*(F) \circ n_*$. There is a natural equivalence of Lie algebras $L_{F\circ P^n}(X) \cong L_F(X^{\otimes n})$.

Example 4.1.15 (Postcomposing with Ω). We can also *post*compose a functor F with the loops functor Ω to define a new functor ΩF. The Goodwillie derivatives are given by function spectra $\partial_*(\Omega F) = \mathrm{Map}(S^1, \partial_*(F))$.

Again by Example 19.4 [AC11], the $\partial_*(\mathrm{id})$-module structure is given by

$$\partial_k(\mathrm{id}) \otimes \mathrm{Map}(S^1, \partial_{n_1}(F)) \otimes \ldots \otimes \mathrm{Map}(S^1, \partial_{n_k}(F)) \to \mathrm{Map}(S^1, \partial_k(\mathrm{id}) \otimes \partial_{n_1}(F) \otimes \ldots \otimes \partial_{n_k}(F))$$

$$\longrightarrow \mathrm{Map}(S^1, \partial_{n_1+\ldots+n_k}(F))$$

where the first map uses the diagonal of S^1.

We observe that there is a natural equivalence e of underlying spectra $L_{\Omega F}(X) \cong \Sigma^{-1} L_F(X)$.

Differentiating the EHP Sequence

In this section, we shall recall the interaction of the EHP sequence with Goodwillie calculus as worked out by Arone-Mahowald [AM99] and Behrens [Beh10].

The EHP sequence of functors $\mathrm{id} \to \Omega\Sigma \to \Omega\Sigma Sq$ gives rise to a sequence of left $\partial_*(\mathrm{id})$-modules

$$\partial_*(\mathrm{id}) \to \partial_*(\Omega\Sigma) \to \partial_*(\Omega\Sigma Sq)$$

Given $X \in \mathrm{Mod}_E^{Cpl(I)}$, we obtain a sequence of spectral Lie algebras $L_{\mathrm{id}}(X) \to L_\Omega(\Sigma X) \to L_\Omega(\Sigma X \otimes X)$. For $X = \Sigma^{2n-1}E$ an odd suspension, we therefore have the following diagram of *module spectra*

$$\begin{array}{ccccc} L_{\mathrm{id}}(\Sigma^{2n-1}E) & \longrightarrow & L_\Omega(\Sigma^{2n}E) & \longrightarrow & L_\Omega(\Sigma^{4n-1}E) \\ \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\ L(\Sigma^{2n-1}E) & & \Sigma^{-1}L(\Sigma^{2n}E) & & \Sigma^{-1}L(\Sigma^{4n-1}E) \end{array}$$

The top sequence is a sequence of Lie algebras. This sequence is in fact a fibre sequence of underlying spectra by Section 4 of [AM99], and it splits into a direct sum of fibre sequences

$$\begin{array}{ccccc} \partial_w(\mathrm{id}) \underset{h\Sigma_w}{\otimes} (\Sigma^{2n-1}E)^{\otimes w} & \longrightarrow & \partial_w(\Omega) \underset{h\Sigma_w}{\otimes} (\Sigma^{2n}E)^{\otimes w} & \longrightarrow & \partial_{\frac{w}{2}}(\Omega) \underset{h\Sigma_{\frac{w}{2}}}{\otimes} (\Sigma^{4n-1}E)^{\otimes \frac{w}{2}} \\ \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\ \mathbb{D}(\Sigma|\Pi_w|^\diamond) \underset{h\Sigma_w}{\otimes} (\Sigma^{2n-1}E)^{\otimes w} & & \Sigma^{-1}\mathbb{D}(\Sigma|\Pi_w|^\diamond) \underset{h\Sigma_w}{\otimes} (\Sigma^{2n}E)^{\otimes w} & & \Sigma^{-1}\mathbb{D}(\Sigma|\Pi_{\frac{w}{2}}|^\diamond) \underset{h\Sigma_{\frac{w}{2}}}{\otimes} (\Sigma^{4n-1}E)^{\otimes \frac{w}{2}} \end{array}$$

where we again use the convention that the right hand spectrum vanishes for w odd.

Free Lie algebras on nonconnective spectra

Let R be any complex oriented ring spectrum and V a *complex* representation of a finite group G. The orientation of E provides an equivalence of naïve G-spectra $E \otimes S^V \xrightarrow{\simeq} E \otimes S^{|V|}$. Applying this to the standard action of Σ_n on $\mathbb{C}^n$, we obtain an equivalence of naïve Σ_n-spectra $E \otimes (S^2)^{\otimes n} \xrightarrow{\simeq} E \otimes (S^{2n})$.

Given any spectrum X, we can apply this to deduce an equivalences of Σ_n-spectra

$$R \otimes \mathbb{D}(\Sigma|\Pi_n|^\diamond) \otimes X^{\otimes n} \cong R \otimes \Sigma^{-2n}\mathbb{D}(\Sigma|\Pi_n|^\diamond) \otimes (\Sigma^2 X)^{\otimes n} \cong R \otimes \Sigma^{-4n}\mathbb{D}(\Sigma|\Pi_n|^\diamond) \otimes (\Sigma^4 X)^{\otimes n} = \dots$$

We deduce that for any integer k, there is an equivalence

$$(R \otimes \mathbb{D}(\Sigma|\Pi_n|^\diamond) \otimes X^{\otimes n})_{h\Sigma_n} \cong (R \otimes \Sigma^{-2kn}\mathbb{D}(\Sigma|\Pi_n|^\diamond) \otimes (\Sigma^{2k} X)^{\otimes n})_{h\Sigma_n}$$

Applying this to the case $R = \mathbb{F}_p$, this makes it possible to compute the $\mathbb{F}_p$-homology of any free spectral Lie algebra and hence solve a case which was left unsolved in [AC15] and [Kja16] (cf. Conjecture 2.14 in [Kja16]). In particular, we obtain a fibre sequence

$$(R \otimes \mathbb{D}(\Sigma|\Pi_w|^\diamond) \otimes (S^{2n-1})^{\otimes w})_{h\Sigma_w} \to (\Sigma^{-1}R \otimes \mathbb{D}(\Sigma|\Pi_w|^\diamond) \otimes (S^{2n})^{\otimes w})_{h\Sigma_w} \to (\Sigma^{-1}R \otimes \mathbb{D}(\Sigma|\Pi_{\frac{w}{2}}|^\diamond) \otimes (S^{4n-1})^{\otimes \frac{w}{2}})_{h\Sigma_{\frac{w}{2}}}$$

for all values of n.

Differentiating the Hilton-Milnor Theorem

Let w be a word in the free spectral Lie algebra on generators $x_1, \dots, x_k$ involving the i^{th} generator n_i times. Given pointed spaces $X_1, \dots, X_k$, we write $w(X_1, \dots, X_k)$ for the space obtained by letting the bracket act as smash product. The *iterated Samelson product* gives rise to a natural transformation $\phi_w : \Omega\Sigma w(X_1, \dots, X_k) \to \Omega\Sigma(X_1 \vee \dots \vee X_k)$. Given integers $i_1, \dots, i_k$, we write $N_w = \sum n_j i_j$. We obtain a transformation of functors from spaces to spaces by considering

$$\Omega\Sigma^{1+N_w} P^{N_w}(-) = \Omega\Sigma w(\Sigma^{i_1}-, \dots, \Sigma^{i_k}-) \circ \Delta(-) \to \Omega\Sigma(\Sigma^{i_1} - \vee \dots \vee \Sigma^{i_k}-) \circ \Delta(-)$$

Taking Goodwillie derivatives gives a map of left $\partial_*(\mathrm{id})$-modules

$$\mathrm{Map}(S^1, \partial_*(\mathrm{id}) \otimes (S^{1+N_w})^*) \circ (N_w)_* = \partial_*(\Omega\Sigma^{1+N}) \circ (N_w)_* \longrightarrow \mathrm{Map}(S^1, \partial_*(\mathrm{id}) \otimes (S^{i_1+1} \vee \dots \vee S^{i_k+1})^*)$$

Here we again used Example 19.4 in [AC11].

We obtain a map of shifted Lie algebras $F_\Omega(S^{1+N_w}) \to F_\Omega(S^{i_1+1} \vee \ldots \vee S^{i_k+1})$. The Hilton-Milnor theorem can be used to prove that the map of spectra $\bigoplus_{w\in B_k} F_\Omega(S^{1+N_w}) \to F_\Omega(S^{i_1+1} \vee \ldots \vee S^{i_k+1})$ defines an equivalence, hence recovering a special case of Corollary 2.3.14. This observation on the Goodwillie layers is originally due to [AK98], and we have refined it to a statement about Goodwillie *towers* in [BH17]. We will not make explicit use of this map since it is not a priori clear how the Samelson products interact with the Lie product on $F_\Omega(X)$ (cf. Question 8.11 in [BR17]).

4.2 Operations

In this section, we will construct operations acting on the homotopy groups Lie algebras in $\mathrm{Alg}_{\Sigma\,\mathrm{Lie}}(\mathrm{Mod}_E^{Cpl(I)})$, i.e. give names to specific elements in homotopy groups $\pi_j(L(\Sigma^{i_1} \oplus \ldots \oplus \Sigma^{i_k}))$ for integers $j, i_1, \ldots, i_k$. We will then compute the various relations between these operations in the following section.

4.2.1 Hecke Operations

We will first construct various *additive* and *unary* operations starting from Rezk's operations acting on $K(h)$-local $\mathbb{E}_\infty$-rings under E. Since Rezk's operations are closely connected to the Hecke algebra for $GL_n(\mathbb{Z}_p)$ (see Section 14 in [Rez06]), we have decided to call the operations constructed in this section the *Hecke operations* on Lie algebras in $\mathrm{Mod}_E^{Cpl(I)}$.

The construction of Hecke operations "starting in odd degrees" relies crucially on Rezk's Koszulness result in [Rez12b] and its reformulation in Lemma 5.6. in [BR15]. We then make use of the EHP sequence to construct Hecke operations "starting in even degrees". A more challenging component of this work is to compute how these operations compose. We will address this question in the following Section 4.3.1– this is where our work from Chapter 3 on the relationship between algebraic and topological Koszul duality is truly needed.

The Monad T

Let $T = T_{\mathrm{Mod}_E^{Cpl(I)}}$ be the augmented monad on $\mathrm{Mod}_E^{Cpl(I)}$ building the free *nonunital* $\mathbb{E}_\infty$-ring (see Definition 4.1.11). It can be constructed as monad corresponding to the ∞-operad $\mathcal{O}_{\mathrm{Comm}}^{nu\ \mathrm{Mod}_E^{Cpl(I)}}$ constructed in Section 4.1.10. Alternatively, we could build T as the monad for the forgetful-free adjunction between modules in $\mathrm{Mod}_E^{Cpl(I)}$ and nonunital commutative algebra objects in $\mathrm{Mod}_E^{Cpl(I)}$.

Remark 4.2.1. The monad T and its later variants $\hat{\mathbb{T}}$ and $\mathcal{L}_{\hat{\mathbb{T}}}$ correspond to *nonunital* $\mathbb{E}_\infty$-rings. Our notation differs slightly from Rezk's notation – he decorates the nonunital functors with a tilde.

The underlying functor of T is given by $T(X) = \bigoplus_{m>0} X^{\otimes m}_{h\Sigma_m}$ where, as always in this section, sum and homotopy coinvariants are computed in the $K(h)$-local setting. We can therefore write $T[m] = X^{\otimes m}_{h\Sigma_m}$ and obtain a direct sum decomposition $T = \bigoplus_{m>0} T[m]$.

Proposition 4.2.2. *The functor T is completed-free in the sense of Definition 3.2.19.*

Proof. Since (completed) coproducts of completed-free modules are evidently completed-free, it suffices to show that $T[m](M) = M^{\otimes m}_{h\Sigma_m}$ is completed-free for any completed-free E-module spectrum $M = \bigoplus_{s\in S} \Sigma^{i_s} E$. We expand "binomially" and write $(\bigoplus_{s\in S} \Sigma^{i_s} E)^{\otimes m}_{h\Sigma_m} \cong \bigoplus_{\{\{m_s\}_{s\in S} \mid \sum m_s = m\}} (\bigotimes_{s\in S} (\Sigma^{i_s} E)^{\otimes m_s}_{h\Sigma_{m_s}})$. Using that m_s vanishes for all but finitely many values of s, we see that $T[m](M)$ is the (completed) coproduct of module spectra of the form $T[m_1](\Sigma^{j_1} E) \otimes \ldots \otimes T[m_k](\Sigma^{j_k} E)$. Proposition 3.17 in [Rez09] and Proposition 5.1.18 in Appendix A imply that all $T[m_i](\Sigma^{j_i} E)$ are finite and free E-module spectra. □

Remark 4.2.3. A similar strategy is used in the proof of Corollary 3.10 in [BF15].

The Analytic Approximation $\hat{\mathbb{T}}$

The functor T gives rise to an analytic approximation monad $\hat{\mathbb{T}}$ on $P_\sigma(\mathrm{Mod}^{Cpl(I)}_{E_*,f})$ by Definition 3.2.21. Write $P_\sigma(\mathrm{Mod}^{Cpl(I)}_{E_*,f})^\heartsuit$ for the full subcategory of $P_\sigma(\mathrm{Mod}^{Cpl(I)}_{E_*,f})$ consisting of all functors which land in discrete spaces, i.e. sets. There is an evident functor $\iota : \mathrm{Mod}^{Cpl(I)}_{E_*} \hookrightarrow P_\sigma(\mathrm{Mod}^{Cpl(I)}_{E_*,f})^\heartsuit$ sending M to $\mathrm{Map}_{\mathrm{Mod}^{Cpl(I)}_{E_*}}(-, M)$. We obtain a functor $\pi_0 : P_\sigma(\mathrm{Mod}^{Cpl(I)}_{E_*,f})^\heartsuit \hookrightarrow P_\sigma(\mathrm{Mod}^{Cpl(I)}_{E_*,f}) \to \mathrm{Mod}^{Cpl(I)}_{E_*}$ in the other direction by postcomposing with the derived functor of the natural inclusion $\mathrm{Mod}^{Cpl(I)}_{E_*,f} \hookrightarrow \mathrm{Mod}^{Cpl(I)}_{E_*}$.

Proposition 4.2.4. *The functors (π_0, ι) form an equivalence of categories.*

Proof. Given an object $X \in P_\sigma(\mathrm{Mod}^{Cpl(I)}_{E_*,f})^\heartsuit$, we use Proposition 4.2.11. in [Lur11a] to pick a simplicial object $X_\bullet$ of completed-free modules whose realisation is X. Then $\pi_0(X)$ is given by the colimit of this diagram in the discrete category $\mathrm{Mod}^{Cpl(I)}_{E_*}$, i.e. there is a reflexive coequaliser $(X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0 \twoheadrightarrow \pi_0(X))$. Given any test module $S \in \mathrm{Mod}^{Cpl(I)}_{E_*,f}$, we use that completed-free modules are projective to compute that $\iota(\pi_0(M))(S) = \mathrm{Map}_{\mathrm{Mod}^{Cpl(I)}_{E_*}}(S, \pi_0(X))$ is given by the reflexive coequaliser

$$\mathrm{Map}_{\mathrm{Mod}^{Cpl(I)}_{E_*}}(S, X_1) \underset{d_1}{\overset{d_0}{\rightrightarrows}} \mathrm{Map}_{\mathrm{Mod}^{Cpl(I)}_{E_*}}(S, X_0) \twoheadrightarrow \iota(\pi_0(X))(S)$$

This coequaliser is equivalent to $\pi_0(|\mathrm{Map}_{\mathrm{Mod}^{Cpl(I)}_{E_*}}(S, X_\bullet)|)$. If X lies in the heart, this establishes that the canonical map $X \to \iota\pi_0 X$ is an equivalence.

Assume conversely that we are given a module $M \in \mathrm{Mod}^{Cpl(I)}_{E_*}$. We pick a simplicial completed-free module $M_\bullet$ over M such that the associated chain complex gives a projective resolution of M. We obtain an associated augmented simplicial diagram $\iota(M_\bullet) \to \iota(M)$ in $P_\sigma(\mathrm{Mod}^{Cpl(I)}_{E_*,f})$. This is in fact a colimit diagram in $P_\sigma(\mathrm{Mod}^{Cpl(I)}_{E_*,f})$ since for any test module $S \in \mathrm{Mod}^{Cpl(I)}_{E_*,f}$, evaluation gives a homotopy colimit in spaces. We can therefore compute $\pi_0(\iota M)$ as colimit of the simplicial module $M_\bullet$ and conclude that the canonical arrow $\pi_0 \iota M \to M$ is an equivalence as well. □

Proposition 4.2.5. *The natural arrow* $\pi_0\hat{\mathbb{T}}(X) \to \pi_0\hat{\mathbb{T}}(\iota\pi_0 X)$ *is an isomorphism for any* $X \in P_\sigma(\mathrm{Mod}_{E_*,f}^{Cpl(I)})$.

Proof. We pick a simplicial diagram $X_\bullet$ in $\mathrm{Mod}_{E_*,f}^{Cpl(I)}$ whose geometric realisation is X. We pick another diagram $Y_\bullet$ in $\mathrm{Mod}_{E_*,f}^{Cpl(I)}$ with an injection $X_\bullet \to Y_\bullet$ which is an isomorphism at level 0 and 1 and such that the simplicial set $Y_\bullet$ has vanishing homotopy groups above dimension 0.

The map in the above statement is then given by $\pi_0(|\hat{\mathbb{T}}(X_\bullet)|) \to \pi_0(|\hat{\mathbb{T}}(Y_\bullet)|)$ and therefore an equivalence. □

We therefore obtain an analytic approximation monad on $\mathrm{Mod}_{E_*}^{Cpl(I)}$ by considering $\pi_0 \circ \hat{\mathbb{T}} \circ \iota$. This monad has been constructed by Barthel-Frankland [BF15] using completely different methods.

Rezk's Rings Δ^{-i}

Rezk uses the monad $\hat{\mathbb{T}}$ (or rather its uncompleted version) to study the operations on the homotopy of (nonunital) $\mathbb{E}_\infty$-rings A under E. In order to recall some of Rezk's results concerning operations on the tangent space $\pi_*(A)/\pi_*(A)^2$, we follow [Rez12b] in a slightly more completed setting and introduce the following definition:

Definition 4.2.6. The linearisation of an arbitrary functor $F : \mathrm{Mod}_{E_*,f}^{Cpl(I)} \to \mathrm{Mod}_{E_*}^{Cpl(I)}$ is defined as the cokernel $\mathcal{L}_F(X) := \mathrm{cok}(F(X \oplus X) \xrightarrow{F(p_1+p_2)-F(p_1)-F(p_2)} F(X))$. Here p_i denote the evident projections.

The functor $\mathcal{L}_F$ is additive and in fact the initial such functor receiving a map from F.

We now apply this construction to the analytic approximation monad $\hat{\mathbb{T}}$ (considered as a functor of ordinary categories $\mathrm{Mod}_{E_*,f}^{Cpl(I)} \to \mathrm{Mod}_{E_*}^{Cpl(I)}$) to obtain the linearisation $\hat{\mathbb{T}} \to \mathcal{L}_{\hat{\mathbb{T}}}$.

Writing $\hat{\mathbb{T}}[n] = h(T[n]|_{\mathrm{Mod}_{E,f}^{Cpl(I)}})$ for $T[n](M) = M_{h\Sigma_n}^{\otimes n}$, we obtain another completed-free functor and a direct sum decomposition $\bigoplus_{n\geq 1} \hat{\mathbb{T}}[n] \cong \hat{\mathbb{T}}$ (where the (completed) sum is computed in the category $\mathrm{Mod}_{E_*}^{Cpl(I)}$ of L-complete E_*-modules).

Applying linearisation, we obtain a corresponding decomposition $\bigoplus_{n\geq 1} \mathcal{L}_{\hat{\mathbb{T}}[n]} \cong \mathcal{L}_{\hat{\mathbb{T}}}$, and the map $\hat{\mathbb{T}} \to \mathcal{L}_{\hat{\mathbb{T}}}$ is induced by taking the (completed) sum of the maps $\hat{\mathbb{T}}[n] \to \mathcal{L}_{\hat{\mathbb{T}}[n]}$.

Proposition 4.2.7. *The functors* $\mathcal{L}_{\hat{\mathbb{T}}[n]}$ *and* $\mathcal{L}_{\hat{\mathbb{T}}}$ *preserve completed-free* E_**-modules.*

Proof. Since completed sums of completed-free functors are completed-free, it suffices to check this for each $\mathcal{L}_{\hat{\mathbb{T}}[n]}$. Since this functor is additive, it suffices to check that $\mathcal{L}_{\hat{\mathbb{T}}[n]}(\Sigma^i E_*)$ is completed-free for each i. This (and more) is established in Proposition 5.7 in [Rez12b]. □

By Corollary 5.5 in [Rez12b], this implies that the functor $\mathcal{L}_{\hat{\mathbb{T}}}$ has the structure of a monad such that the natural map of endofunctors of $\mathrm{Mod}_{E_*,f}^{Cpl(I)}$ given by $\hat{\mathbb{T}} \to \mathcal{L}_{\hat{\mathbb{T}}}$ preserves this structure. We can in fact extend $\mathcal{L}_{\hat{\mathbb{T}}}$

to a realisation-preserving monad of $P_\sigma(\mathrm{Mod}_{E_*,f}^{Cpl(I)})$ with a transformation of extended endofunctors $\hat{\mathbb{T}} \to \mathcal{L}_{\hat{\mathbb{T}}}$. These functors come with an extra piece of additional structure: the topological suspension can be used to define natural transformations $Susp : \Sigma\mathcal{L}_{\hat{\mathbb{T}}}(M) \to \mathcal{L}_{\hat{\mathbb{T}}}(\Sigma M)$ and $Susp : \Sigma\mathcal{L}_{\hat{\mathbb{T}}}[m](M) \to \mathcal{L}_{\hat{\mathbb{T}}}[m](\Sigma M)$.

Definition 4.2.8. The i^{th} *ring of additive cotangent operations* for (nonunital) $K(h)$-local $\mathbb{E}_\infty$-rings under E is given by the ring $\Delta^{-i} = \mathrm{Map}_{\mathrm{Alg}_{\mathcal{L}_{\hat{\mathbb{T}}}}(\mathrm{Mod}_{E_*,f}^{Cpl(I)})}(\mathcal{L}_{\hat{\mathbb{T}}}(\Sigma^i E_*), \mathcal{L}_{\hat{\mathbb{T}}}(\Sigma^i E_*)) = [\mathcal{L}_{\hat{\mathbb{T}}}(\Sigma^i E_*)]_i$.

Here multiplication is defined by composition and addition is defined by using codiagonal maps.

We turn Δ^{-i} into a weight-graded ring by setting $\Delta^{-i}[w] = [\mathcal{L}_{\hat{\mathbb{T}}}[w](\Sigma^i E_*)]_i$.

Remark 4.2.9. The slightly confusing degree convention for the superscript of Δ^i is in agreement with Rezk's work. Our weight-grading differs slightly from Rezk's "logarithmic" choice.

Every scalar $\lambda \in E_j$ gives rise to a map of $\mathcal{L}_{\hat{\mathbb{T}}}$-algebras $\mathcal{L}_{\hat{\mathbb{T}}}(\Sigma^{i+j}E_*) \xrightarrow{\lambda} \mathcal{L}_{\hat{\mathbb{T}}}(\Sigma^i E_*)$. We can use this to define a morphism $E_j \otimes \Delta^{-i} \otimes E_{-j} \to \Delta^{-i-j}$. This in particular gives each of the above rings the structre of an (E_0, E_0)-bimodule. The resulting morphisms $E_j \otimes_{E_0} \Delta^{-i} \otimes_{E_0} E_{-j} \to \Delta^{-i-j}$ are in fact isomorphisms for all j even since E is even periodic.

The following is a completed version of Lemma 3.2 in [BR15]:

Proposition 4.2.10. *If M is a completed-free E_*-module, then there is a weight-grading preserving equivalence $[\mathcal{L}_{\hat{\mathbb{T}}}(M)]_j \cong \Delta^{-j} \otimes_{E_0} M_j$.*

In particular, we can write $[\mathcal{L}_{\hat{\mathbb{T}}}(\Sigma^i E_*)]_j = E_{j-i} \otimes_{E_0} [\mathcal{L}_{\hat{\mathbb{T}}}(\Sigma^i E_*)]_i = E_{j-i} \otimes_{E_0} \Delta^{-i} \cong \Delta^{-j} \otimes_{E_0} E_{j-i}$.

It is shown in 3.10 of [Rez12b] that Δ^{-i} indeed acts on the degree i component of the cotangent space of any $\hat{\mathbb{T}}$-algebra. By Remark 7.5 in [Rez09], the suspensions maps $Susp$ give rise to a diagram of weight-graded rings

$$\ldots \Delta^2 \xrightarrow{\cong} \Delta^1 \hookrightarrow \Delta^0 \xrightarrow{\cong} \Delta^{-1} \hookrightarrow \Delta^{-2} \xrightarrow{\cong} \ldots$$

where the maps alternate between monomorphisms and isomorphisms and all become isomorphisms after tensoring with $\mathbb{Q}$. We can in fact use the rings Δ^i to compute some non-additive derived functors of $\hat{\mathbb{T}}$. The transformation $\hat{\mathbb{T}} \to \mathcal{L}_{\hat{\mathbb{T}}}$ gives rise to a morphism of simplicial functors $\mathrm{Bar}_\bullet(\mathrm{id}, \hat{\mathbb{T}}, \mathrm{id}) \to \mathrm{Bar}_\bullet(\mathrm{id}, \mathcal{L}_{\hat{\mathbb{T}}}, \mathrm{id})$. Evaluating on a module $M \in \mathrm{Mod}_{E_*,f}^{Cpl(I)}$ gives a morphism of simplicial modules $\mathrm{Bar}_\bullet(\mathrm{id}, \hat{\mathbb{T}}, \overline{M}) \to \mathrm{Bar}_\bullet(\mathrm{id}, \mathcal{L}_{\hat{\mathbb{T}}}, \overline{M})$. Here an overline denotes the module M endowed with the trivial $\hat{\mathbb{T}}$- or $\mathcal{L}_{\hat{\mathbb{T}}}$-algebra structure, defined using the augmentation.

If $M = \Sigma^i E_*$ for i an odd integer, then $\overline{\Sigma^i E_*}$ is a $\hat{\mathbb{T}}$-algebra whose underlying strictly commutative nonunital E_*-algebra is free. Lemma 3.8 in [BR15] therefore shows:

Proposition 4.2.11. *For i odd, the canonical arrow* $\mathrm{Bar}_\bullet(\mathrm{id}, \hat{\mathbb{T}}, \overline{\Sigma^i E_*}) \to \mathrm{Bar}_\bullet(\mathrm{id}, \mathcal{L}_{\hat{\mathbb{T}}}, \overline{\Sigma^i E_*})$ *gives a weak equivalence in the nonnegative derived category* $P_\sigma(\mathrm{Mod}_{E_*,f}^{Cpl(I)}) \cong \mathcal{D}^-_{\geq 0}(\mathrm{Mod}_{E_*}^{Cpl(I)})$ *(see Proposition 4.1.2).*

Using Behrens-Rezk's Proposition 4.2.10, we observe:

Proposition 4.2.12. *For i odd, there is a weak equivalence of simplicial E_*-modules*

$$\mathrm{Bar}_\bullet(\mathrm{id}, \mathcal{L}_{\hat{\mathbb{T}}}, \overline{\Sigma^i E_*}) = \Sigma^i E_* \underset{E_0}{\otimes} [\mathrm{Bar}_\bullet(\mathrm{id}, \mathcal{L}_{\hat{\mathbb{T}}}, \overline{\Sigma^i E_*})]_i = \Sigma^i E_* \underset{E_0}{\otimes} \mathrm{Bar}_\bullet(\mathrm{id}, \Delta^i, \overline{E}_0)$$

Constructing Hecke Operations

We will now use the various rings Δ^i to define abelian groups as follows:

Definition 4.2.13. The *group* $(\mathcal{H}^{\mathrm{Lie}})_i^j[w]$ *of Hecke operations on Lie algebras from degree i to degree j* of weight w is defined as

$$(\mathcal{H}^{\mathrm{Lie}})_i^j[w] = \begin{cases} \mathrm{Ext}^a_{\Delta^i}(\overline{E}_0, \overline{E}_{-i+j+a}) & \text{if } w = p^a \\ 0 & \text{if } w \text{ is not a power of } p \end{cases}$$

Here $\overline{E}_0$ and $\overline{E}_{-i+j+a}$ denote the modules E_0 and E_{-i+j+a} endowed with the trivial Δ^i-action.

In the remainder of this section, we will attach a unique additive unary operation to any element in the above groups. In Section 4.3.1, we will then use the techniques from Chapter 3 to endow the collection $(\mathcal{H}^{\mathrm{Lie}})_i^j$ with additional multiplicative structure which encodes how the corresponding operations compose.

In order to access the techniques and notation used in Chapter 3, we shall now define a weighted structure on T on the sense of Definition 3.3.1. We use the notation of [BR15] and define a model for the doubly suspended partition complex $\Sigma|\Pi_n|^\diamond$ as the realisation of the simplicial set

$$(P_n)_s \quad = \quad \{x_0 \leq \cdots \leq x_s \mid \; x_i \text{ a partition of } \underline{\mathrm{n}}, \;\; x_0 \text{ discrete}, \;\; x_s \text{ indiscrete}\} \coprod \{*\}$$

with face and degeneracy map defined in an evident manner.

By a straightforward combinatorial argument explained in [Chi05] and [BR15], we can use this simplicial set to express the simplicial symmetric sequences in S-modules $(\mathbf{O}^{nu}_{\mathrm{Comm}})^{\circ s}_\bullet = (P_\bullet)_s$. Applying the functors m, Φ and $(-)^c$ from Section 4.1.2, we obtain an equivalence of simplicial endofunctors of $\mathrm{Mod}_E^{Cpl(I)}$:

$$T^{\circ s} = m(E \otimes \Phi(\mathbf{O}^{nu}_{\mathrm{Comm}})^{\circ s}) \simeq \bigoplus_{w \geq 1} (P_w)_s \underset{h\Sigma_w}{\otimes} (-)^{\otimes w}$$

The bar construction $\mathrm{Bar}_\bullet(T)$ is thus endowed with a natural direct sum decomposition with components

$$B_s^{[w]}(X) = m(p_w(\mathcal{O}_{\mathrm{Comm}}^{nu\ \ \mathrm{Mod}_E^{Cpl(I)}})^{\circ s}) = (P_w)_s \underset{h\Sigma_w}{\otimes} (-)^{\otimes w}$$

Let $C = \mathrm{KD}(T) = m(\mathrm{KD}(\mathcal{O}_{\mathrm{Comm}}^{nu\ \ \mathrm{Mod}_E^{Cpl(I)}})) \cong \bigoplus_{w\geq 1} C[w]$ for $C[w] = |B_\bullet^{[w]}(X)| \cong m(p_w\,\mathrm{KD}(\mathcal{O}_{\mathrm{Comm}}^{nu\ \ \mathrm{Mod}_E^{Cpl(I)}})) \cong \Sigma|\Pi_w|^\diamond \otimes_{h\Sigma_w} X^{\otimes w}$. The comonadic structure map $C \to C \circ C$ is obtained by applying m to a map of symmetric sequences.

Proposition 4.2.14. *This decomposition $B_\bullet^{[w]} \to \mathrm{Bar}(T)$ defines a weighted structure in the sense of Definition 3.3.1.*

Proof. The functors $B_s^{[w]}$ clearly preserve geometric realisations.

To check that $B_s^{[w]}$ is completed-free, it suffices by Proposition 5.1.18 in Appendix A to check that $\pi_* B_s^{[w]}$ sends completed-free E-modules to completed-free E_*-modules. By construction, $\pi_* B_s^{[w]}(M)$ is a summand of the module $\pi_* T^s(M)$ which we know to be completed-free by Proposition 4.2.2. Since completed-free E_*-modules are the projective objects in the abelian category of L-complete E_*-modules, they are closed under retracts. This establishes that $B_s^{[w]}$ is completed-free.

Condition 2) of Definition 3.3.1 holds true since the structure maps $T \circ T \to T$ and $C \to C \circ C$ are induced by corresponding maps of symmetric sequences. □

The following results are crucial since they in particular establishes that the universal cases are torsion-free:

Theorem 4.2.15. *The functors $C[w]$ and $C = \bigoplus_{w\geq 1} C[w]$ are completed-free (see Definition 3.2.19).*

Proof. It suffices to check the first statement. Fix a completed-free E-module spectrum $M = \bigoplus_{s\in S} \Sigma^{i_s} E$ (where the sum is completed). We can write $C[w](M) = E \otimes (\Sigma|\Pi_w|^\diamond \otimes (\bigoplus_{s\in S} S^{i_s})^{\otimes w})_{h\Sigma_w}$ (where we implicitly work $K(h)$-locally and abuse notation inside the bracket by writing $\otimes$ and $\oplus$ for the smash and wedge product of spectra). Expanding binomially and using Proposition 5.1.18 in Appendix A, it suffices to check that every module of the form $E_*^\wedge((\Sigma|\Pi_w|^\diamond \otimes (S^{j_1})^{\otimes w_1} \otimes \cdots \otimes (S^{j_k})^{\otimes w_k})_{h\Sigma_{w_1}\times\cdots\times\Sigma_{w_k}})$ is completed-free. By the splitting of Young restrictions of the partition complex established in Theorem 2.3.11, it is therefore enough to check that the E_*-modules $E_*^\wedge(\Sigma|\Pi_n|^\diamond \otimes_{\Sigma_n} (S^j)^{\otimes n})$ are completed-free.

If j is odd, then Lemma 5.6 of [BR15] implies that this module is null unless $n = p^k$. If $n = p^k$, then it is completed-free in odd degrees if k is odd and it is completed-free in even degrees if k is even by the same Lemma 5.6 and Theorem 2.13 in [BR15].

For j even, we may assume by periodicity that $j = 0$. Theorem 3.2 in [Aro06] gives a cofibre sequence

$$\Sigma|\Pi_{\frac{n}{2}}|^{\diamond} \underset{h\Sigma_{\frac{n}{2}}}{\otimes} (S^1)^{\otimes \frac{n}{2}} \to \Sigma|\Pi_n|^{\diamond}_{h\Sigma_n} \to |\Pi_n|^{\diamond} \underset{h\Sigma_n}{\otimes} (S^1)^{\otimes n}$$

where we use the convention that the left term is contractible if n is odd.

If p is an odd prime, then one of the outer terms has vanishing completed E-homology and the claim follows. If $p = 2$ and $n = 2^k$, then both of the outer modules must have completed E-homology concentrated in the same parity by Lemma 5.6 and Theorem 2.13 in [BR15]. We therefore obtain a short exact sequence

$$0 \to E_*^{\wedge}(\Sigma|\Pi_{\frac{n}{2}}|^{\diamond} \underset{h\Sigma_{\frac{n}{2}}}{\otimes} (S^1)^{\otimes \frac{n}{2}}) \to E_*^{\wedge}(\Sigma|\Pi_n|^{\diamond}_{h\Sigma_n}) \to E_*^{\wedge}(|\Pi_n|^{\diamond} \underset{h\Sigma_n}{\otimes} (S^1)^{\otimes n}) \to 0$$

The middle module is therefore an extension of projective objects of $\mathrm{Mod}_{E_*}^{Cpl(I)}$ and hence projective itself. □

Essentially the same argument as in the proof of Theorem 4.2.15 (this time making use of the Goodwillie derivatives of the actual EHP-sequence as explained in [AM99] and [Beh12]) implies:

Theorem 4.2.16. *The functors* $L[w] = (\mathbb{D}(\Sigma\Pi_w^{\diamond}) \otimes_{h\Sigma_w} (-)^{\otimes w})$ *and* $L = \bigoplus_w L[w]$ *preserve* $\mathrm{Mod}_{E,f}^{Cpl(I)}$.

We now come to the problem of lifting operations. Rezk's Koszulness proof implies:

Theorem 4.2.17. *The module* $\Sigma^i E_*$ *is p-Koszul for* T *in the sense of Definition 3.3.3 whenever* i *is odd.*

Proof. This result is contained in Theorem 2.13 and the proof of Lemma 5.6 in [BR15]. □

We encourage the reader to recall the meaning of the symbols $\mathbb{Q}, \mathfrak{Q}$, and Q from Section 3.3. We introduce a linearised version $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ by defining $\overline{\mathbb{Q}}_{S^a(i)}^{S^b(j)}$ as

$$\mathrm{Map}_{P_\sigma(\mathrm{Mod}_{E_*,f}^{Cpl(I)})}(\mathrm{KD}(\mathcal{L}_{\hat{\mathbb{T}}})(S^a(\Sigma^i E)), S^b(\Sigma^j E)) = \mathrm{Map}_{P_\sigma(\mathrm{Mod}_{E_*,f}^{Cpl(I)})}(|\,\mathrm{Bar}_\bullet(\mathrm{id}, \mathcal{L}_{\hat{\mathbb{T}}}, \overline{S^a(\Sigma^i E)})|, S^b(\Sigma^j E))$$

The decomposition of $\mathcal{L}_{\hat{\mathbb{T}}}$ into weighted pieces induces a decomposition on the bar construction which we use to define a product decomposition of $\overline{\mathbb{Q}}_{S^a(i)}^{S^b(j)}$ into pieces $\overline{\mathbb{Q}}_{S^a(i)}^{S^b(j)}[w]$. Write $P_i^j[w] = \mathrm{Map}_{\mathrm{Mod}_E^{Cpl(I)}}(\Sigma^j E, L[w](\Sigma^i E))$.

Definition 4.2.18. The *duality transformation* $\mathbb{D} : Q_i^j[w] \to P_{-i}^{-j}[w]$ is defined as the following composite:

$$Q_i^j[w] = \mathrm{Map}_{\mathrm{Mod}_E^{Cpl(I)}}(C[w](\Sigma^i E), \Sigma^j E) \xrightarrow{\mathbb{D}} \mathrm{Map}_{\mathrm{Mod}_E^{Cpl(I)}}(\Sigma^{-j} E, \mathbb{D}C[w](\Sigma^i E)) \xrightarrow{\nu_w^{-1}\circ} P_{-i}^{-j}[w]$$

Here we first use $K(h)$-local Spanier-Whitehead duality and then the transformation ν_w from Section 4.1.3 (which is an equivalence since $K(h)$-local Tate spectra vanish).

For i odd, j any integer, and $a \geq 0$, Lemma 3.3.6 then implies the existence of natural maps:

$$\overline{\mathbb{Q}}^{S^a(j-a)}_{S^0(i)}[w] \to \mathbb{Q}^{S^a(j-a)}_{S^0(i)}[w] \to \mathfrak{Q}^{S^a(j-a)}_{S^0(i)}[w] \to Q^j_i[w] \to P^{-j}_{-i}[w]$$

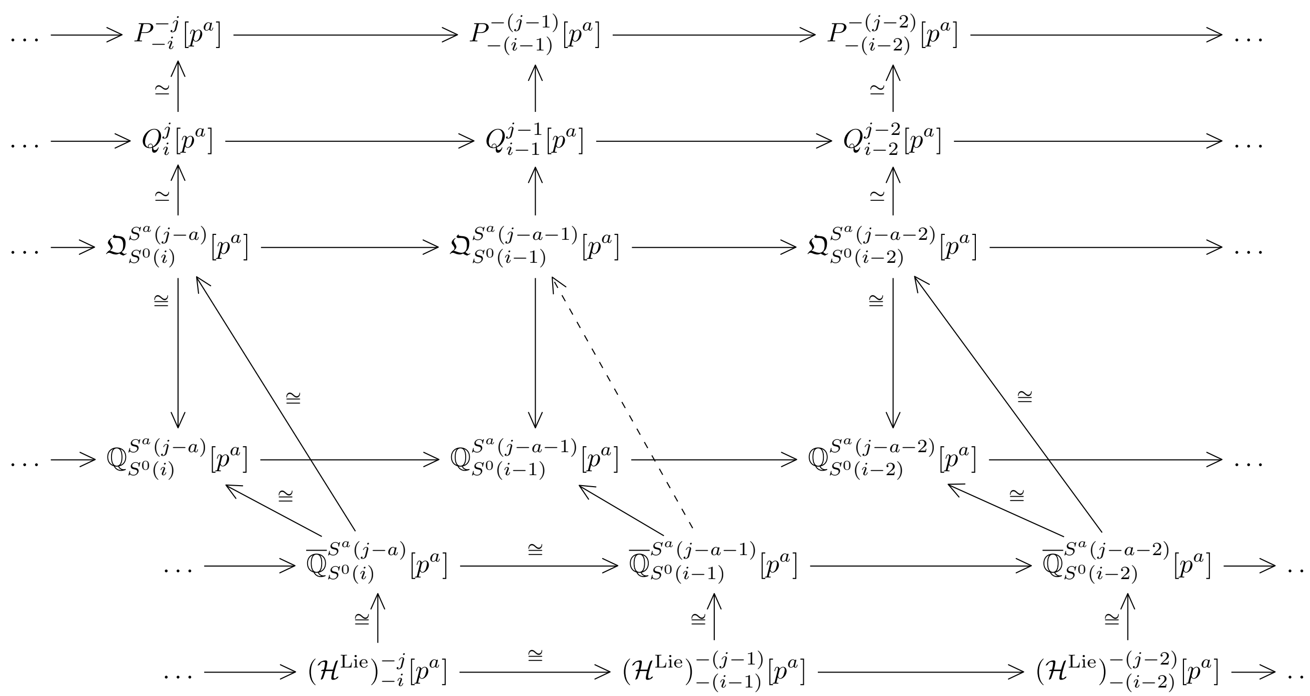
If $w \neq p^a$, then the second map is defined to be zero. If $w = p^a$, then we define it to be the inverse of the isomorphism supplied by Lemma 3.3.6. The same Lemma and Proposition 4.2.11 in fact imply that all maps are isomorphisms in this case. Moreover, we can use Lemma 4.2.12 to deduce a natural identification

$$(\mathcal{H}^{\mathrm{Lie}})^{-j}_{-i}[p^a] = \mathrm{Ext}^a_{\Delta^{-i}}(\overline{E}_0, \overline{E}_{i-j+a}) \cong \mathrm{Map}_{\mathcal{D}^-_{\geq 0}(\mathrm{Mod}^{Cpl(I)}_{E_0})}\left(\mathrm{Bar}_\bullet(\overline{E}_0, \Delta^{-i}, \overline{E}_0)[p^a], S^a(E_{i-j+a})\right)$$

$$\xrightarrow{\simeq} \mathrm{Map}_{\mathcal{D}^-_{\geq 0}(\mathrm{Mod}^{Cpl(I)}_{E_*})}\left(\mathrm{Bar}_\bullet(1, \mathcal{L}_{\hat{\mathbb{T}}}, \overline{\Sigma^i E_*})[p^a], S^a(\Sigma^{j-a} E_*)\right) = \overline{\mathbb{Q}}^{S^a(j-a)}_{S^0(i)}[p^a]$$

Since the suspension morphism $\Delta^{i+1} \to \Delta^i$ is an isomorphism of rings whenever i is odd (see [Rez09]), we deduce that the canonical suspension morphism $(\mathcal{H}^{\mathrm{Lie}})^j_i[p^a] \xrightarrow{\simeq} (\mathcal{H}^{\mathrm{Lie}})^{j+1}_{i+1}[p^a]$ is an isomorphism for i odd.

This allows us to fill in the dotted arrows in the following infinitely long commutative diagram (for i odd):



Theorem 4.2.19. *There are natural maps $(\mathcal{H}^{\mathrm{Lie}})^j_i[w] \to P^j_i[w]$ for all values of $i, j \in \mathbb{Z}$ and $w \in \mathbb{N}$ which are compatible with suspensions and isomorphisms whenever i is odd.*

This concludes our construction of Hecke operations. Their compositions will be the subject of a later section.

Additivity

We will now establish the additivity of the Hecke operations which we have just constructed. For this and later multiplicative considerations, the following Lemma is crucial:

Proposition 4.2.20. *The map* $\iota : (\mathcal{H}^{\mathrm{Lie}})_i^j \to P_i^j$ *is an equivalence for* i *odd and an injection for* i *even. Moreover,* $(\mathcal{H}^{\mathrm{Lie}})_i^j \hookrightarrow P_i^j$ *is a pure subgroup, which means (since* P_i^j *is torsion-free) that if* $\alpha \in P_i^j$ *and* $n \in \mathbb{N}$ *are such that* $n \cdot \alpha$ *lies in the image of* ι*, then* α *lies in the image of* ι*.*

Proof. By constriction of the map $(\mathcal{H}^{\mathrm{Lie}})_i^j \to P_i^j$ in the preceding section, it suffices to check that whenever $i = 2n-1$ is odd, the suspension map $P_{2n-1}^{j-1}[w] \to P_{2n}^j[w]$ is injective and the image is a pure subgroup.

The diagonal map $S^1 \to S^w$ of Σ_w-spaces gives rise to a map $\Sigma E \to (\Sigma E)^{\otimes w}$ of Σ_w-objects in $\mathrm{Mod}_E^{Cpl(I)}$. We smash this map with $\mathbb{D}(\Sigma|\Pi_w|^\diamond) \otimes (\Sigma^{2n-1}E)^{\otimes w}$ to obtain the following map of $K(h)$-local E-module spectra with Σ_w-action:

$$\mathbb{D}(\Sigma|\Pi_w|^\diamond) \otimes (\Sigma^{2n-1}E)^{\otimes w} \otimes \Sigma E \longrightarrow \mathbb{D}(\Sigma|\Pi_w|^\diamond) \otimes (\Sigma^{2n-1}E)^{\otimes w} \otimes (\Sigma E)^{\otimes w}$$

Using the chosen complex orientation on E-theory, we obtain an equivalence of Σ_w-spectra $(\Sigma^2 E)^{\otimes w} \cong \Sigma^{2w}E$. We obtain a square of Σ_w-objects

$$\begin{array}{ccc}
\mathbb{D}(\Sigma|\Pi_w|^\diamond) \otimes (\Sigma^{2n-1}E)^{\otimes w} \otimes \Sigma E & \longrightarrow & \mathbb{D}(\Sigma|\Pi_w|^\diamond) \otimes (\Sigma^{2n-1}E)^{\otimes w} \otimes (\Sigma E)^{\otimes w} \\
\simeq\downarrow & & \downarrow\simeq \\
\mathbb{D}(\Sigma|\Pi_w|^\diamond) \otimes (\Sigma E)^{\otimes w} \otimes \Sigma^{(2n-2)w}E \otimes \Sigma E & \longrightarrow & \mathbb{D}(\Sigma|\Pi_w|^\diamond) \otimes (\Sigma E)^{\otimes w} \otimes \Sigma^{(2n-2)w}E \otimes (\Sigma E)^{\otimes w}
\end{array}$$

We apply the functor $\pi_j(-)_{h\Sigma_w}$ to this diagram and observe a square

$$\begin{array}{ccc}
P_{2n-1}^{j-1} & \longrightarrow & P_{2n}^{j} \\
\simeq\downarrow & & \downarrow\simeq \\
P_1^{j-(2n-2)w-1}[w] & \longrightarrow & P_2^{j-(2n-2)w}[w]
\end{array}$$

It therefore suffices to check that $P_1^{k-1}[w] \to P_2^k[w]$ is injective with pure image.

Applying π_{k-1} to the weight w component of the EHP sequence explained in Section 4.1.3 gives rise to a short exact sequence sequence $0 \to P_1^{k-1}[w] \to P_2^k[w] \to P_3^k\left[\frac{w}{2}\right] \to 0$. We see that the left map is injective. The image is pure since $P_3^k\left[\frac{w}{2}\right]$ is torsion-free (and in fact free) as an E_0-module by Theorem 4.2.16.

∎

We will now establish that all Hecke operations in $Q_i^j[w]$ which we have produced from $\mathcal{H}^{\mathrm{Lie}}$ in fact act additively on the cohomotopy of C-coalgebras. Here, as before, we define the j^{th} cohomotopy group of some module spectrum $M \in \mathrm{Mod}_E^{Cpl(I)}$ by $\pi^j(M) := \pi_0 \operatorname{Map}_{\mathrm{Mod}_E^{Cpl(I)}}(M, \Sigma^j E)$.

We first use that the comonad $C = \bigoplus_w \Sigma|\Pi_w|^\diamond \otimes_{h\Sigma_w} (-)^{\otimes w}$ comes from a symmetric sequence to establish two useful properties of the suspension:

Lemma 4.2.21. *The suspension* $\mathrm{Susp}(\alpha) \in Q_{i-1}^{j-1}[w]$ *of any operation* $\alpha \in Q_i^j[w]$ *is additive.*

Proof. By decomposing Lemma 3.1.5 into the weighted pieces of C, we observe that it suffices to check that for each functor $C[w](X) = \Sigma|\Pi_w|^\diamond \otimes_{h\Sigma_w} X^{\otimes w}$, the two ways of passing from left to right in the following diagram of $K(h)$-local E-module spectra agree:

$$\begin{array}{ccccc} & & C[w](\Sigma^i E) \oplus C[w](\Sigma^i E) & & \\ & & \uparrow & \searrow & \\ \Sigma C[w](\Sigma^{i-1}E \oplus \Sigma^{i-1}E) & \longrightarrow & C[w](\Sigma^i E \oplus \Sigma^i E) & \longrightarrow & C[w](\Sigma^i E) \end{array}$$

By the binomial formula, we have $C[w](\Sigma^i E \oplus \Sigma^i E) \cong \bigoplus_{k=0}^{w} (\Sigma|\Pi_w|^\diamond \otimes (\Sigma^i E)^{\otimes k} \otimes (\Sigma^i E)^{\otimes w-k})_{h\Sigma_k \times \Sigma_{w-k}}$.

For $k \neq 0, w$, the map $f_k : \Sigma C[w](\Sigma^{i-1}E \oplus \Sigma^{i-1}E) \longrightarrow \Sigma|\Pi_w|^\diamond \otimes_{h\Sigma_k \times \Sigma_{w-k}} ((\Sigma^i E)^{\otimes k} \otimes (\Sigma^i E)^{\otimes w-k})$ factors through:

$$(S^1 \otimes \Sigma|\Pi_w|^\diamond \otimes (\Sigma^{i-1}E)^{\otimes k} \otimes (\Sigma^{i-1}E)^{\otimes w-k})_{h\Sigma_k \times \Sigma_{w-k}} \to (S^1 \otimes S^1 \otimes \Sigma|\Pi_w|^\diamond \otimes (\Sigma^{i-1}E)^{\otimes k} \otimes (\Sigma^{i-1}E)^{\otimes w-k})_{h\Sigma_k \times \Sigma_{w-k}}$$

Since the diagonal $S^1 \to S^1 \otimes S^1$ is null, we conclude that the original map f_k is in fact null for $k \neq 0$. This implies the claim. ☐

We remark that a similar strategy is used in the proof of Proposition 3.3 in [Rez12b].

We now use the absence of torsion in universal examples to prove:

Lemma 4.2.22. *If* $\alpha \in Q_i^j[w]$ *is an operation and there exists some positive integer* N *for which* $N \cdot \alpha$ *is additive, then* α *must also be additive.*

Proof. We again consider the "weight w"-component of the triangle from Lemma 3.1.5

$$\begin{array}{ccc} \pi^j C[w](\Sigma^i R) & \xrightarrow{\pi^j C[w](co\Delta)} & \pi^j C[w](\Sigma^i R \oplus \Sigma^i R) \\ & \searrow^{\pi^j co\Delta_{C[w](\Sigma^i R)}} & \uparrow{\scriptstyle \pi^j C[w](p_1) \oplus \pi^j C[w](p_2)} \\ & & \pi_j T(\Sigma^i R) \oplus \pi^j C[w](\Sigma^i R) \end{array}$$

We denote the two possible composites in the triangle by f and g, respectively. Our assumption implies that $N \cdot (f(\alpha) - g(\alpha)) = f(N\alpha) - g(N\alpha) = 0$ by Lemma 3.1.5. By Theorem 4.2.15, the group $\pi^j C[w](\Sigma^i R \oplus \Sigma^i R)$ is torsion-free. This implies $f(\alpha) = g(\alpha)$ and hence α is additive by Lemma 3.1.5. □

Lemma 4.2.23. *Every α in the image of the maps $(\mathcal{H}^{\mathrm{Lie}})^j_i[w] \to Q^{-j}_{-i}[w]$ constructed in the last section is an additive operation.*

Proof. Let α be attached to $\lambda \in (\mathcal{H}^{\mathrm{Lie}})^j_i[w]$. Since the suspension map $\Delta^{-(i-1)} \to \Delta^{-i}$ becomes an isomorphism after tensoring with $\mathbb{Q}$, we can chose a positive integer N such that $N\lambda$ lies in the image of the suspension map $(\mathcal{H}^{\mathrm{Lie}})^{j-1}_{i-1}[w] \to (\mathcal{H}^{\mathrm{Lie}})^j_i[w]$. Since our family of maps $(\mathcal{H}^{\mathrm{Lie}})^j_i[w] \to Q^{-j}_{-i}[w]$ respects suspension by construction, this implies that $N\alpha$ lies in the image of the suspension map as well and hence is additive by Lemma 4.2.21. This in turn implies by Lemma 4.2.22 that α is itself additive. □

Corollary 4.2.24. *Every α in the image of the maps $(\mathcal{H}^{\mathrm{Lie}})^j_i[w] \to P^j_i[w]$ constructed in the last section is an additive operation.*

Proof. This follows from the previous Lemma by applying Spanier-Whitehead duality to the weight w component of the diagram appearing in the criterion in Lemma 3.1.5 and then appealing to Lemma 3.1.3. □

4.2.2 The θ-Operations

In this section, we will construct operations $\Psi_{2n}, \theta_{2n}, s_{2n} \in P^{4n-1}_{2n}[2] = E^{\wedge}_{4n-1}(\Sigma^{-1}(S^{2n})^{\otimes 2}_{h\Sigma_2})$ for every $n \in \mathbb{Z}$.

We begin with a decomposition of the identity map on the sphere spectrum: $S^0 \cong Be_+ \xrightarrow{res} B\Sigma_{2\,+} \to S^0$. Here the first map is given by restriction and the second map is given by collapsing $B\Sigma_2$ to a point. We apply $L_{K(h)}\mathbb{D}(-)$ and use $K(h)$-local vanishing of Tate spectra and duality between restrictions and transfers to obtain a diagram of $K(h)$-local spectra

$$\begin{array}{ccccc} S^0 & \longleftarrow & (S^0)^{h\Sigma_2} & \longleftarrow & S^0 \\ \uparrow\simeq & & \uparrow\simeq & & \uparrow\simeq \\ S_0 & \xleftarrow{tr} & B\Sigma_{2\,+} & \xleftarrow{-\theta} & S^0 \end{array}$$

This defines a canonical element $\theta \in \pi^s_0(B\Sigma_{2+})$ (cf. [Sta16] for a different perspective). Write $s \in \pi^s_0(B\Sigma_{2+})$ for the class given by the restriction $S^0 \to B\Sigma_{2+}$. The double coset formula yields $tr(s) = 2$.

Finally, we set $\Psi := s + 2\theta$ and observe immediately that $tr(\Psi) = 2 + 2 \cdot (-1) = 0$.

We apply the (implicitly $K(h)$-localised) functor $\Sigma^{-1}(E \otimes -)$ to obtain three classes

$$\Psi_0, \theta_0, s_0 \in P_0^{-1}[2] = E^\wedge_{-1}(\Sigma^{-1} B\Sigma_{2\,+})$$

For each integer n, we define the operations

$$\Psi_{2n}, \theta_{2n}, s_{2n} \in P_{2n}^{4n-1}[2] = E^\wedge_{4n-1}(\Sigma^{-1}(S^{2n})^{\otimes 2}_{h\Sigma_2})$$

as the various classes corresponding to Ψ_0, θ_0, s_0 under the isomorphisms

$$\ldots \xrightarrow{\cong} E^\wedge_3(\Sigma^{-1}(S^2)^{\otimes 2}_{h\Sigma_2}) \xrightarrow{\cong} E^\wedge_{-1}(\Sigma^{-1}B\Sigma_{2+}) \xrightarrow{\cong} E^\wedge_{-5}(\Sigma^{-1}(S^{-2})^{\otimes 2}_{h\Sigma_2}) \xrightarrow{\cong} \ldots$$

coming from the equivalence of naïve Σ_2-spectra $E \otimes (S^2)^{\otimes 2} \cong E \otimes S^4$ provided by the Thom isomorphism induced by our chosen orientation on E. Here Σ_2 acts by swapping on the left and trivially on the right.

It is clear that $\Psi_{2n} = s_{2n} + 2\theta_{2n}$ for all n. Moreover, we observe that s_{2n} is given by the image of the fundamental class under the desuspended restriction.

Proposition 4.2.25. *The image of Ψ_{2n} under the transfer $E^\wedge_{4n-1}(\Sigma^{-1}(S^{2n})^{\otimes 2}_{h\Sigma_2}) \to E^\wedge_{4n-1}(\Sigma^{-1}(S^{2n})^{\otimes 2})$ vanishes.*

Proof. We smash the equivalence $E \otimes (S^{2n})^{\otimes 2} \cong E \otimes S^{4n}$ coming from the Thom isomorphism with the transfer map $S^0 \to \Sigma_{2+}$ and then desuspend to obtain the following square of naïve Σ_2-spectra:

$$\begin{array}{ccc}
\Sigma^{-1}E \otimes (S^{2n})^{\otimes 2} & \longrightarrow & \Sigma^{-1}E \otimes (S^{2n})^{\otimes 2} \otimes \Sigma_{2\,+} \\
\downarrow & & \downarrow \\
\Sigma^{-1}E \otimes (S^{4n}) & \longrightarrow & \Sigma^{-1}E \otimes (S^{4n}) \otimes \Sigma_{2\,+}
\end{array}$$

Taking homotopy coinvariants and applying homotopy groups implies the claim as $tr(\Psi) = 0$. ☐

Proposition 4.2.26. *The element Ψ_{2n} lies in the image under the injection $(\mathcal{H}^{\mathrm{Lie}})_{2n}^{4n-1}[2] \hookrightarrow P_{2n}^{4n-1}[2]$.*

Proof. For every even integer $2n$, there is a cofibre sequence of spectra (coming from the EHP sequence)

$$\begin{array}{ccccc}
(S^{2n-1})^{\otimes 2}_{h\Sigma_2} & \longrightarrow & \Sigma^{-1}(S^{2n})^{\otimes 2}_{h\Sigma_2} & \xrightarrow{\Sigma^{-1}tr} & S^{4n-1} \\
\| & & \| & & \\
\Sigma^{2n-1}\mathbb{R}P^\infty_{2n-1} & & \Sigma^{2n-1}\mathbb{R}P^\infty_{2n} & &
\end{array}$$

All three spectra have completed E-homology $E^\wedge_*(-)$ concentrated in odd degree (this is established in

Remark 3.20 and Corollary 3.21 in [Rez09].) Applying $E_*^\wedge(-)$ therefore gives a short exact sequence. Since $tr(\Psi_{2n}) = 0$, we see that Ψ_{2n} lies in the image of the suspension map $P_{2n-1}^{4n-2}[2] \to P_{2n}^{4n-1}[2]$. The claim follows from the construction of the map $\mathcal{H}^{\mathrm{Lie}} \to P$ in Theorem 4.2.19. □

4.2.3 The Lie Bracket

We define $[-,-]_{i,j} \in P_{i,j}^{i+j-1} = \pi_{i+j-1}(\bigoplus_k \mathbb{D}(\Sigma|\Pi_k|^\diamond) \otimes_{h\Sigma_k} (\Sigma^i E \oplus \Sigma^j E)^{\otimes k})$ by the canonical map

$$\Sigma^{i+j-1}E \longrightarrow \Sigma^{-1}E \otimes (\Sigma^i E \otimes \Sigma^j E) \longrightarrow \mathbb{D}(\Sigma|\Pi_2|^\diamond) \underset{h\Sigma_2}{\otimes} (\Sigma^i E \oplus \Sigma^j E)^{\otimes 2} \longrightarrow L(\Sigma^i E \oplus \Sigma^j E)$$

This means that if M is a shifted Lie algebra in $\mathrm{Mod}_E^{Cpl(I)}$ and $x \in \pi_i(M)$, $y \in \pi_j(M)$ are two given classes represented by maps $\Sigma^i E \xrightarrow{\overline{x}} M$ and $\Sigma^j E \xrightarrow{\overline{y}} M$, then the class $[x,y] = [x,y]_{i,j} \in \pi_{i+j-1}(M)$ is represented by

$$\overline{[x,y]}: \quad \Sigma^{i+j-1}E \longrightarrow L(\Sigma^i E \oplus \Sigma^j E) \xrightarrow{L(\overline{x}\oplus\overline{y})} L(M) \longrightarrow M$$

The final map is given by the structure map of M.

We will from now on drop the subscripts from the Lie bracket.

4.3 Relations

In the preceding Section 4.2, we have constructed three families of operations: the additive unary Hecke operations, the non-additive unary θ-operations, and the binary Lie bracket. We can now enquire what happens when we plug these operations into one another, i.e. compose them. The principal aim of this section is to compute the various relations between the compositions of the above operations.

4.3.1 The Action of $\mathcal{H}^{\mathrm{Lie}}$

We first examine how Hecke operations compose. Given an E_0-module M, we again write $\overline{M}$ for the "trivial" Δ^j-module with underlying abelian group M and structure map $\Delta^j \otimes M \to \Delta^j[1] \otimes M = E_0 \otimes M \to M$ defined using the augmentation. We upgrade Definition 4.2.13 multiplicatively, hence producing a *power ring* in the sense of Definition 3.1.7:

Definition 4.3.1. The *power ring* $\mathcal{H}^{\mathrm{Lie}}$ *of Hecke operations on Lie algebras* is given by

$$(\mathcal{H}^{\mathrm{Lie}})_i^j[w] = \begin{cases} \mathrm{Ext}^a_{\Delta^i}(\overline{E}_0, \overline{E}_{-i+j+a}) & \text{if } w = p^a \\ 0 & \text{if } w \text{ is not a power of } p \end{cases}$$

The multiplication map $(H^{Lie})_i^j[p^a] \otimes (\mathcal{H}^{\mathrm{Lie}})_j^k[p^b] \to (\mathcal{H}^{\mathrm{Lie}})_i^k[p^{a+b}]$ is given by the composite:

$$(\mathrm{Ext}^a_{\Delta^i}(\overline{E}_0, \overline{E}_{-i+j+a})) \otimes (\mathrm{Ext}^b_{\Delta^j}(\overline{E}_0, \overline{E}_{-j+k+b})) \to (\mathrm{Ext}^a_{\Delta^i}(\overline{E}_0, \overline{E}_{-i+j+a})) \otimes (\mathrm{Ext}^b_{\Delta^{j+a}}(\overline{E}_0, \overline{E}_{-j+k+b}))$$

$$\to (\mathrm{Ext}^a_{\Delta^i}(\overline{E}_0, \overline{E}_{-i+j+a})) \otimes (\mathrm{Ext}^b_{\Delta^i}(\overline{E}_{-i+j+a}, \overline{E}_{-i+k+a+b})) \to (\mathrm{Ext}^{a+b}_{\Delta^i}(\overline{E}_0, \overline{E}_{-i+k+a+b}))$$

The first map uses the suspension $\Delta^{j+a} \to \Delta^j$, the second map uses the morphism $\mathrm{Ext}^*_{\Delta^\ell}(\overline{E}_0, \overline{E}_r) \to \mathrm{Ext}^*_{\Delta^{\ell-s}}(\overline{E}_s, \overline{E}_{r+s})$ coming from the twisting morphism, and the final map uses the Yoneda product.

The following main result of this section establishes that compositions in algebra and topology are compatible. We therefore obtain an action of the power ring $\mathcal{H}^{\mathrm{Lie}}$ on the homotopy of any Lie algebra in $\mathrm{Mod}_E^{Cpl(I)}$:

Theorem 4.3.2. *For all $i, j, k \in \mathbb{Z}$ and all $v, w \in \mathbb{N}$, the following diagram commutes:*

$$\begin{array}{ccc} (\mathcal{H}^{\mathrm{Lie}})_i^j[v] \times (\mathcal{H}^{\mathrm{Lie}})_j^k[w] & \longrightarrow & (\mathcal{H}^{\mathrm{Lie}})_i^k[vw] \\ \downarrow & & \downarrow \\ P_i^j[v] \times P_j^k[w] & \longrightarrow & P_i^k[vw] \end{array}$$

We prove this result in several smaller steps which we shall assemble to a proof in the end of this section.

Lemma 4.3.3. *For all $i, j, k \in \mathbb{Z}$ and all $a, b \in \mathbb{N}$, the following diagram commutes:*

$$\begin{array}{ccc}
(\mathcal{H}^{\mathrm{Lie}})_i^j[p^a] \times (\mathcal{H}^{\mathrm{Lie}})_j^k[p^b] & \longrightarrow & (\mathcal{H}^{\mathrm{Lie}})_i^k[p^{a+b}] \\
\downarrow & & \downarrow \\
\overline{\mathbb{Q}}_{S^0(-i)}^{S^a(-j-a)}[p^a]\times\overline{\mathbb{Q}}_{S^0(-j)}^{S^b(-k-b)}[p^b] \xrightarrow{\mathrm{id}\times Sh^a} \overline{\mathbb{Q}}_{S^0(-i)}^{S^a(-j-a)}[p^a] \times \overline{\mathbb{Q}}_{S^a(-j-a)}^{S^{a+b}(-k-a-b)}[p^b] & \longrightarrow & \overline{\mathbb{Q}}_{S^0(-i)}^{S^{a+b}(-k-a-b)}[p^{a+b}]
\end{array}$$

Proof. We consider the following diagram:

$$\begin{array}{ccc}
\mathrm{Ext}^a_{\Delta^i}(\overline{E}_0, \overline{E}_{-i+j+a}) \times \mathrm{Ext}^b_{\Delta^j}(\overline{E}_0, \overline{E}_{-j+k+b}) & \longrightarrow & \dots \\
\downarrow & & \\
\overline{\mathbb{Q}}_{S^0(-i)}^{S^a(-j-a)}[p^a] \times \overline{\mathbb{Q}}_{S^0(-j)}^{S^b(-k-b)}[p^b] & \xrightarrow{id\times \mathrm{Sh}^a} & \dots
\end{array}$$

$$\begin{array}{ccccc}
\dots \longrightarrow & \mathrm{Ext}^a_{\Delta^i}(\overline{E}_0, \overline{E}_{-i+j+a}) \times \mathrm{Ext}^b_{\Delta^{j+a}}(\overline{E}_0, \overline{E}_{-j+k+b}) & \longrightarrow & \mathrm{Ext}^{a+b}_{\Delta^i}(\overline{E}_0, \overline{E}_{-i+k+a+b}) \\
& \downarrow & & \downarrow \\
\dots \longrightarrow & \overline{\mathbb{Q}}_{S^0(-i)}^{S^a(-j-a)}[p^a] \times \overline{\mathbb{Q}}_{S^a(-j-a)}^{S^{a+b}(-k-a-b)}[p^b] & \longrightarrow & \overline{\mathbb{Q}}_{S^0(-i)}^{S^{a+b}(-k-a-b)}[p^{a+b}]
\end{array}$$

Here the top left map on the right factor is obtained from the suspension morphisms $\Delta^{j+a} \to \Delta^j$, the top right map on the right factor first uses the twisting isomorphism $\mathrm{Ext}^*_{\Delta^\ell}(\overline{E}_0, \overline{E}_r) \to \mathrm{Ext}^*_{\Delta^{\ell-s}}(\overline{E}_s, \overline{E}_{r+s})$ for any ℓ, r, s, and then the usual Yoneda product in Ext groups.

The left square commutes by the definition of the suspension maps.

For the right square, we unravel how the lower composition map works. We note that we are in the situation of Section 5.4.1 in Appendix D since the additive monad $\mathcal{L}_{\hat{\mathbb{T}}}$ acts additively on the derived category of complexes of completed-free E_*-modules. Its Koszul dual comonad $\mathrm{KD}(\mathcal{L}_{\hat{\mathbb{T}}}) = |\,\mathrm{Bar}_\bullet(1, \mathcal{L}_{\hat{\mathbb{T}}}, \mathbf{triv})|$ therefore has structure map given explicitly by

$$|\,\mathrm{Bar}_\bullet(\mathbf{1}, \mathcal{L}_{\hat{\mathbb{T}}}, \mathbf{triv})| \xleftarrow{\simeq} |\,\mathrm{Bar}_\bullet(\mathbf{1}, \mathcal{L}_{\hat{\mathbb{T}}}, |\,\mathrm{Bar}_\bullet(\mathcal{L}_{\hat{\mathbb{T}}}, \mathcal{L}_{\hat{\mathbb{T}}}, \mathbf{triv})|)| \to |\,\mathrm{Bar}_\bullet(\mathbf{1}, \mathcal{L}_{\hat{\mathbb{T}}}, |\,\mathrm{Bar}_\bullet(\mathbf{1}, \mathcal{L}_{\hat{\mathbb{T}}}, \mathbf{triv})|)|$$

Elements $(\sigma, \tau) \in \overline{\mathbb{Q}}_{S^0(-i)}^{S^a(-j-a)} \times \overline{\mathbb{Q}}_{S^a(-j-a)}^{S^{a+b}(-k-a-b)} = \mathrm{Map}_{\mathcal{D}^-_{\geq 0}(\mathrm{Mod}^{Cpl(I)}_{E_*})}(|\,\mathrm{Bar}_\bullet(\mathbf{1}, \mathcal{L}_{\hat{\mathbb{T}}}, \overline{\Sigma^{-i}E_*})|, S^a(\Sigma^{-j-a}E_*)) \times \mathrm{Map}_{\mathcal{D}^-_{\geq 0}(\mathrm{Mod}^{Cpl(I)}_{E_*})}(|\,\mathrm{Bar}_\bullet(\mathbf{1}, \mathcal{L}_{\hat{\mathbb{T}}}, \overline{S^a(\Sigma^{-j-a}E_*)})|, S^{a+b}(\Sigma^{-k-a-b}E_*))$ can be represented by pairs of maps

$$s : \mathcal{L}^a_{\hat{\mathbb{T}}}(\Sigma^{-i}E_*) \to \Sigma^{-j-a}E_*, \qquad t : \mathcal{L}^b_{\hat{\mathbb{T}}}(\Sigma^{-j-a}E_*) \to \Sigma^{-k-a-b}E_*$$

The map $(h\,\mathrm{KD}(\mathcal{L}_{\hat{\mathbb{T}}}))(h\,\mathrm{KD}(\mathcal{L}_{\hat{\mathbb{T}}}))(\Sigma^{-i}E_*) \xrightarrow{h\,\mathrm{KD}(\mathcal{L}_{\hat{\mathbb{T}}})(\sigma)} (h\,\mathrm{KD}(\mathcal{L}_{\hat{\mathbb{T}}}))(S^a(\Sigma^{-j-a}E_*)) \xrightarrow{\tau} S^{a+b}(\Sigma^{-k-a-b}E_*)$ is obtained by applying the "total complex" construction to the map of double complexes

$$\mathrm{Bar}_*(\mathbf{1}, \mathcal{L}_{\hat{\mathbb{T}}}, \overline{\mathrm{Bar}_*(\mathbf{1}, \mathcal{L}_{\hat{\mathbb{T}}}, \overline{\Sigma^{-i} E_*})}) \to S^{a,b}_{*,*}(\Sigma^{-k-a-b} E_*)$$

obtained by combining s and t in the evident manner. Here $S^{a,b}_{*,*}(\Sigma^{-k-a-b}E_*)$ denotes the double complex where $\Sigma^{-k-a-b}E_*$ is placed in bidegree (a, b).

The composition $\tau \circ \sigma$ of the two operations is thus given by the composite

$$|\,\mathrm{Bar}_*(\mathbf{1}, \mathcal{L}_{\hat{\mathbb{T}}}, \overline{\Sigma^{-i}E_*}| \xleftarrow{\simeq} |\,\mathrm{Bar}_*(\mathbf{1}, \mathcal{L}_{\hat{\mathbb{T}}}, \overline{|\,\mathrm{Bar}_*(\mathcal{L}_{\hat{\mathbb{T}}}, \mathcal{L}_{\hat{\mathbb{T}}}, \overline{\Sigma^{-i}E_*})|})|$$

$$\to |\,\mathrm{Bar}_*(\mathbf{1}, \mathcal{L}_{\hat{\mathbb{T}}}, \overline{|\,\mathrm{Bar}_*(\mathbf{1}, \mathcal{L}_{\hat{\mathbb{T}}}, \overline{\Sigma^{-i}E_*})|})| \to S^{a+b}(\Sigma^{k-a-b}E_*)$$

Our explicit description of how to lift cycles across the first quasi-isomorphism in Lemma 5.4.1 in Appendix D lets us conclude that this composition is given by the class in $\overline{\mathbb{Q}}^{S^{a+b}(-k-a-b)}_{S^0(-i)}$ represented by the map

$$\mathcal{L}^{a+b}_{\hat{\mathbb{T}}}(\Sigma^{-i}E_*) \xrightarrow{\mathcal{L}^b_{\hat{\mathbb{T}}}(s)} \mathcal{L}^b_{\hat{\mathbb{T}}}(\Sigma^{-j-a}E_*) \xrightarrow{t} \Sigma^{-k-a-b}E_*$$

We therefore recover the expected product structure (cf. [Fox88]) and conclude that the right square above commutes (since this is precisely how the usual Yoneda product on the corresponding Ext-groups is defined).

□

The following result establishes the desired link between algebra and topology:

Theorem 4.3.4. *The following diagram commutes for all $i, j \in \mathbb{Z}$ and $a, b \in \mathbb{Z}_{\geq 0}$:*

$$\begin{array}{ccccc}
\overline{\mathbb{Q}}^{S^a(-j-a)}_{S^0(-i)}[p^a] \times \overline{\mathbb{Q}}^{S^b(-k-b)}_{S^0(-j)}[p^b] & \xrightarrow{\mathrm{id} \times Sh^a} & \overline{\mathbb{Q}}^{S^a-j-a}_{S^0(-i)}[p^a] \times \overline{\mathbb{Q}}^{S^{a+b}-k-b-b}_{S^a(-j-a)}[p^b] & \longrightarrow & \overline{\mathbb{Q}}^{S^{a+b}(-k-a-b)}_{S^0(i)}[p^{a+b}] \\
\downarrow & & & & \downarrow \\
Q^{-j}_{-i}[p^a] \times Q^{-k}_{-j}[p^b] & & \longrightarrow & & Q^{-k}_{-i}[p^{a+b}]
\end{array}$$

Proof. We denote all maps from the components of $\overline{\mathbb{Q}}$ to the components of Q by Φ.

If $-i$ is odd, then the claim follows from Theorem 3.5.1 since $\Sigma^{-i}E_*$ is $p-$Koszul.

If $-i$ is even, we fix nonzero classes $\alpha \in \overline{\mathbb{Q}}^{S^a(-j)}_{S^0(-i)}[p^a]$ and $\beta \in \overline{\mathbb{Q}}^{S^b(-k)}_{S^0(-j)}[p^b]$. We can write $M\alpha = \mathrm{Susp}(\lambda)$ and $N\beta = \mathrm{Susp}(\mu)$ as nonzero integral multiples of suspensions. By additivity of α, β (since $\mathcal{L}_{\hat{\mathbb{T}}_n}$ is additive) and $\Phi(\alpha), \Phi(\beta)$ (established in Lemma 4.2.23), we then have:

$$NM(\Phi(\alpha \circ \beta) - \Phi(\alpha) \circ \Phi(\beta)) = \Phi(M\alpha \circ N\beta) - \Phi(M\alpha) \circ \Phi(N\beta)$$

Since suspension of operations respects composition and Φ, we observe

$$(\Phi(\mathrm{Susp}(\lambda)\circ \mathrm{Susp}(\mu)) - \Phi(\mathrm{Susp}(\lambda))\circ \Phi(\mathrm{Susp}(\mu))) = \mathrm{Susp}(\Phi(\lambda\circ\mu) - \Phi(\lambda)\circ\Phi(\mu)) = 0$$

The last vanishing follows from the odd case. By Theorem 4.2.15, the group $Q^{-k}_{-i}[p^{a+b}]$ is torsion-free. The result follows. $\square$

In a final step, we go from operations on the E-theory of (restricted and nilpotent) Lie coalgebras in $K(h)$-local E-modules, i.e. coalgebras over the comonad C from above, to the desired Lie algebras in $\mathrm{Mod}_E^{Cpl(I)}$:

Lemma 4.3.5. *For all degrees $i, j, k \in \mathbb{Z}$ and all weights $v, w \in \mathbb{Z}_{\geq 0}$, the following square commutes:*

$$\begin{array}{ccc} Q^{-j}_{-i}[v] \times Q^{-k}_{-j}[w] & \longrightarrow & Q^{-k}_{-i}[vw] \\ {\scriptstyle \mathbb{D}\times\mathbb{D}}\downarrow & & {\scriptstyle \mathbb{D}}\downarrow \\ P^j_i[v] \times P^k_j[w] & \longrightarrow & P^k_i[vw] \end{array}$$

Here $\mathbb{D}$ denotes the duality transformation from Definition 4.2.18.

Proof. Assume we are given $(\alpha, \beta) \in Q^{-j}_{-i}[v] \times Q^{-k}_{-j}[w]$ represented by maps $a : C[v](\Sigma^{-i}E) \to \Sigma^{-j}E$ and $b : C[w](\Sigma^{-j}E) \to \Sigma^{-k}E$. The composite $\beta\circ\alpha$ is then computed as

$$C[vw](\Sigma^{-i}E) \to (C\circ C)(\Sigma^{-i}E) \xrightarrow{p_w\circ p_v} (C[w]\circ C[v])(\Sigma^{-i}E) \xrightarrow{C[w](a)} C[w](\Sigma^{-j}E) \xrightarrow{b} \Sigma^{-k}E$$

We now use the natural transformations ν and its components ν_j from Section 4.1.3 to write down the following diagram:

$$\begin{array}{ccccccc} \mathbb{D}C[vw](\Sigma^{-i}E) & \longleftarrow & \mathbb{D}(C[w]\circ C[v])(\Sigma^{-i}E) & \longleftarrow & \mathbb{D}C[w](\Sigma^{-j}E) & \longleftarrow & \Sigma^k E \\ {\scriptstyle \nu_{vw}}\uparrow & & {\scriptstyle \nu_w\circ\nu_v}\uparrow & & {\scriptstyle \nu_w}\uparrow & & \| \\ L[vw](\Sigma^i E) & \longleftarrow & (L[w]\circ L[v])(\Sigma^i E)) & \xleftarrow{L[w](\mathbb{D}(a))} & L[w](\Sigma^j E) & \xleftarrow{\mathbb{D}(b)} & \Sigma^k E \end{array}$$

The two right squares commute by definition of the duality transformation $\mathbb{D}$.

For the left square, we abuse notation and identify $\Sigma^{-i}E$ with the symmetric sequence with $\Sigma^{-i}E$ placed in degree 0. By Lemma 4.1.12 and the properties of the transformations ν_j, (for top and bottom square), the following diagram of symmetric sequences commutes up to homotopy:

$$\begin{array}{ccc}
p_v\mathcal{O}_{\Sigma\,\mathrm{Lie}}^{\mathrm{Mod}_E^{Cpl(I)}} \circ p_w\mathcal{O}_{\Sigma\,\mathrm{Lie}}^{\mathrm{Mod}_E^{Cpl(I)}} \circ \mathbb{D}(\Sigma^{-i}E) & \longrightarrow & \mathbb{D}\Big(p_v\,\mathrm{KD}(\mathcal{O}_{\mathrm{Comm}}^{nu\ \mathrm{Mod}_E^{Cpl(I)}}) \circ p_w\,\mathrm{KD}(\mathcal{O}_{\mathrm{Comm}}^{nu\ \mathrm{Mod}_E^{Cpl(I)}}) \circ (\Sigma^{-i}E)\Big) \\
\downarrow & & \downarrow \\
\mathcal{O}_{\Sigma\,\mathrm{Lie}}^{\mathrm{Mod}_E^{Cpl(I)}} \circ \mathcal{O}_{\Sigma\,\mathrm{Lie}}^{\mathrm{Mod}_E^{Cpl(I)}} \circ \mathbb{D}(\Sigma^{-i}E) & \longrightarrow & \mathbb{D}\Big(\mathrm{KD}(\mathcal{O}_{\mathrm{Comm}}^{nu\ \mathrm{Mod}_E^{Cpl(I)}}) \circ \mathrm{KD}(\mathcal{O}_{\mathrm{Comm}}^{nu\ \mathrm{Mod}_E^{Cpl(I)}}) \circ (\Sigma^{-i}E)\Big) \\
\downarrow & \text{Lemma 4.1.12} & \downarrow \\
\mathcal{O}_{\Sigma\,\mathrm{Lie}}^{\mathrm{Mod}_E^{Cpl(I)}} \circ \mathbb{D}(\Sigma^{-i}E) & \longrightarrow & \mathbb{D}\Big(\mathrm{KD}(\mathcal{O}_{\mathrm{Comm}}^{nu\ \mathrm{Mod}_E^{Cpl(I)}}) \circ (\Sigma^{-i}E)\Big) \\
\downarrow & & \downarrow \\
p_{vw}\mathcal{O}_{\Sigma\,\mathrm{Lie}}^{\mathrm{Mod}_E^{Cpl(I)}} \circ \mathbb{D}(\Sigma^{-i}E) & \longrightarrow & \mathbb{D}\Big(p_{vw}\,\mathrm{KD}(\mathcal{O}_{\mathrm{Comm}}^{nu\ \mathrm{Mod}_E^{Cpl(I)}}) \circ (\Sigma^{-i}E)\Big)
\end{array}$$

The commutativity of the left square now follows by taking the "zeroth component" of this diagram. ☐

Proof of Theorem 4.3.2. We combine Lemma 4.3.3, Theorem 4.3.4, and Lemma 4.3.5 to obtain the following commutative diagram:

$$\begin{array}{ccc}
(\mathcal{H}^{\mathrm{Lie}})_i^j[p^a] \times (\mathcal{H}^{\mathrm{Lie}})_j^k[p^b] & \longrightarrow & (\mathcal{H}^{\mathrm{Lie}})_i^k[p^{a+b}] \\
\| & & \| \\
\mathrm{Ext}^a_{\Delta^i}(\overline{E}_0, \overline{E}_{-i+j+a}) \times \mathrm{Ext}^b_{\Delta^j}(\overline{E}_0, \overline{E}_{-j+k+b}) & \longrightarrow & \mathrm{Ext}^{a+b}_{\Delta^i}(\overline{E}_0, \overline{E}_{-i+k+a+b}) \\
\downarrow & & \downarrow \\
\overline{\mathbb{Q}}_{S^0(-i)}^{S^a(-j-a)}[p^a] \times \overline{\mathbb{Q}}_{S^0(-j)}^{S^b(-k-b)}[p^b] & \longrightarrow & \overline{\mathbb{Q}}_{S^0(-i)}^{S^{a+b}(-k-a-b)}[p^{a+b}] \\
\Phi\times\Phi \downarrow & & \Phi \downarrow \\
Q_{-i}^{-j}[p^a] \times Q_{-j}^{-k}[p^b] & \longrightarrow & Q_{-i}^{-k}[p^{a+b}] \\
\mathbb{D}\times\mathbb{D} \downarrow \cong & & \mathbb{D} \downarrow \cong \\
P_i^j[p^a] \times P_j^k[p^b] & \longrightarrow & P_i^k[p^{a+b}]
\end{array}$$

☐

We have thus defined the structure of an $\mathcal{H}^{\mathrm{Lie}}$-module on the homotopy of any Lie algebra in $\mathrm{Mod}_E^{Cpl(I)}$.

4.3.2 Lie Algebra Relations

Now let L be a shifted Lie algebra in $\mathrm{Mod}_E^{Cpl(I)}$.

Proposition 4.3.6. *The bracket $[-,-]$ satisfies the shifted Jacobi identity:*

For $x \in \pi_i(L), y \in \pi_j(L)$, and $z \in \pi_k(L)$, we have $(-1)^{ik}[x,[y,z]] + (-1)^{ji}[y,[z,x]] + (-1)^{kj}[z,[x,y]] = 0.$

Proof. This has been established in joint work with Antolín-Camarena and can be found in his thesis. ☐

Proposition 4.3.7. *If $x \in \pi_i(L)$, $y \in \pi_j(L)$, then $[x,y] = (-1)^{ij}[y,x]$.*

Proof. This follows since the second term of the shifted Lie operad is S^{-1} with the trivial Σ_2-action. ☐

Proposition 4.3.8. *If $x \in \pi_i(L)$ is an element of odd degree i, then $[x,x] = 0$.*

Proof. By antisymmetry, we have $[x,x] = (-1)[x,x]$ which implies $2 \cdot [x,x] = 0$. The claim follows since $[x,x] \in P_i^{2i-1}$ as a unary operation and the group P_i^{2i-1} is torsion-free by Theorem 4.2.16. ☐

Proposition 4.3.9. *If $x \in \pi_i(L)$ is any element, then $[x,[x,x]] = 0$.*

Proof. Assume x has degree i. The Jacobi identity $(-1)^{i^2}[x,[x,x]] + (-1)^{i^2}[x,[x,x]] + (-1)^{i^2}[x,[x,x]] = 0$ implies that $3 \cdot [x,[x,x]] = 0$. The claim follows since the unary operation $[x,[x,x]]$ is an element of P_i^{3i-2}, which is a torsion-free group by Theorem 4.2.16. ☐

Remark 4.3.10. Over $\mathbb{F}_3$, the vanishing of $[x,[x,x]]$ has been observed independently by Kjaer [Kja16] relying on different methods.

4.3.3 Mixed Relations

We will now compute the mixed relations between the Hecke operations, the θ-operation, and Lie bracket. We start by recalling some general background. If we are given operations

$$\alpha = (\alpha_1, \ldots, \alpha_t) \in P_{j_1,\ldots,j_s}^{k_1,\ldots,k_t} = \prod_{b=1}^{t} \pi_{k_b}(L(\Sigma^{j_1}E \oplus \ldots \oplus \Sigma^{j_s}E))$$

$$\beta = (\beta_1, \ldots, \beta_s) \in P_{i_1,\ldots,i_r}^{j_1,\ldots,j_s} = \prod_{a=1}^{s} \pi_{j_a}(L(\Sigma^{i_1}E \oplus \ldots \oplus \Sigma^{i_r}E))$$

then the composite $\alpha(\beta_1(-), \ldots, \beta_s(-))$ corresponds to the map of E-modules

$$\begin{aligned}\Sigma^{k_1}E \oplus \ldots \oplus \Sigma^{k_t}E &\xrightarrow{\alpha_1 \oplus \ldots \oplus \alpha_t} L(\Sigma^{j_1}E \oplus \ldots \oplus \Sigma^{j_s}E)\\ &\xrightarrow{L(\beta_1 \oplus \ldots \oplus \beta_s)} LL(\Sigma^{i_1}E \oplus \ldots \oplus \Sigma^{i_r}E) \to L(\Sigma^{i_1}E \oplus \ldots \oplus \Sigma^{i_r}E)\end{aligned}$$

Moreover, if $\gamma \in P_{i_1,\ldots,i_s}^{j_1,\ldots,j_t}$ and $\delta \in P_{i_{s+1},\ldots,i_r}^{j_{t+1},\ldots,j_u}$ are represented by

$$(\Sigma^{j_1}E \oplus \ldots \oplus \Sigma^{j_t}E) \to L(\Sigma^{i_1}E \oplus \ldots \oplus \Sigma^{i_s}E) \qquad (\Sigma^{j_{t+1}}E \oplus \ldots \oplus \Sigma^{j_u}E) \to L(\Sigma^{i_{s+1}}E \oplus \ldots \oplus \Sigma^{i_r}E)$$

then we can form a new element $(\gamma, \delta) \in P^{j_1,\ldots,j_u}_{i_1,\ldots,i_r}$ corresponding to

$$(\Sigma^{j_1} E \oplus \ldots \oplus \Sigma^{j_r} E) \to L(\Sigma^{i_1} E \oplus \ldots \oplus \Sigma^{i_s} E) \oplus L(\Sigma^{i_{s+1}} E \oplus \ldots \oplus \Sigma^{i_r} E) \to L(\Sigma^{i_1} E \oplus \ldots \oplus \Sigma^{i_r} E)$$

We now fix a (shifted) Lie algebra $M \in \mathrm{Alg}_{\Sigma\,\mathrm{Lie}}(\mathrm{Mod}_E^{Cpl(I)})$.

Proposition 4.3.11. *We have $s_{2n}(x) = [x, x]$ for any $x \in \pi_{2n}(M)$.*

Proof. We have to check that the image of the operation $[-,-]$ in the group $\pi_{4n-1}(L(\Sigma^{2n}E \oplus \Sigma^{2n}E)) = P^{4n-1}_{2n,2n}$ under the map induced by applying L to the codiagonal gives s_{2n}.

The claim follows by observing the commutative diagram

$$\begin{array}{ccccccc}
S^{4n-1} & \longrightarrow & \Sigma^{-1}E \otimes (\Sigma^{2n}E \otimes \Sigma^{2n}E) & \longrightarrow & \Sigma^{-1}E \underset{h\Sigma_2}{\otimes} (\Sigma^{2n}E \oplus \Sigma^{2n}E)^{\otimes 2} & \longrightarrow & L(\Sigma^{2n}E \oplus \Sigma^{2n}E) \\
& \searrow & \downarrow & & \downarrow & & \downarrow \\
& & \Sigma^{4n-1}E & \longrightarrow & \Sigma^{-1}E \underset{h\Sigma_2}{\otimes} (\Sigma^{2n}E)^{\otimes 2} & \longrightarrow & L(\Sigma^{2n}E)
\end{array}$$

where the upper path gives $[x, x]$ and the lower path gives $s_{2n}(x)$. □

By construction of the operations Ψ_{2n}, s_{2n}, and θ_{2n}, this proposition immediately implies:

Corollary 4.3.12. *Given any $x \in \pi_{2n}(M)$, there is an equality $\Psi_{2n}(x) = [x, x] + 2 \cdot \theta_{2n}(x)$.*

We now study the interaction between Lie bracket and Hecke operations:

Proposition 4.3.13. *If $\lambda \in E_{2m}$, $x \in \pi_i(M)$, and $y \in \pi_j(M)$, then $[x, \lambda \cdot y] = \lambda \cdot [x, y]$.*

Proof. We denote the unit and multiplication of the monad L by η and μ. The scalar λ gives elements

$$(\Sigma^{j+2m}E \to \Sigma^j E \xrightarrow{\eta_{\Sigma^j E}} L(\Sigma^j E)) \in P^{j+2m}_{j}, \quad (\Sigma^{i+j+2m-1}E \to \Sigma^{i+j-1}E \xrightarrow{\eta_{\Sigma^{i+j-1}E}} L(\Sigma^{i+j-1}E)) \in P^{i+j+2m-1}_{i+j-1}$$

We consider the following commutative diagram

$$\begin{array}{ccccccc}
\Sigma^{i+j+2m-1}E & \xrightarrow{\simeq} & \Sigma^{-1}E \otimes (\Sigma^i E \otimes \Sigma^{j+2m}E) & \longrightarrow & \Sigma^{-1}E \otimes (\Sigma^i E \oplus \Sigma^{j+2m}E)^{\otimes 2} & \longrightarrow & L(\Sigma^i E \oplus \Sigma^{j+2m}E) \\
\downarrow{\scriptstyle\lambda} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle L(\mathrm{id}\oplus\lambda)} \\
\Sigma^{i+j-1}E & \longrightarrow & \Sigma^{-1}E \otimes (\Sigma^i E \otimes \Sigma^j E) & \longrightarrow & \Sigma^{-1}E \otimes (\Sigma^i E \oplus \Sigma^j E)^{\otimes 2} & \longrightarrow & L(\Sigma^i E \oplus \Sigma^j E) \\
\downarrow{\scriptstyle\eta_{\Sigma^{i+j-1}E}} & & \downarrow & & \downarrow{\scriptstyle\eta_{L_2(\Sigma^i E \oplus \Sigma^j E)}} & & \downarrow{\scriptstyle\eta_{L(\Sigma^i E \oplus \Sigma^j E)}} \\
L(\Sigma^{i+j-1}E) & \longrightarrow & L(\Sigma^{-1}E \otimes (\Sigma^i E \otimes \Sigma^j E)) & \longrightarrow & L(\Sigma^{-1}E \otimes (\Sigma^i E \oplus \Sigma^j E)^{\otimes 2}) & \longrightarrow & LL(\Sigma^i E \oplus \Sigma^j E) \\
& & & & & & \downarrow{\scriptstyle\mu_{\Sigma^i E \oplus \Sigma^j E}} \\
& & & & & & L(\Sigma^i E \oplus \Sigma^j E)
\end{array}$$

The operation $[-,\lambda\cdot -]_{i,j+2m}$ is given by the lower path whereas the operation $\lambda\cdot[-,-]_{i,j}$ can be defined by following the upper path. ☐

Corollary 4.3.14. *If $\lambda \in E_{2m}$ is a scalar, then we have the identity $s_{2n}\cdot\lambda = \lambda^2\cdot s_{2(n-m)}$ in $P^{4n-1}_{2(n-m)}[2]$.*

Proof. This follows immediately by combining Proposition 4.3.11, Proposition 4.3.13, and Proposition 4.3.7. ☐

Proposition 4.3.15. *If $\alpha \in (\mathcal{H}^{\mathrm{Lie}})^k_j[w]$ with $w > 1$, $x \in \pi_i(M)$, and $y \in \pi_j(M)$, then $[x,\alpha(y)] = 0$.*

Proof. If we represent $\alpha \in P^k_j[w]$ by $\Sigma^k E \to L[w](\Sigma^j E) \cong \mathbb{D}(\Sigma|\Pi_w|^\diamond)\otimes_{h\Sigma_w}(\Sigma^j E)^{\otimes w}$, then the operation $[-,\alpha(-)]_{i,k} \in \pi_{i+k-1}(L(\Sigma^i E\oplus\Sigma^j E))$ is represented by the composite

$$\begin{array}{c} S^{i+k-1} \\ \downarrow \\ \mathbb{D}(\Sigma|\Pi_2|^\diamond)\otimes(\Sigma^i E\otimes\Sigma^k E) \\ \downarrow \\ \mathbb{D}(\Sigma|\Pi_2|^\diamond)\otimes\left((\mathbb{D}(\Sigma|\Pi_1|^\diamond)\underset{h\Sigma_1}{\otimes}(\Sigma^i E)^{\otimes 1})\otimes(\mathbb{D}(\Sigma|\Pi_w|^\diamond)\underset{h\Sigma_w}{\otimes}(\Sigma^j E)^{\otimes w})\right) \\ \downarrow \\ \mathbb{D}(\Sigma|\Pi_{1+w}|^\diamond)\underset{h\Sigma_1\times\Sigma_w}{\otimes}(\Sigma^i E\otimes(\Sigma^j E)^{\otimes w}) \\ \downarrow \\ L(\Sigma^i E\oplus\Sigma^j E) \end{array}$$

The restriction of $\mathbb{D}(\Sigma|\Pi_{1+w}|^\diamond)$ to $\Sigma_1\times\Sigma_w$ is freely induced from the trivial subgroup acting on the sphere S^{-w}. This implies that $\mathbb{D}(\Sigma|\Pi_{1+w}|^\diamond)\otimes_{h\Sigma_1\times\Sigma_w}(\Sigma^i E\otimes(\Sigma^j E)^{\otimes w}) \cong \Sigma^{i+w(j-1)}E$.

The operation $(x,y) \mapsto [y,[y,[\ldots,[y,x]\ldots]]]$ (where we repeat y precisely w times) corresponds to a *nonzero* element Q living in $\pi_{i+w(j-1)}\left(\mathbb{D}(\Sigma|\Pi_{1+w}|^\diamond)\otimes_{h\Sigma_1\times\Sigma_w}(\Sigma^i E)\otimes(\Sigma^j E)^{\otimes w}\right) \subset \pi_{i+w(j-1)}(L(\Sigma^i E\oplus\Sigma^j E))$.

Since Q is nonzero and both Q and $[-,\alpha(-)]$ are elements of $\pi_*(\mathbb{D}(\Sigma|\Pi_{w+1}|^\diamond)\otimes_{h\Sigma_1\times\Sigma_w}(\Sigma^i E\otimes(\Sigma^j E)^{\otimes w})) \cong \pi_*(\Sigma^{i+w(j-1)}E)$, we can find scalars $\lambda,\mu\in E_*$ with $\lambda\cdot Q = \mu\cdot[-,\alpha(-)]$ and $\mu \neq 0$.

We now use that α is additive to compute

$$2^w\cdot\mu\cdot[x,\alpha(y)] = 2^w\cdot\lambda Q(x,y) = \lambda\cdot[2y,[2y,[\ldots,[2y,x]\ldots]] = \lambda\cdot Q\cdot(x,2y) = \mu\cdot[x,\alpha(y+y)] = 2\cdot\mu\cdot[x,\alpha(y)]$$

This implies that $(2^w-2)\cdot\mu\cdot[-,\alpha(-)] = 0$. Since E_* is an integral domain and $(2^w-2)\cdot\mu\neq 0$, this implies that $[-,\alpha(-)] = 0$. Observe that, as expected, this proof requires that $w>1$. ☐

The operation θ interacts nontrivially with Lie bracket and sums:

Proposition 4.3.16. *If $x \in \pi_i(M), y \in \pi_{2n}(M)$, then $[x, \theta_{2n}(y)] = [[x, y], y]$.*

Proof. We compute $2 \cdot [x, \theta_{2n}(y)] = [x, 2 \cdot \theta_{2n}(y)] = [x, \Psi_{2n}(y)]] - [x, [y, y]]$. By the Jacobi identity, we have $[x, [y, y]] + [y, [x, y]] + [y, [y, x]] = 0$.

Combining this with antisymmetry and Proposition 4.3.15, we obtain $2 \cdot [x, \theta_{2n}(y)] = 2 \cdot [[x, y], y]$. The claim now follows since $\pi_{i+4n-2}(L(\Sigma^i E \oplus \Sigma^{2n} E)$ is torsion-free by Theorem 4.2.16. ☐

Proposition 4.3.17. *If $x, y \in \pi_{2n}(M)$, then $\theta_{2n}(x + y) = \theta_{2n}(x) + \theta_{2n}(y) - [x, y]$.*

Proof. We compute

$$2 \cdot \theta_{2n}(x+y) = \Psi_{2n}(x+y) - [x+y, x+y] = (\Psi_{2n}(x) - [x, x]) + (\Psi_{2n}(y) - [y, y]) - 2[x, y] = 2 \cdot (\theta_{2n}(x) + \theta_{2n}(y) - [x, y])$$

The claim follows since $\pi_{4n-1}(L(\Sigma^{2n} E \oplus \Sigma^{2n} E))$ is torsion-free by Theorem 4.2.16. ☐

Divisibility Witnesses

In order to descibe the interaction of θ with the Hecke operations, we will first produce various Hecke operations which witness the divisibility of certain operations:

Proposition 4.3.18. *If $\lambda \in E_{2m} \cong (\mathcal{H}^{\text{Lie}})^{2n}_{2(n-m)}[1]$ is a scalar, there there exists a unique operation $\delta^{\lambda}_{2(n-m)} \in (\mathcal{H}^{\text{Lie}})^{4n-1}_{2(n-m)}[2]$ satisfying $\Psi_{2n} \cdot \lambda - \lambda^2 \cdot \Psi_{2(n-m)} = 2 \cdot \delta^{\lambda}_{2(n-m)}$.*

Proof. We have $\Psi_{2n} \cdot \lambda - \lambda^2 \cdot \Psi_{2(n-m)} = (s_{2n} \cdot \lambda + 2 \cdot \theta_{2n} \cdot \lambda) - (\lambda^2 \cdot s_{2(n-m)} + 2 \cdot \lambda^2 \cdot \theta_{2(n-m)})$. By Proposition 4.3.13, the operation $s_{2n} \cdot \lambda - \lambda^2 \cdot s_{2(n-m)}$ vanishes.

We set $\delta^{\lambda}_{2(n-m)} := \theta_{2n} \cdot \lambda - \lambda^2 \cdot \theta_{2(n-m)} \in P^{4n-1}_{2(n-m)}[2]$ and conclude that $\Psi_{2n} \cdot \lambda - \lambda^2 \cdot \Psi_{2(n-m)} = 2 \cdot \delta^{\lambda}_{2(n-m)}$. Since $2 \cdot \delta^{\lambda}_{2(n-m)}$ lies in $(\mathcal{H}^{\text{Lie}})^{4n-1}_{2(n-m)}[2]$, we conclude from Proposition 4.2.20 that $\delta^{\lambda}_{2(n-m)}$ in fact lies in $(\mathcal{H}^{\text{Lie}})^{4n-1}_{2(n-m)}[2]$. The class $\delta^{\lambda}_{2(n-m)}$ is unique by torsion-freeness. ☐

Proposition 4.3.19. *If $\alpha \in (\mathcal{H}^{\text{Lie}})^{2n}_{m}[p^k]$ is an operation with $k > 0$, then there is a unique operation $\epsilon^{\alpha}_{m} \in (\mathcal{H}^{\text{Lie}})^{4n-1}_{m}[2p^a]$ satisfying $\Psi_{2n} \cdot \alpha = 2 \cdot \epsilon^{\alpha}_{m}$*

Proof. We write $\Psi_{2n} \cdot \alpha = s_{2n} \cdot \alpha + 2 \cdot \theta_{2n} \cdot \alpha$.

Since $(s_{2n} \cdot \alpha)(x) = [\alpha(x), \alpha(x)] = 0$ by Proposition 4.3.15 , we conclude that $\Psi_{2n} \cdot \alpha = 2 \cdot \theta_{2n} \cdot \alpha$. We set $\epsilon^{\alpha}_{m} := \theta_{2n} \cdot \alpha$. Since $2 \cdot \theta_{2n} \cdot \alpha$ lies in $(\mathcal{H}^{\text{Lie}})^{4n-1}_{m}[2p^a]$, we conclude once more from Proposition 4.2.20 that ϵ^{α}_{m} in fact lies in $(\mathcal{H}^{\text{Lie}})^{4n-1}_{m}[2p^a]$. The class ϵ^{α}_{m} is again unique by torsion-freeness. ☐

There is an interesting relation between the δ- and the ϵ-operations:

Proposition 4.3.20. *Given $\lambda \in E_{2(n-j)} \cong (\mathcal{H}^{\mathrm{Lie}})^{2n}_{2j}[1]$ and $\alpha \in (\mathcal{H}^{\mathrm{Lie}})^{2j}_{m}$, then $\epsilon^{\lambda\cdot\alpha}_m = \lambda^2 \cdot \epsilon^{\alpha}_m + \delta^{\lambda}_{2j} \cdot \alpha$.*

Proof. We have $2 \cdot \epsilon^{\lambda\cdot\alpha}_m = \Psi_{2n} \cdot \lambda \cdot \alpha = (\lambda^2 \cdot \Psi_{2j} \cdot \alpha + 2 \cdot \delta^{\lambda}_{2j} \cdot \alpha) = (\lambda^2 \cdot 2 \cdot \epsilon^{\alpha}_m + 2 \cdot \delta^{\lambda}_{2j} \cdot \alpha) = 2 \cdot (\lambda^2 \cdot \epsilon^{\alpha}_m + \delta^{\lambda}_{2j} \cdot \alpha)$. The claim follows from the fact that $(\mathcal{H}^{\mathrm{Lie}})^{4n-1}_m$ is torsion-free. □

We can now express how θ interacts with Hecke operations:

Proposition 4.3.21. *If $\lambda \in E_{2m}$ and $x \in \pi_{2(n-m)}(M)$, then $\theta_{2n}(\lambda \cdot x) = \lambda^2 \cdot \theta_{2(n-m)}(x) + \delta^{\lambda}_{2(n-m)}(x)$.*

Proof. This is true by definition of $\delta^{\lambda}_{2(n-m)}$ in Proposition 4.3.18. □

Proposition 4.3.22. *If $\alpha \in (\mathcal{H}^{\mathrm{Lie}})^{2n}_m[w]$ with $w > 1$ and $x \in \pi_m(L)$, then $\theta_{2n}(\alpha(x)) = \epsilon^{\alpha}_m(x)$.*

Proof. This holds true by definition of ϵ^{α}_m in Proposition 4.3.19. □

4.4 Generation

We will now prove our main result describing the operations on the E-homology of $K(h)$-local Lie algebras.

4.4.1 Hecke Lie Algebras

We start with the following notion:

Definition 4.4.1. A *(shifted) Lie algebra* over E_* consists of a graded E_*-module M_* together with E_*-bilinear maps $[-,-] : M_* \otimes_{E_*} M_* \xrightarrow{[-1]} M_*$ satisfying

1. *strict* (shifted) antisymmetry:

$$[a,b] = (-1)^{ij}[b,a] \qquad \text{for all } a, b.$$

$$[a,a] = 0 \qquad \text{for all for all } a \text{ of odd degree.}$$

$$[a,[a,a]] = 0 \qquad \text{for all } a.$$

2. the Jacobi identity: For all $a, b, c \in M_*$, we have:

$$(-1)^{\deg(a)\deg(c)} \cdot [a,[b,c]] \quad + \quad (-1)^{\deg(b)\deg(a)} \cdot [b,[c,a]] \quad + \quad (-1)^{\deg(c)\deg(b)} \cdot [c,[a,b]] \quad = \quad 0$$

Definition 4.4.2. A *Hecke Lie Algebra* consists of a $\mathcal{H}^{\mathrm{Lie}}$-module M together with the structure of a shifted Lie algebra on the underlying E_*-module M_* and maps of sets $\theta_{2n} : M_{2n} \to M_{4n-1}$ such that the following properties hold true:

1. $[x, \alpha(y)] = 0$ for all $\alpha \in (\mathcal{H}^{\mathrm{Lie}})_i^j[w]$ with $w > 1$ and all $x \in M_k$, $y \in M_i$.

2. $\Psi_{2n}(x) = [x,x] + 2 \cdot \theta_{2n}(x)$ for all $x \in M_{2n}$.

Additionally, we impose several additional identities which would all be forced in the torsion-free case:

3. $[x, \theta_{2n}(y)] = [[x,y],y]$ for all $x \in M_m, y \in M_{2n}$.

4. $\theta_{2n}(x+y) = \theta_{2n}(x) + \theta_{2n}(y) - [x,y]$ for all $x, y \in M_{2n}$.

5. $\theta_{2n}(\lambda \cdot x) = \lambda^2 \cdot \theta_{2(n-m)}(x) + \delta^{\lambda}_{2(n-m)}(x)$ for all $\lambda \in E_{2m}$, $x \in M_{2(n-m)}$.

6. $\theta_{2n}(\alpha(x)) = \epsilon^{\alpha}_m(x)$ for all $\alpha \in (\mathcal{H}^{\mathrm{Lie}})^{2n}_m[w]$ with $w > 1$, $x \in M_m$.

This definition simplifies substantially for p an odd prime as Ψ_{2n} vanishes in this case (cf. Remark 1.4.2).

A morphism $L \to L'$ of Hecke Lie algebras is a map f of underlying sets which simultaneously is a morphism of Lie algebras, of $\mathcal{H}^{\mathrm{Lie}}$-modules, and intertwines the θ_*-maps on L and L'.

We write $\mathrm{Lie}^{\mathcal{H}}_{E_*}$ for the resulting category of Hecke Lie algebras.

The notion of a Hecke Lie algebra can be axiomatised as a $\mathbb{Z}$-graded algebraic theory. In Appendix C, we have included an overview of the components of [ARV11] on $\mathbb{Z}$-graded theories which are relevant to us.

We start with the set Σ of symbols

$$\{0_i, -_i(-) \mid i \in \mathbb{Z}\} \cup \{(-) +_i (-) \mid i \in \mathbb{Z}\}$$

$$\cup \qquad \{[-,-]_{i,j} \mid i,j \in \mathbb{Z}\} \cup \{\theta_{2n} \mid n \in \mathbb{Z}\} \cup \{\alpha \in (\mathcal{H}^{\mathrm{Lie}})^j_i \mid i,j \in \mathbb{Z}\}$$

Writing $\mathbb{Z}^*$ for the collection of all finite words in $\mathbb{Z}$, we define a signature function $\sigma : \Sigma \to \mathbb{Z}^* \times \mathbb{Z}$ by:

$$\sigma(0_i) = (\ ,i), \qquad \sigma(-_i) = (i,i), \qquad \sigma(+_i) = (ii,i)$$

$$\sigma([-,-]_{i,j}) = (ij, i+j-1), \qquad \sigma(\theta_{2n}) = (2n, 4n-1), \qquad \sigma(\alpha \in (\mathcal{H}^{\mathrm{Lie}})^j_i) = (i,j)$$

Let Alg_Σ be the category of Σ-algebras in the sense of Definition 5.3.5 in Appendix C – informally speaking, these are $\mathbb{Z}$-graded sets which are acted upon by an operation corresponding to each element of Σ.

Let E be the set of equations which encodes the structure of a Hecke Lie algebra (i.e. the "abelian group axioms" for $+_i, 0_i, -_i$, the shifted Lie algebra axioms for the Lie brackets $[-,-]_{i,j}$ with inputs in degree i and j, the $\mathcal{H}^{\mathrm{Lie}}_{E_*}$-module axioms for the various α, and axioms (1) – (6) in Definition 4.4.2).

Then the category $\mathrm{Lie}^{\mathcal{H}}_{E_*}$ of Hecke Lie algebras is equivalent to the full subcategory $\mathrm{Alg}_\Sigma(E)$ of Alg_Σ consisting of all Σ-algebras on which the operations satisfy the equations in E (see Definition 5.3.6 in Appendix C).

A similar argument can be used to show that the category Mod_{E_*} of graded E_*-modules and grading-preserving maps occurs as $\mathrm{Alg}_{\Sigma'}(E')$ for suitably chosen Σ' and E'.

We obtain natural forgetful functors $\mathrm{Lie}^{\mathcal{H}}_{E_*} \to \mathrm{Mod}_{E_*} \to \mathrm{Set}^{\mathbb{Z}}$ to $\mathbb{Z}$-graded sets.

By Proposition 9.3 in [ARV11], both functors preserves limits, sifted colimits and admit left adjoints.

$$\mathrm{Set}^{\mathbb{Z}} \xrightarrow{\mathrm{Free}_{\mathrm{Mod}_{E_*}}} \mathrm{Mod}_{E_*} \xrightarrow{\mathrm{Free}_{\mathrm{Lie}^{\mathcal{H}}}} \mathrm{Lie}^{\mathcal{H}}_{E_*}$$

We shall abuse notation and also write $\mathrm{Free}_{\mathrm{Lie}^{\mathcal{H}}}$ for the composite of these two functors.

4.4.2 Free Hecke Lie Algebras

We will now construct explicit models for free Hecke Lie algebras on finitely many generators $x_1, \ldots, x_k$ in degrees $i_1, \ldots, i_r$.

Write B_r for the lexicographically ordered set of Lyndon words in letters $x_1, \ldots, x_r$ in the sense of Definition 2.3.1 (with respect to the ordering $x_1 < \ldots < x_r$). Define a degree function $|-|: B_r \to \mathbb{Z}$ inductively by setting $|x_j| = i_j$ and $|w_1 w_2| = |w_1| + |w_2| - 1$ if $w_1 w_2$ is a standard factorisation into Lyndon words. The *length* $\ell(w)$ of a Lyndon word w is defined as the number of letters which occur in w.

We start with the $\mathbb{Z}$-graded abelian group $F_r := F_r(x_1, \ldots, x_r)$ with $F_r^g = \bigoplus_{w \in B_r} F_r(w)^g$ where

$$F_r(w)^g := \begin{cases} (\mathcal{H}^{\mathrm{Lie}})^g_{|w|} & \text{if } |w| \text{ is odd} \\ (\mathcal{H}^{\mathrm{Lie}})^g_{|w|} \oplus (\mathcal{H}^{\mathrm{Lie}})^g_{2|w|-1} & \text{if } |w| \text{ is even} \end{cases}$$

Given w and g, we shall denote the element corresponding to $\alpha \in (\mathcal{H}^{\mathrm{Lie}})^g_{|w|}$ by $\alpha\theta^0 w$. If $|w|$ is even, the element corresponding to $\alpha \in (\mathcal{H}^{\mathrm{Lie}})^g_{2|w|-1}$ will be written as $\alpha\theta^1 w$. We declare that these elements lie in degree g and call them *standard words*. We will often omit $\alpha = 1$ or θ^0 from the notation and write $\theta^1 = \theta$. Observe that if the basic word w lives in degree $|w|$, then the product θw (which is defined whenever $|w|$ is even) lies in degree $2|w| - 1$.

We will now define the structure of a Hecke Lie algebra on F_r.

We first construct the structure of an $\mathcal{H}^{\mathrm{Lie}}$-module on F_r by setting $\lambda \cdot (\alpha\theta^e w) = (\lambda \cdot \alpha)\theta^e w$.

In order to define the structure of a (shifted) Lie algebra on F_r, we first define the *length* of a term $\alpha\theta^e w$ to be the integer $\ell(\alpha\theta^e w) := \mathrm{wt}(\alpha) \cdot (2e) \cdot \ell(w)$ where the weight $\mathrm{wt}(\alpha)$ of the operation $\alpha \in (\mathcal{H}^{\mathrm{Lie}})^j_i[p^a]$ is p^a and $\ell(w)$ is the length of the Lyndon word w.

Every pair $(\sum_i \alpha_i \theta^{e_i} u_i, \sum_j \beta_j \theta^{f_j} v_j)$ gives an element

$$\kappa\Big(\sum_i \alpha_i\theta^{e_i}u_i, \sum_j \beta_j\theta^{f_j}v_j\Big) := \Big(\sum_i \ell(\lambda_i\theta^{e_i}u_i) + \sum_j \ell(\mu_j\theta^{f_j}v_j) \quad , \quad \max(\{u_i\}_i \cup \{v_j\}_j)\Big) \in \mathbb{N} \times B_k$$

We order $\mathbb{N} \times B_k$ by setting $(\ell_1, m_1) < (\ell_2, m_2)$ iff $(\ell_1 < \ell_2)$ or $(\ell_1 = \ell_2$ and $m_1 < m_2)$.

We will now define the product

$$\left[\sum_i \alpha_i\theta^{e_i}u_i, \sum_j \beta_j\theta^{f_j}v_j\right] = \sum_k \gamma_k\theta^{g_k}w_k$$

for any two elements in F_r by recursion on κ.

Our product will have the following crucial property (★):

Every Lyndon word w_k occuring in the above decomposition satisfies $w_k \leq \max(\{u_i\}_i \cup \{v_j\}_j)$.

If all u_i, v_j are pairwise distinct, then this inequality is *strict*.

Assume we are given a pair $(\sum_i \alpha_i\theta^{e_i}u_i, \sum_j \beta_j\theta^{f_j}v_j)$ of Lyndon words.

1. If either of the components of our pair contains more than one summand, we define

$$\left[\sum_i \alpha_i\theta^{e_i}u_i, \sum_j \beta_j\theta^{f_j}v_j\right] := \sum_{i,j} \left[\alpha_i\theta^{e_i}u_i, \beta_j\theta^{f_j}v_j\right]$$

   Each term on the right has shorter length and is therefore already defined by recursion. Moreover, each term on the right satisfies property (★), which implies that our term on the left satisfies (★) as well.

2. If both components only contain a single summand $(\alpha\theta^e u, \beta\theta^f v)$ and $\mathrm{wt}(\alpha) > 1$ or $\mathrm{wt}(\beta) > 1$, we define $[\alpha\theta^e u, \beta\theta^f v] = 0$

3. If both components only contain a single summand $(\alpha\theta^e u, \beta\theta^f v)$ and $\mathrm{wt}(\alpha) = \mathrm{wt}(\beta) = 1$, i.e. $\alpha, \beta \in E_*$, we proceed as follows:

   a) If $u < v$ and u is a letter or it has standard factorisation $u = u' \cdot u''$ with $u'' \geq v$:

      - $[\alpha u, \beta v] := \alpha\beta uv$
      - $[\alpha u, \beta\theta v] := \alpha\beta uvv$
      - $[\alpha\theta u, \beta v] := (-1)^{|v|}\alpha\beta uuv$
      - $[\alpha\theta u, \beta\theta v] := (-1)^{|v|}\alpha\beta uuvv$

      All the above words are indeed Lyndon words:

      This is clear for uv and $uvv = (uv)(v)$. For $uuv = (u'u'')(uv)$, we use that $u < u''$ (since u is a Lyndon word and hence smaller than its proper right factors) and and hence $u'' > uv$. We see that $uuvv = (u, ((uv), v))$ is a Lyndon word since $u'' > uvv$.

      All four words are evidently strictly smaller than $\max(u, v) = v$.

b) If $u < v$ and u has standard factorisation $u = u' \cdot u''$ with $u'' < v$ and $a = |u'|, b = |u''|, c = |v|$:

- $[\alpha u, \beta v] := (-1)^{1+c}\alpha\beta[u', [u'', v]] + (-1)^{ab+ac+c+1}\alpha\beta[u'', [v, u']]$
- $[\alpha u, \beta\theta v] := \alpha\beta[u', [u'', \theta v]] + (-1)^{a(b+1)}\alpha\beta[u'', [\theta v, u']]$
- $[\alpha\theta u, \beta v] := (-1)^{|v|}\alpha\beta[u, [u, v]]$
- $[\alpha\theta u, \beta\theta v] := (-1)^{|v|}\alpha\beta[u, [u, \theta v]]$

We check that this is indeed well defined.

For the first clause of the b), we notice that (u'', v) and (v, u') have shorter length than (u, v) and hence their Lie bracket has already been defined. Write $[u'', v] = \sum_i \delta_i \theta^{a_i} r_i$ and $[v, u'] = \sum_j \epsilon_j \theta^{b_j} s_j$ and recall that $r_i, s_j < \max(u, v) = v$ by induction hypothesis since (u'', v) and (v, u') are pairwise distinct. This implies that the expressions $[[u'', v], u'] = \sum_i \delta_i[\theta^{a_i} r_i, u']$ and $[[v, u'], u''] = \sum_j \epsilon_j \theta^{b_j}[s_j, u'']$ are already defined since $\max(r_i, u') < \max(u, v)$ and $\max(s_j, u'') < \max(u, v)$. We note that the Lyndon words occuring in the final result all are at most as big as $\max(\max(r_i, u'), u'')$ and $\max(\max(s_j, u''), u')$ respectively and hence strictly less than $\max(u, v) = v$.

Similarly, our definition of $[u, \theta(v)]$ makes sense since $[\theta v, u']$ and $[u'', \theta v]$ have already been defined (since they have shorter length) and can be written as sums containing Lyndon words strictly less than v. As before, this implies that the Lie brackets $[u'', [\theta v, u']]$ and $[u', [u'', \theta v]]$ are well-defined and only involve Lyndon words strictly smaller than v.

A similar argument shows that the remaining two cases are well-defined and only involve Lyndon words strictly smaller than $v = \max(u, v)$.

c) If $u = v = w$, we set

- $[\alpha w, \beta w] := \begin{cases} \alpha\beta(\Psi w - 2\theta w) & \text{if } |w| \text{ even} \\ 0 & \text{if } |w| \text{ odd} \end{cases}$
- $[w, \theta(w)] := 0$
- $[\theta w, w] := 0$
- $[\theta w, \theta w] := 0$

4. If $u > v$, set $[\alpha\theta^e u, \beta\theta^f v] = (-1)^{ij}[\beta\theta^f v, \alpha\theta^e u]$ where $\alpha\theta^e u \in F_r^i$ and $\beta\theta^f v \in F_r^j$.

We have defined a product $[-, -] : F_r \times F_r \to F_r$. The map $\theta_{2*} : L_{2*} \to L_{2*-1}$ is given by

11. $\theta_{2n} \cdot (\alpha\theta^e w) = \epsilon_m^\alpha \theta^e w$ if $\alpha \in ((\mathcal{H})^{\text{Lie}})_m^{2n}[w]$ for $w > 1$.

12. $\theta_{2n} \cdot (\lambda w) = \lambda^2 \theta_{2(n-m)} w + \delta_{2(n-m)}^\lambda w$ if $\lambda \in E_{2m}$ and $|w| = 2(n - m)$.

Using induction, we can verify that the operations specified above indeed make F_r into a Hecke Lie algebra.

Proposition 4.4.3. *The Hecke Lie algebra $F_r = F_r(x_1, \dots, x_r)$ is isomorphic to the free Hecke Lie algebra $\mathrm{Free}_{\mathrm{Lie}^{\mathcal{H}}}(x_1, \dots, x_r)$ on the graded set $\{x_1, \dots, x_r\}$ where $|x_j| = i_j$.*

Proof. Let L be a Hecke Lie algebra containing elements $y_j \in L_{i_j}$ for $j = 1, \dots, r$. Using obvious notation, we define a map $F_r \to L$ by sending $\alpha\theta^e w$ to $\alpha \cdot \theta^e_{|w|} w(y_1, \dots, y_r)$ and extending additively. We check by recursion that this indeed gives a well-defined map of Hecke Lie algebras, and it is clear that there is at most one such extension. Hence F_r satisfies the desired universal property. □

4.4.3 The Main Theorem

In this section, we will prove the main theorem of this thesis:

Theorem 4.4.4.

1. *The homotopy groups of any Lie algebra in $K(h)$-local E-module spectra naturally carry the structure of a Hecke Lie algebra.*
2. *Given a flat E-module spectrum M, the canonical map $\mathrm{Free}_{\mathrm{Lie}^{\mathcal{H}}}(\pi_*(M)) \to \pi_*\left(\mathrm{Free}_{\Sigma\,\mathrm{Lie}}(L_{K(h)}(M))\right)$ induces an isomorphism after completion.*

Here we call an E-module spectrum M *flat* if it can be written as a filtered colimit of finitely generated free E-module spectra (i.e. finite coproducts of shifts of E)

Proof. Let L be a Lie algebra in $\mathrm{Mod}_E^{Cpl(I)}$. The homotopy groups $\pi_*(L)$ form a shifted Lie algebra in the sense of Definition 4.4.1 by Section 4.2.3 and Section 4.3.2. We have constructed a canonical action of the power ring $\mathcal{H}^{\mathrm{Lie}}$ on $\pi_*(L)$ in Section 4.2.1 and Section 4.3.1. The construction of the operation θ is carried out in Chapter 4.2.2. These operations satisfy the axioms of a Hecke Lie algebra by Section 4.3.3.

In order to prove the second claim, we proceed step by step.

One generator in odd degree:

Let $M = \Sigma^j E$ with j an *odd* integer. By Proposition 4.4.3 and Theorem 4.2.19, we have in degree g:

$$[\mathrm{Free}_{\mathrm{Lie}^{\mathcal{H}}}(\Sigma^j E_*)]_g = (\mathcal{H}^{\mathrm{Lie}})^g_j = \bigoplus_{w \in \mathbb{N}} (\mathcal{H}^{\mathrm{Lie}})^g_j[w] \xrightarrow{\cong} \bigoplus_{w \in \mathbb{N}} P^g_j[w] = \bigoplus_{w \in \mathbb{N}} L[w](\Sigma^j E)_g \to [L(\Sigma^i E)]_g$$

Since we can check isomorphisms of E_*-modules degreewise, this implies that the first of the following two maps is an isomorphism of (uncompleted) E_*-modules:

$$\mathrm{Free}_{\mathrm{Lie}^{\mathcal{H}}}(\Sigma^j E_*) \xrightarrow{\cong} \bigoplus_{w \in \mathbb{N}} L[w](\Sigma^j E) \to L(\Sigma^i E)$$

We apply completion to obtain the desired result for j odd.

One generator in even degree:

Now assume that $M = \Sigma^j E$ with $j \geq 2$ an *even* integer. We invite the reader to recall our recollections on Goodwillie calculus and the EHP sequence in Section 4.1.3 . We can relate operations on the homotopy of $L_{\Omega F}(X)$ to operations on the homotopy of $L_F(X)$ for $F \in [\mathbf{sSet}_*, \mathbf{sSet}_*]$ any pointed simplicial homotopy functor via the following commutative square:

$$\begin{array}{ccc} P_i^j \times \pi_i(L_{\Omega F}(X)) & \longrightarrow & \pi_j(L_{\Omega F}(X)) \\ \Big\downarrow {\scriptstyle \mathrm{Susp} \times e} & & \Big\downarrow {\scriptstyle e} \\ P_{i+1}^{j+1} \times \pi_{i+1}(L_F(X)) & \longrightarrow & \pi_{j+1}(L_F(X)) \end{array}$$

Here *Susp* denotes the suspension of operations and e uses the equivalence of spectra $L_{\Omega F}(X) = \Sigma^{-1} L_F(X)$.

We define a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_* L_{\mathrm{id}}(\Sigma^{2n-1}E) & \xrightarrow{E} & \pi_* L_{\Omega}(\Sigma^{2n}E) & \xrightarrow{H} & \pi_* L_{\Omega}(\Sigma^{4n-1}E) & \longrightarrow & 0 \\ & & \cong \Big\uparrow \mathrm{id} & & \cong \Big\uparrow e^{-1} & & \cong \Big\uparrow e^{-1} & & \\ & & \pi_* L(\Sigma^{2n-1}E) & & \Sigma^{-1}\pi_* L(\Sigma^{2n}E) & & \Sigma^{-1}\pi_* L(\Sigma^{4n-1}E) & & \\ & & \Big\uparrow & & \Big\uparrow & & \Big\uparrow & & \\ 0 & \longrightarrow & \mathrm{Free}_{\mathrm{Lie}^{\mathcal{H}}}(x_{2n-1}) & \xrightarrow{E_L} & \Sigma^{-1}\,\mathrm{Free}_{\mathrm{Lie}^{\mathcal{H}}}(x_{2n}) & \xrightarrow{H_L} & \Sigma^{-1}\,\mathrm{Free}_{\mathrm{Lie}^{\mathcal{H}}}(x_{4n-1}) & \longrightarrow & 0 \end{array}$$

Here x_k denotes a variable in degree k. The lower vertical maps are all defined by sending some x_k to a fundamental class ι_k, extending to a map of Hecke Lie algebras, and possibly shifting degrees in the end. We abuse notation and denote all vertical maps from bottom to top by τ and the two right upper vertical maps by e^{-1}.

The map E_L is obtained by applying the rule $\alpha x_{2n-1} \mapsto \mathrm{Susp}(\alpha) x_{2n}$. The map H_L is defined by the rules $\alpha x_{2n} \mapsto 0$ and $\alpha\theta x_{2n} \mapsto \alpha x_{4n-1}$. The lower sequence is short exact by explicit inspection using our concrete description of free Hecke Lie algebras in Proposition 4.4.3.

The left square commutes: it is clear that the fundamental classes match up. We deduce from the previous commutative square that both ways of sending round αx_{2n-1} agree since the top row consists of maps of Hecke Lie algebras and hence $E(\tau(\alpha x_{2n-1})) = E(\alpha \cdot \iota_{2n-1}) = \alpha \cdot \tau(x_{2n}) = e^{-1}(\mathrm{Susp}(\alpha) \cdot \iota_{2n}) = \tau(E_L(\alpha x_{2n-1}))$.

In order to check that the right square commutes, we first use the EHP sequence at the end of Section 4.1.3 for weight $w = 1$ to observe that the (desuspended) fundamental class $\tau(x_{2n})$ in the top middle indeed goes to 0 along the top right map H. Since H is a map of Hecke Lie algebras, it follows from the naturality of the operations that for $\alpha = \mathrm{Susp}(\beta)$ a suspension, we have $H(\tau(\alpha x_{2n})) = H(\beta \cdot e^{-1}(\iota_{2n})) = \beta \cdot H(\tau(x_{2n}))) = 0$. Since some nonzero multiple of any additive Hecke operation is a suspension and the top right group is torsion-free, this implies $H(\tau(\alpha x_{2n})) = 0$ for all α. The right square therefore commutes on these operations.

The $(w = 2)$-component of the EHP sequence is given by the Takayasu cofibre sequence

$$\Sigma^{-1}(S^{2n-1})^{\otimes 2}_{h\Sigma_2} \to \Sigma^{-2}(S^{2n})^{\otimes 2}_{h\Sigma_2} \to S^{4n-2}$$

where the right map is a desuspended transfer (cf. [Beh12]). The operation $\tau(\theta x_{2n})$ goes to the fundamental class $\tau(x_{4n-1})$ on the top right since θ_{2n} is defined as a lift of the unit under the desuspended transfer

$$\Sigma^{-1}\mathbb{D}(\Sigma|\Pi_2|^{\diamond}) \underset{h\Sigma_2}{\otimes} (S^{2n})^2 \to \Sigma^{-1}\mathbb{D}(\Sigma|\Pi_1|^{\diamond}) \underset{h\Sigma_2}{\otimes} (S^{4n-1})$$

This shows that the right square commutes on the element θx_{2n}. Given an operation α which can be written as $\mathrm{Susp}(\beta)$, the naturality of Hecke operations shows the right square commutes on all elements $\alpha\theta x_{2n}$:

$$H(\tau(\alpha\theta x_{2n})) = H(\beta \cdot \tau(\theta x_{2n})) = \beta \cdot e^{-1}(\iota_{4n}) = e^{-1}(\alpha \cdot \iota_{4n}) = \tau(\alpha x_{4n}) = \tau(H_L(\alpha\theta x_{2n}))$$

For a general operation α, we pick a positive integer N with $N \cdot \alpha = \mathrm{Susp}(\beta)$ a suspension and compute:

$$N \cdot H(\tau(\alpha\theta x_{2n})) = H(\tau((N \cdot \alpha)\theta x_{2n})) = \tau(H_L((N \cdot \alpha)\theta x_{2n})) = N \cdot \tau(H_L(\alpha\theta x_{2n}))$$

The torsion-freeness of the top right group thus implies that the right square commutes on all $\alpha\theta x_{2n}$.

We have therefore produced a map of short exact sequences in E_*-modules. We now apply L^0-completion. The left and the right vertical map then become isomorphisms by the first step of this proof. The lower sequence remains exact after (derived) completion since the functor L^0 is right exact and the higher derived functors of ordinary completion vanish on free modules. The top sequence remains exact since it already consists of L^0-complete modules. This implies the theorem for $j \geq 2$.

For j a general even number, we pick an isomorphism of E-module spectra $\alpha : \Sigma^2 E \to \Sigma^j E$ and notice that the following diagram implies the result for general even j.

$$\begin{array}{ccc}
\pi_* L(\Sigma^2 E) & \xrightarrow{\cong} & \pi_* L(\Sigma^j E) \\
\uparrow & & \uparrow \\
\mathrm{Free}_{\mathrm{Lie}^{\mathcal{H}}}(x_2) & \xrightarrow{\cong} & \mathrm{Free}_{\mathrm{Lie}^{\mathcal{H}}}(x_j)
\end{array}$$

Finitely many generators: Given generators $x_{i_1}, \ldots, x_{i_k}$ in degrees $i_1, \ldots, i_k$, there is a commutative square:

$$\begin{array}{ccc}
\bigoplus_{w \in B_k} \mathrm{Free}_{\mathrm{Lie}^{\mathcal{H}}}(x_{\sum_t i_t |w|_t - |w| + 1}) & \longrightarrow & \mathrm{Free}_{\mathrm{Lie}^{\mathcal{H}}}(x_{i_1}, \ldots, x_{i_k}) \\
\downarrow & & \downarrow \\
\bigoplus_{w \in B_k} \pi_*(\mathrm{Free}_{\Sigma\,\mathrm{Lie}}(\Sigma^{\sum_t i_t |w|_t - |w| + 1} E)) & \longrightarrow & \pi_*(\mathrm{Free}_{\Sigma\,\mathrm{Lie}}(\Sigma^{i_1} E \oplus \ldots \oplus \Sigma^{i_k} E))
\end{array}$$

where the generator in the summand corresponding to a Lie word w is sent to the class on the right which corresponds to said Lie word. Here $|w|$ denotes the length of a word and $|w|_i$ stands for the number of occurrences of the i^{th} letter.

The top map is an equivalence by inspection using our explicit models for free Hecke Lie algebras in Proposition 4.4.3. The lower map is an equivalence after completion by Corollary 2.3.14. The left map is an equivalence after completion by our previous considerations in this proof. It therefore follows that the right map induces an isomorphism after completion.

Flat E-modules: Now assume M is a flat E-module spectrum. Write M as a filtered (homotopy) colimit $\mathrm{colim}_{d \in D} \; M_d \;\xrightarrow{\simeq}\; M$ with all M_d finite and free E-module spectra. Here the colimit is computed in uncompleted E-module spectra. For the length of this proof, we indicate localised colimits by $\widehat{\mathrm{colim}}$. We have a natural map $\pi_*(M) \to \pi_*(L_{K(h)}(M)) \to \pi_*(\mathrm{Free}_{\Sigma\,\mathrm{Lie}}(L_{K(h)}(M)))$, and this map fits into a commutative diagram:

$$\begin{array}{ccccc}
\operatorname*{colim}_{d \in D} \mathrm{Free}_{\mathrm{Lie}^{\mathcal{H}}}(\pi_*(M_d)) & & \longrightarrow & & \mathrm{Free}_{\mathrm{Lie}^{\mathcal{H}}}(\pi_*(M)) \\
\downarrow & & & & \\
\operatorname*{colim}_{d \in D} \pi_*(\mathrm{Free}_{\Sigma\,\mathrm{Lie}}(M_d)) & & & & \downarrow \\
\downarrow & & & & \\
\pi_*(\operatorname*{colim}_{d \in D} \mathrm{Free}_{\Sigma\,\mathrm{Lie}}(M_d)) & \longrightarrow & \pi_*(\operatorname*{\widehat{colim}}_{d \in D} \mathrm{Free}_{\Sigma\,\mathrm{Lie}}(M_d)) & \longrightarrow & \pi_*(\mathrm{Free}_{\Sigma\,\mathrm{Lie}}(L_{K(h)}(M)))
\end{array}$$

The right lower horizontal arrow is an equivalence since the functor $\mathrm{Free}_{\Sigma\,\mathrm{Lie}}$ preserves sifted colimits computed in $K(h)$-local E-module spectra. Each component $L_{K(h)}(\mathbb{D}(\Sigma|\Pi_w|^{\diamond}) \otimes_{h\Sigma_w} M_d^{\otimes w})$ is both finite and free as in Theorem 4.2.16, and we can therefore write $\widehat{\operatorname{colim}_{d\in D}} \mathrm{Free}_{\Sigma \mathrm{Lie}}(M_d)$ as the $K(h)$-localisation of a flat E-module spectrum. This implies that the bottom left map is an equivalence after L^0-completion (cf. Corollary 3.8 in [Rez09]).

The top horizontal arrow is an equivalence since the monad $\mathrm{Free}_{\mathrm{Lie}^{\varkappa}}$ on the category of E_*-modules is defined in terms of the $\mathbb{Z}$-graded algebraic theory of Hecke Lie algebras. The left top vertical arrow becomes an isomorphism after completion by our work in the last step of this proof. The bottom left arrow is an equivalence since E is compact in Mod_E. It follows that the right vertical map gives an isomorphism after (derived) completion. □

Remark 4.4.5. The general strategy of this proof goes back at least to [Goe90].

Chapter 5

Appendix

Our Appendix contains four parts. In part A, we review completion in algebra and topology (following [GM95] and [Lur16]) and develop the theory of completed-free modules. This is a key ingredient to the main computation of this thesis and to our simplification and generalisation of algebraic approximation monads. In Appendix B, we recall the basics for the theory of tensored ∞-categories (following [Lura]). In Appendix C, we review the theory of $\mathbb{Z}$-graded algebraic theories as treated in [ARV11].

The final Appendix D contains more original work. Here, we discuss the relation between Lurie's Koszul duality in monoidal ∞-categories (see [Lur11b]) and more classical instances of Koszul duality, namely the Yoneda product on Ext-groups and Ching's operadic Bar construction via tree grafting [Chi05] (cf. [Sal98]).

5.1 Appendix A: Completeness

We first review the theory of complete modules in Section 5.1.1 and module spectra in Section 5.1.2 and then develop the theory of completed-free modules in Section 5.1.3.

5.1.1 Completion of Graded Modules

We fix a graded commutative ring R_* with ideal $I \subset R_0$ and write $\mathrm{Mod}^*_{R_*}$ for the category of graded R_*-modules and homogeneous maps (of arbitrary degree) between them. Given an ideal $I \subset R_0$, we consider the endofunctor on $\mathrm{Mod}^*_{R_*}$ given by ordinary I-adic completion:

$$M_* \quad \mapsto \quad Cpl_I(M_*) := \varprojlim_n M_*/I^n$$

The functor Cpl_I is additive. We follow Greenlees-May and use (graded) homological algebra to define its left derived functors L^I_*. Ordinary completion is neither right nor left exact in general, and the canonical map $L^I_0(M_*) \to Cpl_I(M_*)$ is therefore usually *not* an equivalence.

If R_* is Noetherian, we can compute these derived functors more explicitly. For this, we recall terminology:

If $I = (x)$ is principal, we define a (cohomologically graded) complex $K^\bullet(x)$ of graded R_*-modules with R_* in degree 0, $R_*[\frac{1}{x}]$ in degree 1, and the evident map between them as differential.

More generally, if we are given generators $x_1, \ldots, x_k \in I$, we define

$$K^\bullet(x_1, \ldots, x_k) = K^\bullet(x_1) \otimes \cdots \otimes K^\bullet(x_k)$$

By Corollary 1.2 in [GM95], the complex $K^\bullet(x_1, \ldots, x_k)$ only depends on I up to quasi-isomorphisms.

Writing $PK^\bullet_I$ for a projective replacement of the complex $K^\bullet_I$, we can now define the local homology of an R_*-module M_* as $H^I_*(R_*, M_*) := H_*(\mathrm{Map}(PK^\bullet(I), M_*))$.

Greenlees and May prove that for R_* Noetherian, there is a canonical isomorphism $H^I_*(R_*, M_*) \cong L^I_*(M_*)$.

Using this identification, the canonical map of complexes $PK^\bullet(I) \to R_*$ gives rise to a factorisation

$$M_* \to L^I_0(M_*) \to Cpl_I(M_*)$$

Definition 5.1.1. An R_*-module M_* is said to be *L-complete* (with respect to I) if the first of these arrows is an equivalence. Write $\mathrm{Mod}^{*Cpl(I)}_{R_*} \xrightarrow{\iota} \mathrm{Mod}^*_{R_*}$ for the full subcategory spanned by all such modules. The inclusion ι admits a left adjoint L_0. We write $\mathrm{Mod}^{Cpl(I)}_{R_*} \subset \mathrm{Mod}^{*Cpl(I)}_{R_*}$ and $\mathrm{Mod}_{R_*} \subset \mathrm{Mod}^*_{R_*}$ for the subcategories where maps are required to *preserve* degree.

5.1.2 Completion of Module Spectra

We will now recall the topological version of these constructions. We follow Lurie's modern formulation of the old theory by Greenlees and May [GM95]. Assume R is an $\mathbb{E}_2$-ring containing a finitely generated ideal $I \subset \pi_0(R)$. We write Mod_R for the stable ∞-category of *left* R-module spectra. It is endowed with a monoidal structure given by the relative smash product which we shall simply write as $\otimes$.

Given two modules $C, D \in \mathrm{Mod}_R$, the construction $M \mapsto \mathrm{Map}(M \otimes C, D)$ is representable by a left R-module spectrum $\underline{\mathrm{Map}}(C, D)$ (see Example D.7.1.2. in [Lur16]). Recall the following terminology (Chapter 6 in [Lur16]):

Definition 5.1.2. A module spectrum $M \in \mathrm{Mod}_R$ is said to be

- I-nilpotent if for any $m \in \pi_*(M)$ and any $\lambda \in I$, we have $\lambda^n m = 0$ for n large.

- I-local if $\mathrm{Map}(N, M) \cong *$ for any I-nilpotent module spectrum N.
- I-complete if $\mathrm{Map}(N, M) \cong *$ for any I-local module spectrum N.

We write $\mathrm{Mod}_R^{Cpl(I)}$ and $\mathrm{Mod}_R^{Loc(I)}$ for the full subcategories of Mod_R spanned by I-complete and I-local (left) R-module spectra.

By Proposition 2.3.3.11. in [Lur16], the pair $\left(\mathrm{Mod}_R^{Loc(I)}, \mathrm{Mod}_R^{Cpl(I)}\right)$ forms a semiorthogonal decomposition of the ∞-category Mod_R. The inclusion of the full subcategory $\mathrm{Mod}_R^{Cpl(I)} \hookrightarrow \mathrm{Mod}_R$ of *I-complete R-module spectra* admits an (accessible) left adjoint $(-)_I^\wedge : \mathrm{Mod}_R \to \mathrm{Mod}_R^\wedge$, namely (derived) completion.

Work by Hovey and Strickland [HS99] implies that these ingredients describe a situation of interest in the chromatic context:

Lemma 5.1.3. *For $R = E$ Morava E-theory at height h and $I \subset E_0$ the unique maximal ideal, a module spectrum $M \in \mathrm{Mod}_E$ is I-complete if and only if it is $K(h)$-local. Moreover, I-completion is given by $K(h)$-localisation.*

If our ideal $I \subset \pi_0(R)$ is *finitely generated*, then we can follow give an explicit definition of completion:

Definition 5.1.4. (Greenlees-May)
If $I = (x)$ is principal, we define $K(x) = fib(R \to \mathrm{colim}(R \xrightarrow{x} R \xrightarrow{x} \dots)) \in \mathrm{Mod}_R$.

If I is generated by $x_1, \dots, x_n$, we define $K(x_1, \dots, x_n) := K(x_1) \otimes \dots \otimes K(x_n) \in \mathrm{Mod}_R$.

We can use these modules to give an explicit description of completion:

Proposition 5.1.5. *If $M \in \mathrm{Mod}_R$ is a module spectrum, then the natural map*

$$M \cong \underline{\mathrm{Map}}(R, M) \to \underline{\mathrm{Map}}(K(x_1, \dots, x_n), M)$$

exhibits the mapping spectrum $\underline{\mathrm{Map}}(K(x_1, \dots, x_n), M)$ as the I-adic completion of the module spectrum M.

Proof. We define $C \in \mathrm{Mod}_R$ to be the cofibre of the map $K(x_1, \dots, x_n) \to R$. Assume N is an I-nilpotent module. We have a cofibre sequence

$$\mathrm{Map}(N, \underline{\mathrm{Map}}(C, M)) \longrightarrow \mathrm{Map}(N, \underline{\mathrm{Map}}(R, M)) \longrightarrow \mathrm{Map}(N, \underline{\mathrm{Map}}(K(x_1, \dots, x_n), M))$$

The first term is equivalent to $\mathrm{Map}(N \otimes C, M)$. By 4.1.12. in DAG XII, there is a map of cofibre sequences

$$\begin{array}{ccccc} N \otimes K(x_1, \dots, x_n) & \longrightarrow & N \otimes R & \longrightarrow & N \otimes C \\ \simeq \downarrow & & \downarrow \simeq & & \downarrow \simeq \\ \Gamma_I(N) & \longrightarrow & N & \longrightarrow & L_I(N) \end{array}$$

where Γ_I and L_I denote the right and left adjoint to the inclusion of I-nilpotent and I-local modules into all modules respectively (Note that L_I is *not* related to the functor L_0 from the previous section).

Since N is nilpotent, $L_I(N) \cong N \otimes C$ is null and hence $\mathrm{Map}(N, \underline{\mathrm{Map}}(C, M))$ must vanish. Since this is true for all nilpotent N, the module spectrum $\underline{\mathrm{Map}}(C, M)$ is in fact I-local. The fibre sequence

$$\underline{\mathrm{Map}}(C, M) \to \underline{\mathrm{Map}}(R, M) \to \underline{\mathrm{Map}}(K(x_1, \dots, x_n), M)$$

then establishes the Proposition. □

Remark 5.1.6. This proof is very close to the techniques used in Section 4 of [Lurb].

Definition 5.1.7. An $\mathbb{E}_2$-ring R is said to be *Noetherian* if $\pi_*(R)$ is Noetherian.

One can filter the spectra $K(x_1, \dots, x_k)$ and use this to set up a spectral sequence (cf. [GM95]):

Theorem 5.1.8. *Let R be a Noetherian $\mathbb{E}_2$-ring with ideal $I \subset \pi_0(R)$. Given a left R-module spectrum $M \in \mathrm{Mod}_R$, there is a spectral sequence of graded R_*-modules*

$$E^2_{s,t} = H^I_s(R_*, \pi_*(M))_t \Rightarrow \pi_{s+t}(M^\wedge_I) \qquad d^r : E^{s,t}_r \to E^{s-r,t+r-1}_r$$

5.1.3 Completed Free Modules

We introduce the following terminology:

Definition 5.1.9. A module spectrum M over an $\mathbb{E}_2$-ring R is said to be *free* if it is of the form $M = \bigoplus_{s \in S} \Sigma^{i_s} R$ for some set S and some family of integers $\{i_s\}_{s \in S}$.

We write $\mathrm{Mod}_{R,f}$ for the full subcategory of Mod_R spanned by all such modules.

Definition 5.1.10. A graded module M_* over a graded ring R_* is said to be *free* if it is of the form $M = \bigoplus_{s \in S} \Sigma^{i_s} R_*$ for some set S and some family of integers $\{i_s\}_{s \in S}$.

We write $\mathrm{Mod}^*_{R_*,f}$ for the full subcategory of $\mathrm{Mod}^*_{R_*}$ spanned by all such modules.

Here $\Sigma^i R_*$ denotes the i^{th} shift of R_* satisfying $(\Sigma^i R_*)_s = R_{s-i}$.

Remark 5.1.11. These free modules are also sometimes called quasi-free in the literature (cf. [Lur14]).

Proposition 5.1.12. *Let R be an $\mathbb{E}_2$-ring. The functor π_* sends free R-modules to free R_*-modules.*

Proof. This holds since R is compact in ∞-category Mod_R by Example D.7.3.2. in [Lur16]. □

We will now introduce a completed variant of freeness:

Definition 5.1.13. A left module spectrum over an $\mathbb{E}_2$-ring R with ideal $I \subset \pi_0(R)$ is said to be *completed-free* if it is the completion $M_I^\wedge$ of a free R-module spectrum M.
We write $\mathrm{Mod}_{R,f}^{Cpl(I)}$ for the full subcategory spanned by all such modules.

Definition 5.1.14. A graded module over a graded commutative ring R_* with ideal $I \subset R_0$ is said to be *completed-free* if it is the completion $L_0(M_*) \cong Cpl_I(M_*)$ of a free R_*-module M_*.
We write $\mathrm{Mod}_{R_*,f}^{*Cpl(I)} \subset \mathrm{Mod}_{R_*}^{*Cpl(I)}$ for the full subcategory of all such modules.

Remark 5.1.15. Completed-free R_*-modules are often called pro-free in the literature (cf. [HS99]).

Proposition 5.1.16. *If $M \in \mathrm{Mod}_{R,f}$ is a free module spectrum over a Noetherian $\mathbb{E}_2$-ring R with ideal $I \subset \pi_0(R)$, then there is an isomorphism $\pi_*(M_I^\wedge) \cong L_0(\pi_*(M)) \cong Cpl_I(\pi_*(M)) = \varprojlim \pi_*(M)/I^n$.
In particular, the functor π_* sends completed-free module spectra to completed-free modules.*

Remark 5.1.17. This was observed for the case where R is Morava E-theory with its unique maximal ideal by Hovey as Corollary 2.4 in [Hov04].

Proof of 5.1.16. Since R is Noetherian, the local homology group $H_s^I(R_*, \pi_*(M))$ is given by the derived functor $L_s(\pi_*(M))$. Since $\pi_*(M)$ is free (in the graded sense), these functors vanish for $s > 0$ and we have $L_0(\pi_*(M)) \cong Cpl_I(\pi_*(M))$. The Greenlees-May spectral sequence hence implies the desired result. □

For the rest of this section, we fix a Noetherian $\mathbb{E}_2$-ring R. We can detect whether or not a given module spectrum is completed-free on the level of homotopy groups:

Proposition 5.1.18. *If $M \in \mathrm{Mod}_R$ is a module spectrum with $\pi_*(M)$ completed-free, then $M \in \mathrm{Mod}_{R,f}^{Cpl(I)}$ is a completed-free R-module spectrum.*

Proof. Assume that we are given a isomorphism $Cpl_I(\bigoplus_{s\in S} \Sigma^{i_s} R_*) \xrightarrow{\cong} \pi_*(M)$. We can lift the various grading-preserving maps $\Sigma^{i_s} R_* \to \pi_*(M)$ to maps $\Sigma^{i_s} R \to M$. The resulting map $(\bigoplus_{s\in S} \Sigma^{i_s} R)_I^\wedge \to M$ induces an equivalence on homotopy groups and is therefore itself an equivalence. □

We introduce the following notation:

Notation 5.1.19. Let $\mathrm{Mod}_{R_*,f} \subset \mathrm{Mod}^*_{R_*,f}$ and $Mod^{Cpl(I)}_{R_*,f} \subset \mathrm{Mod}^{*Cpl(I)}_{R_*,f}$ be the respective (non-full) subcategories containing only those homogeneous maps which *preserve* the grading.

The Nerve functor $N : \mathrm{Cat} \to \mathbf{sSet}$ has a right adjoint which we shall denote by h. It assigns to an ∞-category its homotopy category. Homotopy classes of maps out of completed-free module spectra are determined by their effect on homotopy:

Lemma 5.1.20. *If $M \in \mathrm{Mod}^{Cpl(I)}_{R,f}$ and $N \in \mathrm{Mod}^{Cpl(I)}_{R}$, then the following map is bijective:*

$$\pi_* : \mathrm{Map}_{h\,\mathrm{Mod}^{Cpl(I)}_R}(M,N) \to \mathrm{Map}_{\mathrm{Mod}^{Cpl(I)}_{R_*}}(\pi_* M, \pi_* N)$$

Proof. Let $M = \left(\bigoplus_{i\in I} \Sigma^{k_i} R\right)^{\wedge}_I$ be a completed-free module spectrum. We use Proposition 5.1.16 to see that $\pi_*(M^{\wedge}_I) = Cpl_I(\pi_*(M))$ and thus obtain the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Map}_{h\,\mathrm{Mod}^{Cpl(I)}_R}(M,N) & \longrightarrow & Map_{\mathrm{Mod}^{Cpl(I)}_{R_*}}\Big(\pi_*(M), \pi_*(N)\Big) \\
\downarrow & & \downarrow \\
\mathrm{Map}_{h\,\mathrm{Mod}_R}\Big(\bigoplus_{i\in I} \Sigma^{k_i} R, N\Big) & \longrightarrow & \mathrm{Map}_{\mathrm{Mod}_{R_*}}\Big(\bigoplus_{i\in I} \Sigma^{k_i} R_*, \pi_*(N)\Big) \\
\downarrow & & \downarrow \\
\prod_{i\in I} \mathrm{Map}_{h\,\mathrm{Mod}_R}\Big(\Sigma^{k_i} R, N\Big) & \xrightarrow{\cong} & \prod_{i\in I} \mathrm{Map}_{\mathrm{Mod}_{R_*}}\Big(\Sigma^{k_i} R_*, \pi_*(N)\Big)
\end{array}$$

The vertical maps are evidently bijective. The lowest map is a bijection since both sides are readily identified with the set $\prod_{i\in I} \pi_{k_i}(N)$. □

The following result is *crucial* in our definition of approximation functors in Chapter 3:

Corollary 5.1.21. *Taking homotopy groups induces an equivalence $\pi_* : h\,\mathrm{Mod}^{Cpl(I)}_{R,f} \xrightarrow{\cong} \mathrm{Mod}^{Cpl(I)}_{R_*,f}$.*

Proof. The functor lands in the subcategory on the right by Proposition 5.1.16. The same proposition straightforwardly implies essential surjectivity. The functor is fully faithful by Lemma 5.1.20.

□

5.2 Appendix B: Monoidal ∞-categories

5.2.1 Algebras and Modules

We follow [Lura] and recall some of the basic notions in the theory of monoidal and tensored ∞-categories. Only the final section on restrictions is (easy) original work.

Definition 5.2.1. A *monoidal ∞-category* is a coCartesian fibration $\mathcal{C}^{\circledast} \to N(\Delta)^{op}$ in $\widehat{\mathbf{sSet}}_{/N(\Delta)^{op}}$ such that $\mathcal{C}^{\circledast}_{[n]} \to \mathcal{C}^{\circledast}_{\{0,1\}} \times \cdots \times \mathcal{C}^{\circledast}_{\{n-1,n\}}$ determines an equivalence to $(\mathcal{C}^{\circledast}_{[1]})^{\times n}$ for all n.

We mark every edge in $N(\Delta)^{op}$ and consider the combinatorial model category $\mathbf{A} = (\widehat{\mathbf{sSet}}^{+})_{/N(\Delta)^{op}}$ of (not necessarily small) marked simplicial sets over $N(\Delta)^{op}$, endowed with the coCartesian model structure defined in Remark 3.1.3.9 in [Lur09]. The category $\mathbf{A}^{o}$ of fibrant-cofibrant objects is given by coCartesian fibrations over $N(\Delta)^{op}$ with coCartesian edges marked. We will write $\widehat{Cat}^{coCart}_{\infty/N(\Delta)^{op}} = N(\mathbf{A}^{o})$ for the simplicial nerve of $\mathbf{A}^{o}$.

Definition 5.2.2. The ∞-category of monoidal ∞-categories $\widehat{Cat}^{Mon}_{\infty}$ is given by the full subcategory of $\widehat{Cat}^{coCart}_{\infty/N(\Delta)^{op}}$ spanned by all monoidal ∞-categories.

Definition 5.2.3. Given a monoidal ∞-category $\mathcal{C}^{\circledast} \xrightarrow{p} N(\Delta)^{op}$, the ∞-category $Alg(\mathcal{C})$ of algebra objects consists of the full subcategory of $\mathrm{Fun}_{N(\Delta)^{op}}(N(\Delta)^{op}, \mathcal{C}^{\circledast})$ spanned by all sections which send every convex morphism to a p-coCartesian morphism.

Definition 5.2.4. A *tensored ∞-category* is a morphism $\mathcal{M}^{\circledast} \to \mathcal{C}^{\circledast} \to N(\Delta)^{op}$ in $\widehat{\mathbf{sSet}}_{/N(\Delta)^{op}}$ such that

- $\mathcal{C}^{\circledast}$ is a monoidal ∞-category
- The structure map $M^{\circledast} \to N(\Delta)^{op}$ is a coCartesian fibration.
- The map $M^{\circledast} \to C^{\circledast}$ is a categorical fibration which preserves coCartesian edges.
- For each n, the inclusion $\{n\} \subset [n]$ induces an equivalence $\mathcal{M}^{\circledast}_{[n]} \to \mathcal{C}^{\circledast}_{[n]} \times \mathcal{M}^{\circledast}_{\{n\}}$.

The category $\mathbf{A}^{[1]}$ inherits the structure of a simplicial model category with the injective model structure. By Proposition 4.2.4.4. in [Lur09], there is an equivalence of ∞-categories $N((\mathbf{A}^{[1]})^{o}) \xrightarrow{\cong} \mathrm{Fun}(\Delta^1, \widehat{Cat}_{\infty/N(\Delta)^{op}})$.

The following proposition appears in Section 2.6. of [Lura]:

Proposition 5.2.5. *The fibrant-cofibrant objects in* $\mathbf{A}^{[1]}$ *are precisely given by triangles*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ & \searrow & \downarrow \\ & & N(\Delta)^{op} \end{array}$$

satisfying the following properties:

- $\mathcal{C} \to N(\Delta)^{op}$,$\mathcal{D} \to N(\Delta)^{op}$ *are coCartesian fibrations of simplicial sets with coCartesian edges marked.*
- *The map* f *is a categorical fibration of simplicial sets.*

Definition 5.2.6. The ∞-category of tensored ∞-categories, denoted by $\widehat{\mathrm{CatMod}}$, is given by the full subcategory of $N((\mathbf{A}^{[1]})^o)$ spanned by all tensored ∞-categories.

Definition 5.2.7. Let $\mathcal{M}^\circledast \xrightarrow{q} \mathcal{C}^\circledast \xrightarrow{p} N(\Delta)^{op}$ be a tensored ∞-category.
The ∞-category $\mathcal{M}od(\mathcal{M})$ of *modules* is given by the full subcategory of $\mathrm{Map}_{N(\Delta)^{op}}(N(\Delta)^{op}, \mathcal{M}^\circledast)$ spanned by all functors F for which

- the composition $q \circ F$ is an algebra object of $\mathcal{C}^\circledast$
- if $\alpha : [m] \to [n]$ is a convex map in Δ with $\alpha(m) = n$, then $F(\alpha)$ is a $(p \circ q)$-coCartesian edge.

Composition with q gives a natural map $\mathcal{M}od(\mathcal{M}) \xrightarrow{U} \mathrm{Alg}(\mathcal{C})$. Following Remark 2.1.8 in [Lura], we define:

Definition 5.2.8. Given an algebra object $A \in \mathrm{Alg}(\mathcal{C})$, the ∞-category of (left) A-modules $\mathrm{Mod}_A(\mathcal{M})$ is given by the fibre of U over A.

If $\mathcal{M}^\circledast \to \mathcal{N}^\circledast$ is a morphism of $\mathcal{C}^\circledast$-tensored ∞-categories, every $A \in \mathrm{Alg}(\mathcal{C})$ gives rise to a functor $\mathrm{Mod}_A(\mathcal{M}) \to \mathrm{Mod}_A(\mathcal{N})$.

5.2.2 On coAlgebras and coModules

We have an involution $(op) : \widehat{Cat}_\infty^{Mon} \to \widehat{Cat}_\infty^{Mon}$ obtained by observing that the functor

$$\widehat{Cat}^{coCart}_{\infty/N(\Delta)^{op}} \xrightarrow{St} \mathrm{Fun}(N(\Delta^{op}), \widehat{Cat}_\infty) \xrightarrow{(op)\circ(-)} \mathrm{Fun}(N(\Delta^{op}), \widehat{Cat}_\infty) \xrightarrow{Un} \widehat{Cat}^{coCart}_{\infty/N(\Delta)^{op}}$$

sends monoidal ∞-categories to monoidal ∞-categories. Here St and Un denote straightening and unstraightening respectively. Here op is the "opposite" involution of $\widehat{Cat}_\infty$.

Definition 5.2.9. The ∞-category of coalgebras in a monoidal ∞-category $\mathcal{C}$ is $\mathrm{coAlg}(\mathcal{C}) := \mathrm{Alg}(\mathcal{C}^{op})^{op}$.

We can define an involution $(op) : \widehat{\mathrm{CatMod}} \to \widehat{\mathrm{CatMod}}$ by observing that the following functor sends tensored ∞-categories to tensored ∞-categories.

$$\mathrm{Fun}(\Delta^1, \widehat{Cat}^{coCart}_{\infty/N(\Delta)^{op}}) \xrightarrow{St\circ(-)} \mathrm{Fun}(\Delta^1, \mathrm{Fun}(N(\Delta)^{op}, \widehat{Cat}_\infty))$$

$$\xrightarrow{(op)\circ(-)} \mathrm{Fun}(\Delta^1, \mathrm{Fun}(N(\Delta)^{op}, \widehat{Cat}_\infty)) \xrightarrow{Un} \mathrm{Fun}(\Delta^1, \widehat{Cat}^{coCart}_{\infty/N(\Delta)^{op}})$$

Definition 5.2.10. Let $\mathcal{M}^{\circledast} \xrightarrow{q} \mathcal{C}^{\circledast} \xrightarrow{p} N(\Delta)^{op}$ be a tensored ∞-category. The ∞-category $co\mathcal{M}od(\mathcal{M})$ of *comodules* is given by $co\mathcal{M}od(\mathcal{M}) = (\mathcal{M}od(\mathcal{M}^{op}))^{op}$.

There is a natural map $co\mathcal{M}od(\mathcal{M}) \xrightarrow{U} \mathrm{coAlg}(\mathcal{C})$. Given a coalgebra object $C \in \mathrm{coAlg}(\mathcal{C})$, the ∞-category of (left) C-comodules $\mathrm{coMod}_C(\mathcal{M})$ is given by the fibre of U over C.

5.2.3 Constructing tensored ∞-categories

Let S be a category. We invite the reader to recall the relative Nerve functor $N_f(S) : \widehat{\mathbf{sSet}}^S \to (\widehat{\mathbf{sSet}})_{/S}$ defined in Definition 3.2.5.2. of [Lur09]. Let $K \subset (\widehat{\mathbf{sSet}}^{\Delta^{op}})^{[1]}$ be the full subcategory spanned by all transformations $F \to G$ of functors $\Delta^{op} \to \widehat{\mathbf{sSet}}$ for which $F(i) \to G(i)$ is a categorical fibration between ∞-categories for all $i \in \Delta^{op}$. Using Lemma 3.2.5.11 in [Lur09], the relative nerve gives a map $K \to (\mathbf{A}^{[1]})^o$.

We now introduce a strict variant of tensored ∞-categories:

Definition 5.2.11. The category $\widehat{\mathbf{CatMod}}^{ord}$ of strictly (left) tensored ∞-categories has objects $(\mathcal{C}, \mathcal{M})$, where $\mathcal{C}$ is an ∞-category endowed with the structure of a simplicial monoid and $\mathcal{M}$ is an ∞-category endowed with a (left) monoid action $\mathcal{C} \times \mathcal{M} \to \mathcal{M}$ by $\mathcal{C}$. Morphisms are defined in the evident way.

Remark 5.2.12. The functor $(-)^{op} : \widehat{\mathbf{sSet}} \to \widehat{\mathbf{sSet}}$ is strictly monoidal with respect to the cartesian product and sends ∞-categories to ∞-categories. Thus, it induces an endofunctor $(-)^{op} : \widehat{\mathbf{CatMod}}^{ord} \to \widehat{\mathbf{CatMod}}^{ord}$.

There is a natural functor of categories $\tau : \widehat{\mathbf{CatMod}}^{ord} \to K \subset \left(\widehat{\mathbf{sSet}}^{\Delta^{op}}\right)^{[1]}$ sending $(\mathcal{C}, \mathcal{M})$ to

$$\begin{array}{ccccccc} \cdots & \mathcal{C} \times \mathcal{C} \times \mathcal{M} & \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} & \mathcal{C} \times \mathcal{M} & \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} & \mathcal{M} \\ & \downarrow & & \downarrow & & \downarrow \\ \cdots & \mathcal{C} \times \mathcal{C} & \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} & \mathcal{C} & \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} & [0] \end{array}$$

Composing with the relative nerve hence gives a functor of simplicial categories $\theta : \widehat{\mathbf{CatMod}}^{ord} \to (\mathbf{A}^{[1]})^o$. Applying the simplicial nerve and observing that all objects "on the right" indeed satisfy the axioms for a tensored ∞-category, we have constructed a functor $\Theta : N(\widehat{\mathbf{CatMod}}^{ord}) \longrightarrow \widehat{\mathrm{CatMod}}$.

5.2.4 Restriction of Tensored ∞-Categories

We will now review how to pull back tensored structures along monoidal functors.

Definition 5.2.13. A morphism of tensored ∞-categories in $(\mathbf{A}^{[1]})^o$ given by

$$\begin{array}{ccc} \mathcal{M}^{\circledast} & \longrightarrow & \mathcal{N}^{\circledast} \\ \downarrow & & \downarrow \\ \mathcal{C}^{\circledast} & \xrightarrow{F^{\circledast}} & \mathcal{D}^{\circledast} \\ & \searrow & \downarrow \\ & & N(\Delta)^{op} \end{array}$$

is called *restrictive* if the top square is a pullback in $\widehat{\mathbf{sSet}}_{/N(\Delta^{op})}$.

Since $\mathcal{C}^{\circledast}$, $\mathcal{D}^{\circledast}$, and $\mathcal{N}^{\circledast}$ are fibrant objects of $\mathbf{A}$ and $\mathcal{N}^{\circledast} \to \mathcal{D}^{\circledast}$ is a fibration in $\mathbf{A}$ (by Proposition 2.6.4 in [Lura]), this happens precisely if the top square is a *homotopy pullback* in $\mathbf{A}$, i.e. a pullback in the ∞-category $\widehat{Cat}^{cocart}_{\infty/N(\Delta)^{op}} = N(\mathbf{A}^o)$.

We also say that the morphism *exhibits $\mathcal{M}$ as restriction of $\mathcal{N}$ along the monoidal functor $F : \mathcal{C} \to \mathcal{D}$.*

Proposition 5.2.14. *The functor $(op) : \widehat{\mathrm{CatMod}} \to \widehat{\mathrm{CatMod}}$ preserves restrictive morphisms.*

Proof. We think of a 1-morphism $G \in \mathrm{Fun}(\Delta^1, \widehat{\mathrm{CatMod}}) \subset \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, \widehat{Cat}^{coCart}_{\infty/N(\Delta)^{op}}))$ as an element of $\Delta^1 \times \Delta^1 \to \widehat{Cat}^{coCart}_{\infty/N(\Delta)^{op}}$. Then $(op)(G)$ corresponds to $\Delta^1 \times \Delta^1 \to \widehat{Cat}^{coCart}_{\infty/N(\Delta)^{op}} \xrightarrow{(op)} \widehat{Cat}^{coCart}_{\infty/N(\Delta)^{op}}$. Since (op) is an equivalence, G is a pullback square if and only if $(op)(G)$ is one. $\square$

Lemma 5.2.15. *Let $F : \mathcal{C} \to \mathcal{D}$ be a map of simplicial monoids which are ∞-categories. Assume that $\mathcal{M}$ is an ∞-category with a (left) action by the monoid $\mathcal{D}$. We write $\mathcal{M}_{\mathcal{C}}$ for $\mathcal{M}$ with the $\mathcal{C}$-action obtained by restriction along F. Applying θ to the morphism $\tilde{F} = (\mathcal{C}, \mathcal{M}_{\mathcal{C}}) \to (\mathcal{C}, \mathcal{M})$ in $\widehat{\mathbf{CatMod}}^{ord}$ yields a morphism*

$$\begin{array}{ccc} \mathcal{M}_{\mathcal{C}}^{\circledast} & \longrightarrow & \mathcal{M}^{\circledast} \\ \downarrow & & \downarrow \\ \mathcal{C}^{\circledast} & \xrightarrow{F^{\circledast}} & \mathcal{D}^{\circledast} \\ & \searrow & \downarrow \\ & & N(\Delta)^{op} \end{array}$$

which exhibits $\mathcal{M}_{\mathcal{C}}$ as restriction of $\mathcal{M}$ along F.

Proof. The morphism $\tau(\tilde{F}) \in ((\widehat{\mathbf{sSet}}^{\Delta^{op}})^{[1]})^{[1]} \cong (\widehat{\mathbf{sSet}}^{\Delta^{op}})^{[1]\times[1]}$ is a pullback of functors $\Delta^{op} \to \widehat{\mathbf{sSet}}$ as it is evidently a pointwise pullback. As remarked in 3.2.5.5. of [Lur09], the relative nerve preserves limits, and we therefore see that $\theta(\tilde{F}) \in (\mathbf{A}^{[1]})^{[1]} \cong \mathbf{A}^{[1]\times[1]}$ is a pullback square in $\mathbf{A} = (\widehat{\mathbf{sSet}}^{+})_{/N(\Delta)^{op}}$. □

Now assume that

$$\begin{array}{ccc} \mathcal{M}^{\circledast} & \longrightarrow & \mathcal{N}^{\circledast} \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{C}^{\circledast} & \xrightarrow{F^{\circledast}} & \mathcal{D}^{\circledast} \\ & \searrow \quad \swarrow & \\ & N(\Delta)^{op} & \end{array}$$

is a morphism of tensored ∞-categories which exhibits $\mathcal{M}$ as obtained from $\mathcal{N}$ by restriction along F.

Lemma 5.2.16. *Given an algebra $C \in \mathrm{Alg}(\mathcal{C})$, the functor $\mathrm{Mod}_C(\mathcal{M}) \xrightarrow{\simeq} \mathrm{Mod}_{F(C)}(\mathcal{N})$ is an equivalence.*

Proof. By the universal property of pullbacks, we have a natural isomorphism

$$\mathrm{Fun}_{N(\Delta)^{op}}(N(\Delta)^{op}, \mathcal{M}^{\circledast}) \underset{\mathrm{Fun}_{N(\Delta)^{op}}(N(\Delta)^{op}, \mathcal{C}^{\circledast})}{\times} \{C\} \xrightarrow{\simeq} \mathrm{Fun}_{N(\Delta)^{op}}(N(\Delta)^{op}, \mathcal{N}^{\circledast}) \underset{\mathrm{Fun}_{N(\Delta)^{op}}(N(\Delta)^{op}, \mathcal{D}^{\circledast})}{\times} \{F(C)\}$$

The claim follows by observing that this isomorphism identifies modules in the sense of Definition 5.2.7. □

5.3 Appendix C: $\mathbb{Z}$-Graded Algebraic Theories

We recall the theory of many-sorted algebraic theories from [ARV11] in the special case where $S = \mathbb{Z}$:

Definition 5.3.1. An *algebraic theory* is a small category $\mathcal{T}$ with finite products. A *morphism $\mathcal{T}_1 \to \mathcal{T}_2$ of algebraic theories* is a functor which preserves finite products. An algebra over an algebraic theory $\mathcal{T}$ is a functor $\mathcal{T} \to \mathrm{Set}$ which preserves finite products. We write $\mathrm{Alg}_{\mathcal{T}}$ for the category of $\mathcal{T}$-algebras.

Remark 5.3.2. Following [ARV11], we do *not* require every object to be a Cartesian power of a fixed object. Hence this notion is more general than the notion of a Lawvere theory.

Example 5.3.3. Write $\mathbb{Z}^*$ for the collection of (possibly empty) words of finite length in $\mathbb{Z}$. We turn $\mathbb{Z}^*$ into a category by declaring morphisms $s_1 \dots s_n \to t_1 \dots t_k$ to be functions $f : \{1, \dots, k\} \to \{1, \dots, n\}$ with $s_{f(i)} = t_i$ for all i. Then $\mathbb{Z}^*$ is the free completion of $\mathbb{Z}$ under finite products. We can consider $\mathbb{Z}^*$ as an algebraic theory. Algebras over $\mathbb{Z}^*$ are the same as functors $\mathbb{Z} \to \mathrm{Set}$.

Definition 5.3.4. A $\mathbb{Z}$-graded algebraic theory consists of a pair $(\mathcal{P}, F)$ of algebraic theory $\mathcal{P}$ with objects the words over $\mathbb{Z}$ and $F : \mathbb{Z}^* \to \mathcal{T}$ a morphism of theories which is the identity map on objects.

A morphism $(\mathcal{P}_1, F_1) \to (\mathcal{P}_2, F_2)$ of $\mathbb{Z}$-graded theories is a map of theories $M : \mathcal{P}_1 \to \mathcal{P}_2$ with $M \circ F_1 = F_2$.

An algebra over a $\mathbb{Z}$-graded algebraic theory $(\mathcal{P}, F)$ consists of a functor $\mathcal{P} \to \mathrm{Set}$ which preserves finite products. We write $\mathrm{Alg}_{\mathcal{P}}$ for the category of $\mathcal{P}$-algebras.

There is a forgetful functor $\mathrm{Alg}_{\mathcal{P}} \to \mathrm{Set}^{\mathbb{Z}}$. By Proposition 14.8 in [ARV11], this functor is faithful, conservative, preserves and reflects limits, sifted colimits, monomorphisms, and regular epimorphisms.

We can define $\mathbb{Z}$-graded algebraic theories by specifying operations and relations between them.

For this, assume Σ is a set together with a so-called "arity function" $\Sigma \to \mathbb{Z}^* \times \mathbb{Z}$.

Definition 5.3.5. A Σ-*algebra* is a $\mathbb{Z}$-graded set X together with a map $\sigma_X : X_{j_1} \times \ldots \times X_{j_n} \to X_i$ for each $\sigma \in \Sigma$ with arity $(j_1 \ldots l_n, i)$. There is an evident notion of morphisms of Σ-algebras, and the resulting category shall be denoted by Alg_{Σ}.

Given a $\mathbb{Z}$-graded set X, we write $F_{\sigma}(X)$ for the free Σ-algebra on the set X.

An *equation* in Σ is given by an expression $\forall x_1 \ldots \forall x_n : (t = t')$ where x_i is a $\mathbb{Z}$-graded variable of degree s_i and $t, t' \in F_{\Sigma}(\{x_1, \ldots, x_n\})$. We say a Σ-algebra X *satisfies a set of equations* E if whenever $(t = t') \in E$ is an equation in the graded variables $x_1, \ldots, x_n$ and $f : \{x_1, \ldots, x_n\} \to X$ is a grading-preserving map with canonical extension $\overline{f} : F_{\Sigma}(\{x_1, \ldots, x_n\}) \to X$, then $\overline{f}(t) = \overline{f}(t')$.

Definition 5.3.6. Write $\mathrm{Alg}_{\Sigma}(E)$ for the full subcategory consisting of all Σ-algebras satisfying the equations E.

We can construct a $\mathbb{Z}$-graded algebraic theory whose category of algebras is equivalent to $\mathrm{Alg}_{\Sigma}(E)$ as follows:

We start with the category $\mathcal{P}_{\Sigma}$ which is *opposite to* the category whose objects are all $F_{\Sigma}(\{x_1, \ldots, x_k\})$ for graded variables $x_1, \ldots, x_k$ and whose morphisms are maps of Σ-algebras. Every equation $(t = t')$ in E gives rise to two arrows $\underline{t}, \underline{t'} : F_{\Sigma}(\{x_1, \ldots, x_k\}) \rightrightarrows F_{\Sigma}(\{y\})$. We follow Definition 10.4 in [ARV11] and define an equivalence relation $\simeq_E$ on every Hom-set in the category $\mathcal{P}_{\Sigma}$ by requiring:

1. $\underline{t} \simeq_E \underline{t'}$ whenever $(t = t')$ is an equation in E.

2. If $u, v : x \to y$ and $s, t : y \to z$ have $u \simeq_E v$ and $s \simeq_E t$, then $s \circ u \simeq_E t \circ v$.

3. If $u_i, v_i : x \to y_i$ have $u_i \simeq_E v_i$ for $i = 1, \ldots, n$, then $u_1 \times \ldots \times u_n \simeq_E v_1 \times \ldots \times v_n$.

We form a new category $\mathcal{P}_{\Sigma}/E$ by identifying equivalent morphisms in each hom-set (individually).

By Proposition 14.23 in [ARV11], $\mathcal{P}_{\Sigma}/E$ defines a $\mathbb{Z}$-graded algebraic theory whose category of algebras $\mathrm{Alg}(\mathcal{P}_{\Sigma}/E)$ is canonically equivalent to $\mathrm{Alg}_{\Sigma}(E)$.

The forgetful functor $\mathrm{Alg}_{\Sigma}(E) \to \mathrm{Set}^{\mathbb{Z}}$ is therefore again faithful, conservative, preserves and reflects limits, sifted colimits, monomorphisms, and regular epimorphisms.

5.4 Appendix D: Monadic Koszul Duality

Let $\mathcal{D}$ be an ∞-category containing geometric realisations and assume $T \to 1$ is an augmented realisation-preserving monad. By Lemma 4.5.12. in [Lura], the restriction $\mathrm{triv} : \mathcal{D} \cong \mathrm{Alg}_1(\mathcal{D}) \to \mathrm{Alg}_T(\mathcal{D})$ admits a left adjoint Q and we therefore obtain a comonad $C = Q \circ \mathrm{triv} \in \mathrm{coAlg}(\mathrm{End}(\mathcal{D}))$. The comonad C is Koszul dual to T in the sense of [Lur16]. We can now take homotopy categories to obtain an adjunction of ordinary categories $hQ : h\,\mathrm{Alg}_T(\mathcal{D}) \rightleftarrows h\mathcal{D} : h\,\mathrm{triv}$ and a corresponding comonad $hC = hQ \circ h\,\mathrm{triv} \in \mathrm{coAlg}(\mathrm{End}(h\mathcal{D}))$.

Now suppose that $\mathbf{D}$ is a cofibrantly generated simplicial or topological model category in which all objects are fibrant. Write $\mathcal{W}$ for the class of weak equivalences in $\mathbf{D}$. Let $\mathbf{T} : \mathbf{D} \to \mathbf{D}$ be a simplicial or topological endofunctor which is endowed with the structure of an augmented monad and which preserves cofibrant objects and weak equivalences between them. Assume that the left derived functor $L\mathbf{T} = \mathbf{T} \circ Q$ preserves geometric realisations (i.e. Δ^{op}-indexed homotopy colimits). Here Q denotes the cofibrant replacement functor which attaches to an object A the cofibrant domain QA of a *trivial fibration* $QA \to A$.

By Remark 4.5 in [SS00], we can apply Lemma 2.3 in [loc.cit] and deduce the existence of a model category structure on $\mathbf{T}$-algebras whose fibrations and weak equivalences are defined at the level of underlying objects. We have a natural forget-free Quillen adjunction $\mathrm{Free}_{\mathbf{T}} : \mathbf{D} \leftrightarrows \mathrm{Alg}_{\mathbf{T}}(\mathbf{D}) : \mathrm{Forget}_{\mathbf{T}}$. Following [MG16] and [Hin16] generalising Proposition 5.2.4.6. in [Lur09], we obtain an adjunction on underlying ∞-categories

$$\mathrm{Free}_T : N(\mathbf{D})[\mathcal{W}^{-1}] \leftrightarrows N(\mathrm{Alg}_{\mathbf{T}}(\mathbf{D}))[\mathcal{W}^{-1}] : \mathrm{Forget}_T$$

whose constituent functors are induced by $L\,\mathrm{Free}_{\mathbf{T}} = \mathrm{Free}_{\mathbf{T}} \circ Q$ and $R\,\mathrm{Forget}_{\mathbf{T}} = \mathrm{Forget}_{\mathbf{T}}$ respectively. Here we have inverted weak equivalences by taking fibrant replacement of marked simplicial sets (cf. Remark 1.3.4.2. in [Lur14]), which is related to the Hammock localisation through Proposition 1.2.1 of [Hin16].

The associated monad $T = \mathrm{Forget}_T \circ \mathrm{Free}_T$ on $\mathcal{D} := N(\mathbf{D})[\mathcal{W}^{-1}]$ is induced by $\mathbf{T} \circ Q$ and thus preserves geometric realisations. This implies that Forget_T also preserves geometric realisations. Since Forget_T is also conservative, the Barr-Beck-Lurie theorem (Theorem 4.7.4.5 in [Lur14]) implies that the canonical functor $N(\mathrm{Alg}_{\mathbf{T}}(\mathbf{D}))[\mathcal{W}^{-1}] \to \mathrm{Alg}_T(\mathcal{D})$ is an equivalence of ∞-categories.

Using the augmentation $\mathbf{T} \to \mathbf{1}$, we can define a functor $\mathrm{Alg}_{\mathbf{T}}(\mathbf{D}) \leftarrow \mathbf{D} : \mathbf{triv}$. This functor is right Quillen and hence admits a left adjoint $\mathbf{V} : \mathrm{Alg}_{\mathbf{T}}(\mathbf{D}) \to \mathbf{D}$. As before, we obtain a corresponding adjunction of ∞-categories $V : N(\mathrm{Alg}_{\mathbf{T}}(\mathbf{D}))[\mathcal{W}^{-1}] \leftrightarrows N(\mathbf{D})[\mathcal{W}^{-1}] : \mathrm{triv}$ whose constituent functors are induced by $L\mathbf{V} = V \circ Q$ and $R\mathbf{triv} = \mathbf{triv}$ respectively. After composing with the equivalence $N(\mathrm{Alg}_{\mathbf{T}}(\mathbf{D}))[\mathcal{W}^{-1}] \xrightarrow{\simeq} \mathrm{Alg}_T(\mathcal{D})$, we recover the adjunction discussed in the beginning of this section. Passing to homotopy categories gives rise to an adjunction $hV : h\operatorname{Alg}_T(\mathcal{D}) \leftrightarrows h\mathcal{D} : h\mathrm{triv}$ and we can therefore identify the comonad hC induced by C on the homotopy category $h\mathcal{D}$ with the comonad $L\mathbf{V} \circ R\mathbf{triv} = \mathbf{V} \circ Q \circ \mathbf{triv}$ with structure map $\mathbf{V} \circ Q \circ \mathbf{triv} \cong \mathbf{V} \circ Q \circ \mathrm{id} \circ \mathbf{triv} \to \mathbf{V} \circ Q \circ (\mathbf{triv} \circ \mathbf{V} \circ Q) \circ \mathbf{triv}$.

The final map uses the natural arrow $X \overset{\simeq}{\leftarrow} QX \to (\mathbf{triv} \circ \mathbf{V})(QX)$ in the homotopy category, where the second map is the unit of the original adjunction between $\mathbf{triv}$ and $\mathbf{V}$. We will now give concrete descriptions of this comonad in two cases of interest to us.

5.4.1 Additive Monads

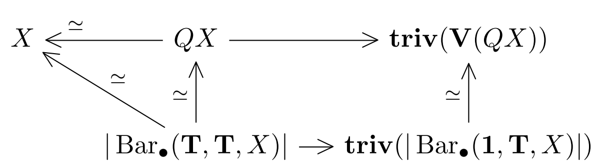
Let $\mathbf{A}$ be an abelian category with enough projectives. We write $\mathrm{Ch}_{\geq 0}(\mathbf{A})$ for the simplicial model category of nonnegatively graded chain complexes in $\mathbf{A}$, where weak equivalences are quasi-isomorphisms and fibrations are levelwise epimorphisms – all objects are fibrant. It is well-known that geometric realisations in the underlying ∞-category $\mathcal{D}^-_{\geq 0}(\mathbf{A})$ can be computed "via the total complex" (see Proposition 19.9. of [Dug08]). Given a simplicial chain complex $X_\bullet \in \mathrm{Fun}(\Delta^{op}, \mathrm{Ch}_{\geq 0}(\mathbf{A}))$, we can first take the alternating face maps in the simplicial direction to obtain a double complex $(Y_{i,j} = (X_j)_i,\ d_h : Y_{*,*} \to Y_{*-1,*},\ d_v : Y_{*,*} \to Y_{*,*-1})$ and then form a single complex $|X_\bullet| = \bigoplus_{i+j=n} Y_{i,j}$ with "total" differential $D = d_v + (-1)^j d_h$.

Now assume we are handed an augmented simplicial monad $\mathbf{T}$ acting additively on $\mathbf{A}$ and preserving cofibrant objects. The natural extension of $\mathbf{T}$ to $\mathrm{Ch}_{\geq 0}(\mathbf{A})$ preserves weak equivalences between cofibrant chain complexes, and our above analysis shows that $L\mathbf{T}$ also preserves geometric realisations.

We can therefore apply the discussion in the last section to obtain a model category $\operatorname{Alg}_{\mathbf{T}}(\mathrm{Ch}_{\geq 0}(\mathbf{A}))$ whose underlying ∞-category is canonically identified with $\operatorname{Alg}_T(\mathrm{Ch}_{\geq 0}(\mathbf{A}))$ for T a monad on $\mathrm{Ch}_{\geq 0}(\mathbf{A})$ whose underlying functor is induced by $L\mathbf{T}$.

The functor $\operatorname{Alg}_{\mathbf{T}}(\mathrm{Ch}_{\geq 0}(\mathbf{A})) \leftarrow \mathrm{Ch}_{\geq 0}(\mathbf{A}) : \mathbf{triv}$ has a left adjoint $\mathbf{V}$ which computes indecomposables. The right derived functor $R\mathbf{triv}$ is just given by $\mathbf{triv}$. In order to compute the left derived functor of $\mathbf{V}$, we first use the universal properties of $\mathbf{V}$ and $\mathbf{T}$ to observe that the value of $\mathbf{V}$ on free algebras $\mathbf{T}Y$ is Y.

Using extra degeneracies and the definition of the model structure on $\mathbf{T}$-algebras, we conclude that the map $|\operatorname{Bar}_\bullet(\mathbf{T}, \mathbf{T}, X)| \to X$ is a weak equivalence from a cofibrant object. By Corollary 2.12. in [GS07], there is a homotopy-unique weak equivalence $|\operatorname{Bar}_\bullet(\mathbf{T}, \mathbf{T}, X)| \to QX$ over X. We can therefore compute $L\mathbf{V}(X) = \mathbf{V}QX \overset{\simeq}{\leftarrow} |\operatorname{Bar}_\bullet(\mathbf{1}, \mathbf{T}, X)|$. The unit $\mathrm{id} \to \mathbf{triv} \circ \mathbf{V} \circ Q$ can be understood by the following diagram:



We can therefore describe the comultiplication on the comonad $hC = \mathbf{V} \circ Q \circ \mathbf{triv}$ via the following diagram:

$$\begin{array}{ccccc}
(\mathbf{V} \circ Q) \circ \mathbf{triv}(A) & \xleftarrow{\simeq} & (\mathbf{V} \circ Q) \circ Q \circ \mathbf{triv}(A) & \longrightarrow & (\mathbf{V} \circ Q) \circ \mathbf{triv} \circ \mathbf{V} \circ Q \circ \mathbf{triv}(A) \\
\simeq \uparrow & & \simeq \uparrow & & \simeq \uparrow \\
|\operatorname{Bar}_\bullet(\mathbf{1}, \mathbf{T}, \mathbf{triv}(A))| & \xleftarrow{\simeq} & |\operatorname{Bar}_\bullet(\mathbf{1}, \mathbf{T}, |\operatorname{Bar}_\bullet(\mathbf{T}, \mathbf{T}, \mathbf{triv}(A))|)| & \longrightarrow & |\operatorname{Bar}_\bullet(\mathbf{1}, \mathbf{T}, |\operatorname{Bar}_\bullet(\mathbf{1}, \mathbf{T}, \mathbf{triv}(A))|)|
\end{array}$$

where the lower right horizontal map makes use of the augmentation.

We will now spell this map out in detail. For this, let $m : \mathbf{T}^2 \to \mathbf{T}$ be the monadic multiplication, $\eta : 1 \to \mathbf{T}$ the unit, and $\epsilon : \mathbf{T} \to 1$ the augmentation transformation. Given a $\mathbf{T}$-algebra X in $\mathbf{A}$ (which can be thought of as a complex in degree 0) with multiplication map $a : \mathbf{T}X \to X$, we can write the monadic Bar construction $|\operatorname{Bar}_\bullet(\mathbf{T}, \mathbf{T}, X)|$ as the following augmented simplicial chain complex:

$$\ldots \longrightarrow \mathbf{T}^4X \xrightarrow{m_{\mathbf{T}^2X} - \mathbf{T}m_{\mathbf{T}X} + \mathbf{T}^2m_X - \mathbf{T}^3a} \mathbf{T}^3X \xrightarrow{m_{\mathbf{T}X} - \mathbf{T}m_X + \mathbf{T}^2a} \mathbf{T}^2X \xrightarrow{m_X - \mathbf{T}a} \mathbf{T}X \overset{a}{\dashrightarrow} X$$

Applying the functor $\mathbf{V}$ and using that the augmentation $\mathbf{T} \to \mathbf{1}$ is a map of monads gives an expression for $|\operatorname{Bar}_\bullet(1, \mathbf{T}, X)|$ as $\ldots \xrightarrow{\epsilon_{\mathbf{T}^2X} - m_{\mathbf{T}X} + \mathbf{T}m_X - \mathbf{T}^2a} \mathbf{T}^2X \xrightarrow{\epsilon_{\mathbf{T}X} - m_X + \mathbf{T}a} \mathbf{T}X \xrightarrow{\epsilon_X - a} X$. For X endowed with trivial multiplication, the augmented simplicial chain complex $\operatorname{Bar}_\bullet(1, \mathbf{T}, |\operatorname{Bar}_\bullet(\mathbf{T}, \mathbf{T}, X)|) \dashrightarrow \operatorname{Bar}_\bullet(1, \mathbf{T}, X)$ therefore yields the following double complex (we include the augmentation on the right):

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
\ldots \mathbf{T}^5X & \xrightarrow{\mathbf{T}^2m_{\mathbf{T}X} - \mathbf{T}^3m_X + \mathbf{T}^4\epsilon_X} & \mathbf{T}^4X & \xrightarrow{\mathbf{T}^2m_X - \mathbf{T}^3\epsilon_X} & \mathbf{T}^3X & \overset{\mathbf{T}^2\epsilon_X}{\dashrightarrow} & \mathbf{T}^2X \\
\downarrow {\scriptstyle \epsilon_{\mathbf{T}^4X} - m_{\mathbf{T}^3X} + \mathbf{T}m_{\mathbf{T}^2X}} & & \downarrow {\scriptstyle \epsilon_{\mathbf{T}^3X} - m_{\mathbf{T}^2X} + \mathbf{T}m_{\mathbf{T}X}} & & \downarrow {\scriptstyle \epsilon_{\mathbf{T}^2X} - m_{\mathbf{T}X} + \mathbf{T}m_X} & & \downarrow {\scriptstyle \epsilon_{\mathbf{T}X} - m_X + \mathbf{T}\epsilon_X} \\
\ldots \mathbf{T}^4X & \xrightarrow{\mathbf{T}m_{\mathbf{T}X} - \mathbf{T}^2m_X + \mathbf{T}^3\epsilon_X} & \mathbf{T}^3X & \xrightarrow{\mathbf{T}m_X - \mathbf{T}^2\epsilon_X} & \mathbf{T}^2X & \overset{\mathbf{T}\epsilon_X}{\dashrightarrow} & \mathbf{T}X \\
\downarrow {\scriptstyle \epsilon_{\mathbf{T}^3X} - m_{\mathbf{T}^2X}} & & \downarrow {\scriptstyle \epsilon_{\mathbf{T}^2X} - m_{\mathbf{T}X}} & & \downarrow {\scriptstyle \epsilon_{\mathbf{T}X} - m_X} & & \downarrow {\scriptstyle \epsilon_X - \epsilon_X} \\
\ldots \mathbf{T}^3X & \xrightarrow{m_{\mathbf{T}X} - \mathbf{T}m_X + \mathbf{T}^2\epsilon_X} & \mathbf{T}^2X & \xrightarrow{m_X - \mathbf{T}\epsilon_X} & \mathbf{T}X & \overset{\epsilon_X}{\dashrightarrow} & X
\end{array}$$

Taking total complexes (and remembering to insert the correct signs) gives rise to the map of complexes

$$\begin{array}{ccccc}
\ldots \quad \mathbf{T}^3X \oplus \mathbf{T}^3X \oplus \mathbf{T}^3X & \longrightarrow & \mathbf{T}^2X \oplus \mathbf{T}^2X & \longrightarrow & \mathbf{T}X \\
\begin{bmatrix} 0 & 0 & \mathbf{T}^2\epsilon_X \end{bmatrix} \downarrow & & \begin{bmatrix} 0 & \mathbf{T}\epsilon_X \end{bmatrix} \downarrow & & \begin{bmatrix} \epsilon_X \end{bmatrix} \downarrow \\
\ldots \longrightarrow \mathbf{T}^2X & \xrightarrow[\epsilon_{\mathbf{T}X} - m_X + \mathbf{T}\epsilon_X]{} & \mathbf{T}X & \xrightarrow[\epsilon_X - \epsilon_X]{} & X
\end{array}$$

The n^{th} differential $d_n : \mathbf{T}^nX \to \mathbf{T}^{n-1}X$ in the lower row is given by

$$d_n = \epsilon_{\mathbf{T}^{n-1}X} - m_{\mathbf{T}^{n-2}X} + \mathbf{T}m_{\mathbf{T}^{n-3}X} + \ldots + (-1)^{n-1}\mathbf{T}^{n-2}m_X + (-1)^n\mathbf{T}^{n-1}\epsilon_X$$

In the top row, the n^{th} differential $D_n : \mathbf{T}^{n+1} \oplus \ldots \oplus \mathbf{T}^{n+1} \to \mathbf{T}^n \oplus \ldots \oplus \mathbf{T}^n$ is given by the $n \times (n+1)$-matrix

$$\begin{bmatrix} m_{\mathbf{T}^{n-1}X} - \mathbf{T}m_{\mathbf{T}^{n-2}X} + ... + (-1)^n \mathbf{T}^n \epsilon_X & \epsilon_{\mathbf{T}^n X} - m_{\mathbf{T}^{n-1}X} & 0 & ... \\ 0 & -\mathbf{T}m_{\mathbf{T}^{n-2}X} + ... + (-1)^n \mathbf{T}^n \epsilon_X & \epsilon_{\mathbf{T}^n X} - m_{\mathbf{T}^{n-1}X} + \mathbf{T}m_{\mathbf{T}^{n-2}X} & ... \\ ... & ... & ... & ... \end{bmatrix}$$

For example, D_2 is given by the matrix

$$\begin{bmatrix} m_{\mathbf{T}X} - \mathbf{T}m_X + \mathbf{T}^2\epsilon_X & \epsilon_{\mathbf{T}^2 X} - m_{\mathbf{T}X} & 0 \\ 0 & -\mathbf{T}m_X + \mathbf{T}^2\epsilon_X & \epsilon_{\mathbf{T}^2 X} - m_{\mathbf{T}X} + \mathbf{T}m_X \end{bmatrix}.$$

In order to relate the comultiplication on $|\operatorname{Bar}_\bullet(1, \mathbf{T}, 1)|$ to the Yoneda product on Tor groups, we prove:

Lemma 5.4.1. *Suppose we are given a morphism $A \xrightarrow{a} \mathbf{T}^n X$ for which*

$$d_n \circ a = (\epsilon_{\mathbf{T}^{n-1}X} - m_{\mathbf{T}^{n-2}X} + \mathbf{T}m_{\mathbf{T}^{n-3}X} + \ldots + (-1)^{n-1}\mathbf{T}^{n-2}m_X + (-1)^n \mathbf{T}^{n-1}\epsilon_X)a = 0 \qquad (\bigstar)$$

We can then consider the map: $A \xrightarrow{a} \mathbf{T}^n X \xrightarrow{\eta_{\mathbf{T}^n X} + \mathbf{T}\eta_{\mathbf{T}^{n-1}X} + \ldots + \mathbf{T}^n \eta_X} \mathbf{T}^{n+1}X \oplus \ldots \oplus \mathbf{T}^{n+1}X$.

This map provides a lift of a in the above map of chain complexes and composing it with D_n gives zero.

Proof. The first claim is clear. For the second, observe that the k^{th} component of this composite is given by

$$(-1)^k(\mathbf{T}^k m_{\mathbf{T}^{n-k-1}X})(\mathbf{T}^k \eta_{\mathbf{T}^{n-k}X})a + (-1)^{k+1}(\mathbf{T}^{k+1}m_{\mathbf{T}^{n-k-2}X})(\mathbf{T}^k\eta_{\mathbf{T}^{n-k}X})a + \ldots + (-1)^n(\mathbf{T}^n\epsilon_X)(\mathbf{T}^k\eta_{\mathbf{T}^{n-k}X})a$$
$$+(\epsilon_{\mathbf{T}^n X})(\mathbf{T}^{k+1}\eta_{\mathbf{T}^{n-k-1}X})a - (m_{\mathbf{T}^{n-1}X})(\mathbf{T}^{k+1}\eta_{\mathbf{T}^{n-k-1}X})a + \ldots + (-1)^{k-1}(\mathbf{T}^k m_{\mathbf{T}^{n-k-1}X})(\mathbf{T}^{k+1}\eta_{\mathbf{T}^{n-k-1}X})a.$$

Observe that $(\mathbf{T}^n\epsilon_X)(\mathbf{T}^k\eta_{\mathbf{T}^{n-k}X}) = (\mathbf{T}^k\eta_{n-k-1X})(\mathbf{T}^{n-1}\epsilon_X)$. We use ($\bigstar$), we transform the above sum to:

$$= (-1)^k(\mathbf{T}^k m_{\mathbf{T}^{n-k-1}X})(\mathbf{T}^k\eta_{\mathbf{T}^{n-k}X})a + \ldots + (-1)^{n-1}(\mathbf{T}^{n-1}m_X)(\mathbf{T}^k\eta_{\mathbf{T}^{n-k}X})a$$
$$-(\mathbf{T}^k\eta_{\mathbf{T}^{n-k-1}X})(\epsilon_{\mathbf{T}^{n-1}X}) + (\mathbf{T}^k\eta_{\mathbf{T}^{n-k-1}X})(m_{\mathbf{T}^{n-2}X}) + \ldots + (-1)^n(\mathbf{T}^k\eta_{\mathbf{T}^{n-k-1}X})(\mathbf{T}^{n-2}m_X)$$
$$+(\epsilon_{\mathbf{T}^n X})(\mathbf{T}^{k+1}\eta_{\mathbf{T}^{n-k-1}X})a - (m_{\mathbf{T}^{n-1}X})(\mathbf{T}^{k+1}\eta_{\mathbf{T}^{n-k-1}X})a + \ldots + (-1)^{k-1}(\mathbf{T}^k m_{\mathbf{T}^{n-k-1}X})(\mathbf{T}^{k+1}\eta_{\mathbf{T}^{n-k-1}X})a$$

By naturality of ϵ, we have $(\mathbf{T}^k\eta_{\mathbf{T}^{n-k-1}X})(\epsilon_{\mathbf{T}^{n-1}X}) = (\epsilon_{\mathbf{T}^n X})(\mathbf{T}^{k+1}\eta_{\mathbf{T}^{n-k-1}X})$. The naturality of η gives that for $j \geq k$, we have $(\mathbf{T}^k\eta_{\mathbf{T}^{n-k-1}X})(\mathbf{T}^j m_{\mathbf{T}^{n-j-2}X}) = (\mathbf{T}^{j+1}m_{\mathbf{T}^{n-j-2}X})(\mathbf{T}^k\eta_{\mathbf{T}^{n-k}X})$ and the naturality of m gives that for $j < k$, we have $(\mathbf{T}^k\eta_{\mathbf{T}^{n-k-1}X})(\mathbf{T}^j m_{\mathbf{T}^{n-j-2}X}) = (\mathbf{T}^j m_{\mathbf{T}^{n-j-1}X})(\mathbf{T}^{k+1}\eta_{\mathbf{T}^{n-k-1}X})$.

The above expression becomes:

$$= (-1)^k(\mathbf{T}^k m_{\mathbf{T}^{n-k-1}X})(\mathbf{T}^k\eta_{\mathbf{T}^{n-k}X})a + \ldots + (-1)^{n-1}(\mathbf{T}^{n-1}m_X)(\mathbf{T}^k\eta_{\mathbf{T}^{n-k}X})a$$
$$-(\epsilon_{\mathbf{T}^n X})(\mathbf{T}^{k+1}\eta_{\mathbf{T}^{n-k-1}X})a + (m_{\mathbf{T}^{n-1}X})(\mathbf{T}^{k+1}\eta_{\mathbf{T}^{n-k-1}X})a + \ldots + (-1)^{k-1}(\mathbf{T}^{k-1}m_{\mathbf{T}^{n-k}X})(\mathbf{T}^{k+1}\eta_{\mathbf{T}^{n-k-1}X}))a + \ldots$$

$+(-1)^k(\mathbf{T}^{k+1}m_{\mathbf{T}^{n-k-2}X})(\mathbf{T}^k\eta_{\mathbf{T}^{n-k}X})a+\ldots+(-1)^n(\mathbf{T}^{n-1}m_X)(\mathbf{T}^k\eta_{\mathbf{T}^{n-k}X})a+$

$+(\epsilon_{\mathbf{T}^nX})(\mathbf{T}^{k+1}\eta_{\mathbf{T}^{n-k-1}X})a-(m_{\mathbf{T}^{n-1}X})(\mathbf{T}^{k+1}\eta_{\mathbf{T}^{n-k-1}X})a+\ldots+(-1)^{k-1}(\mathbf{T}^k m_{\mathbf{T}^{n-k-1}X})(\mathbf{T}^{k+1}\eta_{\mathbf{T}^{n-k-1}X})a$

This simplifies to $(-1)^k(\mathbf{T}^k m_{\mathbf{T}^{n-k-1}X})(\mathbf{T}^k\eta_{\mathbf{T}^{n-k}X})a+(-1)^{k-1}(\mathbf{T}^k m_{\mathbf{T}^{n-k-1}X})(\mathbf{T}^{k+1}\eta_{\mathbf{T}^{n-k-1}X})a=0$. □

5.4.2 Operadic Koszul Duality

We will now establish a link between Ching's Koszul duality for operads via tree grafting as developed in [Chi05] (cf. also [Sal98]) and Lurie's ∞-categorical monadic Koszul duality from [Lur11b].

Operadic from Monadic Koszul Duality

Let $\mathbf{C}$ be either the symmetric monoidal model category $(\mathbf{Sp},\otimes,S)$ of S-modules (see [EKMM97]) with the smash product or the symmetric monoidal model category $(\mathbf{Top}_*,\wedge,S)$ of pointed compactly generated Hausdorff spaces with the smash product. Write $\mathcal{C}$ for the underlying ∞-category of $\mathbf{C}$. As in Section 4.1.2, $\mathbf{SSeq}(\mathbf{C}):=\mathrm{Fun}(\mathrm{Fin}^{\cong},\mathbf{C})$ carries the structure of a model category using the projective model structure on functors. We write $\mathbf{SSeq}(\mathbf{C})^c$ for the full subcategory spanned by all cofibrant objects.

The model category $\mathbf{SSeq}(\mathbf{C})$ carries a monoidal structure $\circ$ called the composition product and a symmetric monoidal structure $\otimes$ given by Day convolution. Write $\mathbf{1}$ for the unit of the composition product.

Since Day convolution on $\mathbf{SSeq}(\mathbf{C})$ preserves cofibrant symmetric sequences and weak equivalences between them, the underlying ∞-category $\mathrm{SSeq}(\mathcal{C})=N(\mathbf{SSeq}(\mathbf{C})^c)[\mathcal{W}^{-1}]$ inherits the structure of a symmetric monoidal ∞-category by Proposition 4.1.3.4. in [Lur14]. More formally, we obtain a coCartesian fibration $\mathrm{SSeq}(\mathcal{C})^{\otimes}\to N(\mathrm{Fin}_*)$ satisfying the conditions of Definition 2.0.0.7 in [loc.cit], where Fin_* denotes the category of finite pointed sets and pointed maps between them.

One can organise the collection of all symmetric monoidal ∞-categories and all symmetric monoidal functors between them into an ∞-category $\mathrm{Cat}_{\infty}^{\mathrm{Comm},\otimes}$, see Remark 1.3.11 in [Lur07]. We define a subcategory $\mathrm{Cat}_{\infty,\mathrm{Pr}^L}^{\mathrm{Comm},\otimes}$ whose objects are the symmetric monoidal ∞-categories $\mathcal{C}^{\otimes}\to N(\mathrm{Fin}_*)$ for which the underlying ∞-category $\mathcal{C}$ is presentable and the symmetric monoidal structure is compatible with small colimits in the sense of Definition 2.2.17 in [Lur07]. Morphisms in $\mathrm{Cat}_{\infty,\mathrm{Pr}^L}^{\mathrm{Comm},\otimes}$ are symmetric monoidal functors which preserve small colimits. By Remark 4.8.1.9 in [Lur14], there is an equivalence $\mathrm{Cat}_{\infty,\mathrm{Pr}^L}^{\mathrm{Comm},\otimes}\cong\mathrm{CAlg}(\mathrm{Pr}^L)$. The symmetric monoidal ∞-category $\mathrm{SSeq}(\mathcal{C})$ is now simply a specific object in $\mathrm{Cat}_{\infty,\mathrm{Pr}^L}^{\mathrm{Comm},\otimes}$.

Let $\mathrm{End}^{c,w,\otimes}(\mathbf{SSeq}(\mathbf{C}))^{rev}$ be the full subcategory spanned by endofunctors which preserve homotopy colimits, Day convolutions, and (weak equivalences between) cofibrant objects, considered as a monoidal category by reversing functor composition. Write $\mathbf{SSeq}(\mathbf{C})^{\mathbf{c}}$ for the category of Σ-cofibrant objects, i.e. sequences M for which $M_{\underline{n}}$ is Σ_n-cofibrant for $n \neq 1$ and with $M_{\underline{1}}$ either cofibrant or equal to S^0. We define functors

$$\mathbf{\Phi} : \mathbf{SSeq}(\mathbf{C})^{\mathbf{c}} \leftrightarrows \mathrm{End}^{c,w,\otimes}(\mathbf{SSeq}(\mathbf{C}))^{rev} : \mathbf{ev}_{\mathbf{1}_c}$$

by declaring that $\mathbf{ev}_{\mathbf{1}_c}$ evaluates on a cofibrant replacement $\mathbf{1}_c$ of the unit symmetric sequence and $\mathbf{\Phi}$ sends a symmetric sequence A to the endofunctor $\mathbf{\Phi}(A) = (- \circ A)$. Since both functors send weak equivalences to weak equivalences, we can pass to underlying ∞-categories and obtain a diagram

$$\begin{array}{ccc} N(\mathbf{SSeq}(\mathbf{C})^{\mathbf{c}}) & \underset{\mathbf{ev}_{\mathbf{1}_c}}{\overset{\mathbf{\Phi}}{\rightleftarrows}} & N(\mathrm{End}^{c,w,\otimes}(\mathbf{SSeq}(\mathbf{C})))^{rev} \\ \downarrow & & \downarrow \\ \mathrm{SSeq}(\mathcal{C}) & \underset{\mathrm{ev}_1}{\overset{\Phi}{\rightleftarrows}} & \mathrm{End}_{\mathrm{Cat}^{\mathrm{Comm},\otimes}_{\infty,\mathrm{Pr}^L,st}}(\mathrm{SSeq}(\mathcal{C}))^{rev} \end{array}$$

The two squares commute by definition. We have silently introduced Φ and ev_1 on the bottom and identified the underlying ∞-categories of $\mathbf{SSeq}(\mathbf{C})^c$ and $\mathbf{SSeq}(\mathbf{C})^{\mathbf{c}}$. The argument presented in Section 4.1.2 shows that ev_1 is an equivalence. Since $\mathrm{ev}_1 \circ \Phi \cong \mathrm{id}$, we know that Φ is an inverse. We define a monoidal structure on $\mathrm{SSeq}(\mathcal{C})$ which makes ev_1 monoidal. The functor $\mathbf{\Phi}$ is monoidal by inspection. The right vertical functor is evidently monoidal. This implies that the left vertical functor is monoidal as well.

We shall now consider composition with a given symmetric sequence *from the left.* Write $\mathrm{End}^w(\mathbf{SSeq}(\mathbf{C}))$ for the collection of endofunctors which preserve cofibrant objects and weak equivalences between them and consider the "Schur functor" $\mathbf{S} : \mathbf{SSeq}(\mathbf{C})^{\mathbf{c}} \to \mathrm{End}^w(\mathbf{SSeq}(\mathbf{C}))$ which attaches to a Σ-cofibrant symmetric sequence A the endofunctor $S_A = (A \circ -)$. On the level of ∞-categories, we obtain a diagram

$$\begin{array}{ccc} N(\mathbf{SSeq}(\mathbf{C})^{\mathbf{c}}) & \xrightarrow{\mathbf{S}} & N(\mathrm{End}^w(\mathbf{SSeq}(\mathbf{C}))) \\ \downarrow & & \downarrow \\ \mathrm{SSeq}(\mathcal{C}) & \xrightarrow{S} & \mathrm{End}(\mathrm{SSeq}(\mathcal{C})) \end{array}$$

The top horizontal map is monoidal by inspection, the right vertical map is evidently monoidal, and the left vertical map is monoidal as argued above. The lower horizontal map is clearly monoidal. Since S preserves realisations, Example 4.4.19. in [Lur11b] gives a commutative square linking the Koszul duality functors:

$$\begin{array}{ccc} \mathrm{Alg}^{aug}(\mathrm{SSeq}(\mathcal{C})) & \xrightarrow{S} & \mathrm{Alg}^{aug}(\mathrm{End}(\mathrm{SSeq}(\mathcal{C}))) \\ {\scriptstyle \mathrm{KD}}\downarrow & & {\scriptstyle \mathrm{KD}}\downarrow \\ \mathrm{coAlg}^{aug}(\mathrm{SSeq}(\mathcal{C})) & \xrightarrow{S} & \mathrm{coAlg}^{aug}(\mathrm{End}(\mathrm{SSeq}(\mathcal{C}))) \end{array}$$

Here we slightly abused notation in the labeling of the horizontal maps. We can therefore compute the left action of the Koszul dual $\mathrm{KD}(\mathcal{O})$ of an operad $\mathcal{O}$ in terms of the comonad $\mathrm{KD}(S_{\mathcal{O}})$ Koszul dual to the

“Schur monad” $S_{\mathcal{O}}$ associated to $\mathcal{O}$. The upshot of this is that we can now apply our reasoning from the beginning of this Appendix D (see p.133) to understand the comultiplication map on $\mathrm{KD}(\mathcal{O})$ explicitly.

For this, suppose $\mathbf{O} \in \mathrm{Alg}^{aug}(\mathbf{SSeq}(\mathbf{C})^c)$ is an augmented cofibrant operad. Write $\mathcal{O} \in \mathrm{Alg}^{aug}(\mathrm{SSeq}(\mathcal{C}))$ for the ∞-operad associated to $\mathbf{O}$. By our reasoning above, the augmented monad $S_{\mathcal{O}}$ on $\mathrm{SSeq}(\mathcal{C})$ is induced by the augmented monad $\mathbf{S_O} = (\mathbf{O} \circ -)$. The monad $\mathbf{S_O}$ acts topologically on the cofibrantly generated topological model category $\mathbf{SSeq}(\mathbf{C})$ and preserves cofibrant objects and weak equivalences between them (by the analogue of Proposition 4.1.6).

As explained on p.133, the work by Schwede-Shipley can therefore be used to endow the category $\mathrm{Alg}_{\mathbf{S_O}}(\mathbf{SSeq}(\mathbf{C}))$ with a model category structure whose weak equivalences and fibrations are defined pointwise, and the functor $\mathbf{SSeq}(\mathbf{C}) \xrightarrow{\mathbf{triv}} \mathrm{Alg}_{\mathbf{S_O}}(\mathbf{SSeq}(\mathbf{C}))$ induced by the augmentation admits a left adjoint $\mathbf{V}$. The monad $S_{\mathcal{O}}$ preserves geometric realisations and the comonad $h\,\mathrm{KD}(S_{\mathcal{O}}) \cong hS_{\mathrm{KD}(\mathcal{O})}$ on the homotopy category $h\,\mathrm{SSeq}(\mathcal{C})$ is given by $\mathbf{V} \circ Q \circ \mathbf{triv}$ with structure map $\mathbf{V} \circ Q \circ \mathbf{triv} \to \mathbf{V} \circ Q \circ (\mathbf{triv} \circ \mathbf{V} \circ Q) \circ \mathbf{triv}$, where Q denotes the cofibrant replacement functor (along trivial fibrations) in the model category $\mathrm{Alg}_{\mathbf{S_O}}(\mathbf{SSeq}(\mathbf{Sp}))$.

For every simplicial object $M_\bullet$ in $\mathbf{SSeq}(\mathbf{C})$, we can take the levelwise geometric realisation and define

$$|M_\bullet|_A = \int^{n \in \Delta} \Delta^n_+ \otimes (M_n)_A$$

Given a left $\mathbf{S_O}$-algebra M, we can form the augmented simplicial $\mathbf{S_O}$-algebra $\mathrm{Bar}_\bullet(\mathbf{S_O}, \mathbf{S_O}, M) \to M$. Since this diagram admits an extra degeneracy, the resulting map $|\,\mathrm{Bar}_\bullet(\mathbf{S_O}, \mathbf{S_O}, M)| \to M$ gives a weak equivalence (see for example Corollary 4.5.2. in [Rie14]). The $\mathbf{S_O}$-algebra $|\,\mathrm{Bar}_\bullet(\mathbf{S_O}, \mathbf{S_O}, M)|$ is readily seen to be cofibrant in the model structure defined by Schwede and Shipley [SS00]. By Corollary 2.12. in [GS07], we therefore again have a homotopy-unique weak equivalence $|\,\mathrm{Bar}_\bullet(\mathbf{S_O}, \mathbf{S_O}, M)| \xrightarrow{\simeq} QM$. We can therefore compute $\mathbf{V} \circ Q \circ \mathbf{triv}(M) \simeq |\,\mathrm{Bar}_\bullet(\mathbf{1}, \mathbf{S_O}, \mathbf{1})| \circ M$.

The cocomposition map for the comonad $h\,\mathrm{KD}(S_{\mathcal{O}}) \cong hS_{\mathrm{KD}(\mathcal{O})}$ is therefore equivalent to the transformation obtained by applying $\mathbf{S}_{(-)}$ to the following sequence of Σ-cofibrant symmetric sequences

$$|\,\mathrm{Bar}_\bullet(\mathbf{1}, \mathbf{O}, \mathbf{1})| \xleftarrow{\simeq} |\,\mathrm{Bar}_\bullet(\mathbf{1}, \mathbf{O}, |\,\mathrm{Bar}_\bullet(\mathbf{O}, \mathbf{O}, \mathbf{1})|)| \to |\,\mathrm{Bar}_\bullet(\mathbf{1}, \mathbf{O}, |\,\mathrm{Bar}_\bullet(\mathbf{1}, \mathbf{O}, \mathbf{1})|)|$$

This in turn implies that the comultiplication map $\mathrm{KD}(\mathcal{O}) \to \mathrm{KD}(\mathcal{O}) \circ \mathrm{KD}(\mathcal{O})$ is homotopic to the map $|\,\mathrm{Bar}_\bullet(\mathbf{1}, \mathbf{O}, \mathbf{1})| \to |\,\mathrm{Bar}_\bullet(\mathbf{1}, \mathbf{O}, \mathbf{1})| \circ |\,\mathrm{Bar}_\bullet(\mathbf{1}, \mathbf{O}, \mathbf{1})|$ constructed above. We will soon (see p.142) write down this map as explicitly as possible and compare it to Ching's construction.

Useful Notions from Ching's Work

We first recall several definitions from Ching's thesis [Chi05].

Definition 5.4.2. Given a finite set A, an A-labelled (generalised) tree consists of a finite poset T containing a unique minimal element (the "root") together with a surjection ι from A to the set of maximal elements of T (the "leaves") such that T satisfies the following additional conditions:

1. If u, v, and t are any elements of T with $u \leq t$ and $v \leq t$, then $u \leq v$ or $v \leq u$.
2. If u, t are elements in T with $u > t$ and $t \neq r$, then there exists a v with $v > t$ and $u \nleq v$.

Elements of T which are neither roots nor leaves are called *vertices*. An *edge* in a tree is an inequality $(u < v)$ such that there does not exist a t with $u < t < v$. An *incoming edge* for some vertex t is an edge of the form $(t < v)$. Given $t \in T$, we write $i(t)$ for the set of incoming edges.

There is an evident notion of isomorphism of A-labelled trees. We define a poset $\mathrm{Tree}(A)$ whose objects are isomorphism classes of A-labelled trees. We declare that $T' \leq T$ if T' can be obtained from T by a sequence of the following two elementary operations:

- Edge collapses: Given an edge $(u < v)$ with v a vertex (i.e. neither a root nor a leaf), we can form a new A-labelled tree by removing v (hence "collapsing" the edge $(u < v)$).
- Bud collapses: Given a vertex b for which all $u > b$ are leaves, we can remove all those $u > b$ and take the evident induced surjection from A to the set of maximal elements of the new tree.

Definition 5.4.3. A *weighting* on a tree $T \in \mathrm{Tree}(A)$ consists of an assignment of nonnegative numbers to every edge of T such that the "distance" from the root to any leaf is 1. We write $w(T)$ for the space of all weightings (with topology induced by realising this as a subspace of $[0,1]^{\{edges\}}$).

Definition 5.4.4. Given a reduced operad $\mathbf{O}$ in $\mathbf{Top}_*$ with a right module $\mathbf{R}$ and a left module $\mathbf{L}$ satisfying $\mathbf{L}_{\underline{0}} = \mathbf{R}_{\underline{0}} = 0$, we define $B(\mathbf{R}, \mathbf{O}, \mathbf{L})_A = \int^{T \in \mathrm{Tree}(A)} w(T)_+ \wedge (\mathbf{R}, \mathbf{P}, \mathbf{L})_A(T)$ for

$$(\mathbf{R}, \mathbf{O}, \mathbf{L})_A(T) := \mathbf{R}(i(r)) \wedge \bigwedge_{vertices\ v \in T} \mathbf{O}(i(v)) \wedge \bigwedge_{leaves\ l \in T} \mathbf{L}(\iota^{-1} l)$$

Here ι denotes the labelling. We recall Proposition 7.10 from [Chi05]:

Proposition 5.4.5. *Let $\mathbf{O}$ be a reduced operad in $\mathbf{Top}_*$ with right module $\mathbf{R}$ and left module $\mathbf{L}$ satisfying $\mathbf{L}_{\underline{0}} = \mathbf{R}_{\underline{0}} = 0$. Then there is an isomorphism of symmetric sequences $B(\mathbf{R}, \mathbf{O}, \mathbf{L}) \cong |\mathrm{Bar}_\bullet(\mathbf{R}, \mathbf{O}, \mathbf{L})|$.*

Extension to Multisimplicial Bar Constructions

Assume now that we are given reduced operads $\mathbf{O}_1, \ldots, \mathbf{O}_n \in \mathrm{Alg}(\mathbf{SSeq}(\mathbf{Top}_*))$, a right $\mathbf{O}_1$-module $\mathbf{R}$, a left $\mathbf{O}_n$-module $\mathbf{L}$, and $(\mathbf{O}_i, \mathbf{O}_{i+1})$-bimodules $\mathbf{B}_i$ with $(\mathbf{L})_{\underline{0}} = (\mathbf{R})_{\underline{0}} = (\mathbf{B}_i)_{\underline{0}} = 0$ for all i.

We can then form a multisimplicial object $\mathrm{Bar}_\bullet(\mathbf{R}, \mathbf{O}_1, \mathbf{B}_1, \ldots, \mathbf{B}_{n-1}, \mathbf{O}_n, \mathbf{L}) : (\Delta^{op})^{\times n} \to \mathbf{Top}_*$ by generalising the usual Bar construction and define its geometric realisation as the coend

$$|\,\mathrm{Bar}_\bullet(\mathbf{R}, \mathbf{O}_1, \mathbf{B}_1, \ldots, \mathbf{B}_{n-1}, \mathbf{O}_n, \mathbf{L})|_A := \int^{(i_1,\ldots,i_n)\in\Delta^{\times n}} \Big((\mathbf{R}\circ\mathbf{O}_1^{\circ i_1}\circ\mathbf{B}_1\circ\ldots\circ\mathbf{B}_{n-1}\circ\mathbf{O}_n^{\circ i_n}\circ\mathbf{L})_A \wedge \Delta_+^{i_1}\wedge\ldots\wedge\Delta_+^{i_n}\Big).$$

We will now generalise Ching's construction to this multisimplicial case:

Definition 5.4.6. Given a finite set A, an *A-labelled n-stage tree* T consists of a poset T containing a unique minimal element r (the "root") together with a map of posets $T \xrightarrow{d} [n] = \{0 < \ldots < n\}$ and a surjection ι from the set A to the set of maximal elements of T (the "leaves") such that the following conditions hold true:

1. If u, v, and t are any elements of T with $u \le t$ and $v \le t$, then $u \le v$ or $v \le u$.
2. If $u > t$ and t is not a leaf in the tree $T_{\le d(t)} := d^{-1}([0 < \ldots < d(t)])$, then there is a v with $v > t$ and $u \not\le v$.
3. $T_0 = d^{-1}(0) = \{r\}$, all leaves of T are mapped to n, and if $(u < v)$ is an edge, then $d(v) \le d(u) + 1$.

Here we used the notation $T_{\le k} = d^{-1}([0 < \ldots < k])$ and $T_k = d^{-1}(k)$.

We define a poset $\mathrm{Tree}_n(A)$ whose objects are isomorphism classes of A-labelled n-stage trees and where we declare that $T' \le T$ if T' can be obtained from T by a sequence of the following elementary moves:

- Edge collapses: Given an edge $e = (u < v)$ with v not a leaf in $T_{\le d(v)}$, we form a new A-labelled tree T/e by removing v. We rename the vertex u in T as $u \circ v$ in T/e.
- Bud collapses: Given a vertex b for which all outgoing edges $(b < u_1), \ldots, (b < u_k)$ have u_j a leaf in $T_{\le d(b)}$, we can remove all $u_i > b$ and thereby obtain a new tree $T_{//b}$. We obtain an evident surjection from A to the set of maximal elements of the new tree. We denote the new vertex by $\{u_j\}$.

Remark 5.4.7. We observe that for $n = 1$, we have $\mathrm{Tree}_1(A) = \mathrm{Tree}(A)$ and thus recover Definition 5.4.2.

Definition 5.4.8. A *weighting* on an $n-$stage tree $T \in \mathrm{Tree}(A)$ consists of an assignment of nonnegative numbers to all edges of T such that for any k, the "distance" from the root r to any leaf of $T_{\le k}$ is exactly k.

Definition 5.4.9. Given reduced operads $\mathbf{O}_1, \ldots, \mathbf{O}_n \in \mathrm{Alg}(\mathbf{SSeq}(\mathbf{Top}_*))$, a right $\mathbf{O}_1$-module $\mathbf{R}$, a left $\mathbf{O}_n$-module $\mathbf{L}$, and $(\mathbf{O}_i, \mathbf{O}_{i+1})$-bimodules $\mathbf{B}_i$ with $(\mathbf{L})_{\underline{0}} = (\mathbf{R})_{\underline{0}} = (\mathbf{B}_i)_{\underline{0}} = 0$ for all i. We define:

$$B(\mathbf{R}, \mathbf{O}_1, \mathbf{B}_1, \ldots, \mathbf{O}_n, \mathbf{L})_A = \int^{T\in\mathrm{Tree}_n(A)} w(T)_+ \wedge (\mathbf{R}, \mathbf{O}_1, \mathbf{B}_1, \ldots, \mathbf{O}_n, \mathbf{L})_A(T)$$

where $(\mathbf{R}, \mathbf{O}_1, \mathbf{B}_1, \ldots, \mathbf{O}_n, \mathbf{L})_A : \mathrm{Tree}_n(A)^{op} \to \mathbf{Top}_*$ is the functor defined on objects as

$$\mathbf{R}(i(r)) \wedge \bigwedge_{vertices\ v \in T_1} \mathbf{O}_1(i(v)) \wedge \bigwedge_{leaves\ l \in T_1} \mathbf{B}_1(i(l)) \wedge \ldots \wedge \bigwedge_{vertices\ v \in T_n} \mathbf{O}_n(i(v)) \wedge \bigwedge_{leaves\ l \in T_n} \mathbf{L}(\iota^{-1}(l))$$

Here $i(v)$ denotes the edges coming into a vertex v and ι denotes the surjection from A to the leaves of T. We define the morphism $(\mathbf{R}, \mathbf{O}_1, \mathbf{B}_1, \ldots, \mathbf{O}_n, \mathbf{L})_A(T) \to (\mathbf{R}, \mathbf{O}_1, \mathbf{B}_1, \ldots, \mathbf{O}_n, \mathbf{L})_A(T')$ corresponding to $T' \leq T$:

- If $T' = T/e \leq T$ is obtained by collapsing an edge $e = (l < v)$ (with v not a leaf in $T_{\leq d(v)}$) and l is a leaf of $T_{\leq d(l)}$, then the morphism is defined using the map $\mathbf{B}_{d(l)}(i(l)) \wedge \mathbf{O}_{d(l)+1}(i(v)) \to \mathbf{B}_{d(l)}(i(l \circ v))$ coming from the right $\mathbf{O}_{d(l)+1}$-module structure on $\mathbf{B}_{d(l)}$. Here we use the convention that $\mathbf{B}_0 = \mathbf{R}$.

- If $T' = T/e \leq T$ is obtained by collapsing an edge $e = (u < v)$ (with v not a leaf in $T_{\leq d(v)}$) and u is not a leaf of $T_{\leq d(u)}$ (and thus $d(u) = d(v) = d$), then the morphism in question is defined using the map $\mathbf{O}_d(i(u)) \wedge \mathbf{O}_d(i(v)) \to \mathbf{O}_d(i(u \circ v))$ coming from the operad structure on $\mathbf{O}_d$.

- If $T' = T//b \leq T$ for b a vertex for which all edges $(b < u_1), \ldots, (b < u_k)$ have u_j a leaf in $T_{\leq d(b)}$, define the required map using the morphism $\mathbf{O}_{d(b)}(i(b)) \wedge \mathbf{B}_{d(b)}(i(u_1)) \wedge \ldots \wedge \mathbf{B}_{d(b)}(i(u_k)) \to \mathbf{B}_{d(b)}(i(\{u_j\}))$. If $d(b) = n$, we use the convention that $\mathbf{B}_n = \mathbf{L}$ and $i(u_r) = \iota^{-1}(u_r)$.

We extend this assignment to composite morphisms to obtain a well-defined functor.

Points in $B(\mathbf{R}, \mathbf{O}_1, \mathbf{B}_1, \ldots, \mathbf{B}_{n-1}, \mathbf{O}_n, \mathbf{L})_A$ will be called "weighted decorated trees" and are given by weighted n-stage trees T together with elements in $\mathbf{O}_k(i(v))$ attached to all vertices v in T_k which *are not* leaves in $T_{\leq k}$ and elements in $\mathbf{B}_k(i(v))$ attached to all vertices v in T_k which *are* leaves in $T_{\leq k}$ (where we again use the convention that $\mathbf{B}_0 = \mathbf{R}$, $\mathbf{B}_n = \mathbf{L}$, and $i(v) = \iota^{-1}(v)$ for v a leaf of a tree T). The coend then identifies trees for which some edges have length zero with smaller trees and uses the module and operad structures to modify decorations accordingly. Proposition 7.10 in [Chi05] has the following generalisation:

Proposition 5.4.10. *Given reduced operads* $\mathbf{O}_1, \ldots, \mathbf{O}_n \in \mathrm{Alg}(\mathbf{SSeq}(\mathbf{Top}_*))$*, a right* $\mathbf{O}_1$*-module* $\mathbf{R}$*, a left* $\mathbf{O}_n$*-module* $\mathbf{L}$*, and* $(\mathbf{O}_i, \mathbf{O}_{i+1})$*-bimodules* $\mathbf{B}_i$ *with* $(\mathbf{L})_{\underline{0}} = (\mathbf{R})_{\underline{0}} = (\mathbf{B}_i)_{\underline{0}} = 0$ *for all* i*. Then*

$$B(\mathbf{R}, \mathbf{O}_1, \mathbf{B}_1, \ldots, \mathbf{B}_{n-1}, \mathbf{O}_n, \mathbf{L}) \cong |\operatorname{Bar}_\bullet(\mathbf{R}, \mathbf{O}_1, \mathbf{B}_1, \ldots, \mathbf{B}_{n-1}, \mathbf{O}_n, \mathbf{L})|.$$

Explicit Comonadic Comultiplication

Let $\mathbf{O}$ be a reduced operad in $\mathbf{Top}_*$ with its canonical augmentation $\mathbf{O} \to \mathbf{1}$. We now use the language of n-stage trees (Definition 5.4.6) to give an explicit description of the following sequence (cf. p.139):

$$|\operatorname{Bar}_\bullet(\mathbf{1}, \mathbf{O}, \mathbf{1})| \xleftarrow{\simeq} |\operatorname{Bar}_\bullet(\mathbf{1}, \mathbf{O}, |\operatorname{Bar}_\bullet(\mathbf{O}, \mathbf{O}, \mathbf{1})|)| \to |\operatorname{Bar}_\bullet(\mathbf{1}, \mathbf{O}, |\operatorname{Bar}_\bullet(\mathbf{1}, \mathbf{O}, \mathbf{1})|)|$$

Using Proposition 5.4.10 for the top vertical arrows, we observe the following commutative diagram

$$\begin{array}{ccccc}
B(\mathbf{1},\mathbf{O},\mathbf{1})_A & \xleftarrow{\alpha} & B(\mathbf{1},\mathbf{O},\mathbf{O},\mathbf{O},\mathbf{1})_A & \xrightarrow{\delta} & B(\mathbf{1},\mathbf{O},\mathbf{1},\mathbf{O},\mathbf{1})_A \\
\cong\downarrow & & \cong\downarrow & & \cong\downarrow \\
|\operatorname{Bar}_\bullet(\mathbf{1},\mathbf{O},\mathbf{1})|_A & \xleftarrow{\simeq} & |\operatorname{Bar}_\bullet(\mathbf{1},\mathbf{O},\mathbf{O},\mathbf{O},\mathbf{1})|_A & \longrightarrow & |\operatorname{Bar}_\bullet(\mathbf{1},\mathbf{O},\mathbf{1},\mathbf{O},\mathbf{1})|_A \\
\cong\downarrow & & \cong\downarrow & & \cong\downarrow \\
|\operatorname{Bar}_\bullet(\mathbf{1},\mathbf{O},\mathbf{1})|_A & \xleftarrow{\simeq} & |\operatorname{Bar}_\bullet(\mathbf{1},\mathbf{O},|\operatorname{Bar}_\bullet(\mathbf{O},\mathbf{O},\mathbf{1})|)|_A & \rightarrow & |\operatorname{Bar}_\bullet(\mathbf{1},\mathbf{O},|\operatorname{Bar}_\bullet(\mathbf{1},\mathbf{O},\mathbf{1})|)|_A
\end{array}$$

The map α sends a weighted decorated A-labelled 2-stage tree T to the basepoint whenever $T_{\leq 1}$ has fewer than $|A|$ leaves or some v with $d_v = 2$ is decorated by the basepoint. If neither of these cases holds, then the map α sends T to the weighted decorated tree $T_{\leq 1}$, considered as an element of $B(\mathbf{1},\mathbf{O},\mathbf{1})_A$ in the natural way. Here we use that in the second case, every leaf in $T_{\leq 1}$ must be decorated by an element of $\mathbf{O}(1) = S^0 = \mathbf{1}(1)$. The map δ on the right of the above diagram uses the augmentation $\mathbf{O} \to \mathbf{1}$. The right column is equivalent to $B(\mathbf{1},\mathbf{O},\mathbf{1}) \circ B(\mathbf{1},\mathbf{O},\mathbf{1}) \xrightarrow{\cong} |\operatorname{Bar}_\bullet(\mathbf{1},\mathbf{O},\mathbf{1})| \circ |\operatorname{Bar}_\bullet(\mathbf{1},\mathbf{O},\mathbf{1})| \xrightarrow{\cong} |\operatorname{Bar}_\bullet(\mathbf{1},\mathbf{O},\mathbf{1})| \circ |\operatorname{Bar}_\bullet(\mathbf{1},\mathbf{O},\mathbf{1})|$. We illustrate these maps for A a set of size 2 by the following picture:

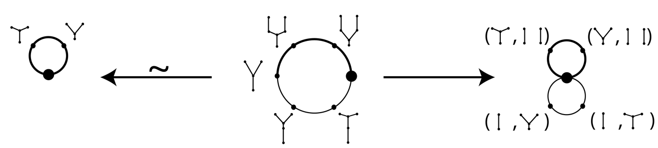


Figure 10: The monadic comultiplication on spaces of trees at weight 2.

The left map collapses the bottom semicircle to a point, the right map pinches. We will now produce a homotopy inverse β to α. For our illustrated case, β will wrap the left circle once round the middle circle.

In general, assume that we are given element $T \in B(\mathbf{1},\mathbf{O},\mathbf{1})_A$, i.e. an A-labelled weighted tree decorated with elements in $\mathbf{O}(i(v))$ attached to all internal vertices v. Write d_v for the "distance" from the root r to v. We produce an element in $B(\mathbf{1},\mathbf{O},\mathbf{O},\mathbf{O},\mathbf{1})_A$ by "scaling by 2 and slicing in the middle". More precisely:

1. For any edge $(u < w)$ in T with $d_u < \frac{1}{2} < d_w$, we add a new vertex v to the poset T satisfying $u < v < w$ and all implied relations. We assign the weight $\frac{1}{2} - d_u$ to the edge $(u < v)$. The edge $(v < w)$ receives the weight $d_w - \frac{1}{2}$. Observe that $d_v = \frac{1}{2}$ by construction.

2. Decorate the new vertices by the non-basepoint in $\mathbf{O}(1) = S^0$, thus obtaining a weighted decorated tree T'.

3. We consider the function $T' \to [0 < 1 < 2]$ which sends a point v to k minimal with $d_v \leq \frac{k}{2}$

4. We multiply all weights by 2.

We now want to produce a homotopy between the identity and $\alpha \circ \beta$. For every $t \in (0,1]$, we define a self-map γ_t on $B(\mathbf{1},\mathbf{O},\mathbf{1})_A$ by asserting that its effect on a weighted decorated A-labelled tree T is given by executing the following steps:

1. If there are fewer than $|A|$ (leaf) edges $(u < w)$ with $d_u < t \leq d_w$ or some leaf is decorated by the basepoint, send T to the basepoint. Otherwise add a vertex v for each such edge satisfying $u < v < w$ (with all implied relations). Give $(u < v)$ and $(v < w)$ the weights $t - d_u$ and $d_w - t$, respectively.

2. Remove all leaves w and rescale the weights on the remaining edges by a factor of $\frac{1}{t}$, hence obtaining a weighted tree T' carrying a naturally induced A-labeling.

3. Decorate each new leaf $v \in T'$ by the non-basepoint of $S^0 = \mathbf{O}(1)$. The decorations of the remaining vertices are the same as in T.

We observe that $\gamma_{\frac{1}{2}} = \alpha \circ \beta$ and $\gamma_1 = \mathrm{id}$. Since γ_t varies continuously in t, we have defined a homotopy $\mathrm{id} \simeq \alpha \circ \beta$ and therefore conclude that our map β is a homotopy inverse to α. Since this map varies naturally in A, we have produced maps of reduced symmetric sequences

$$B(\mathbf{1},\mathbf{O},\mathbf{1}) \xrightarrow{\beta} B(\mathbf{1},\mathbf{O},\mathbf{O},\mathbf{O},\mathbf{1}) \xrightarrow{\delta} B(\mathbf{1},\mathbf{O},\mathbf{1},\mathbf{O},\mathbf{1}) \cong B(\mathbf{1},\mathbf{O},\mathbf{1}) \circ B(\mathbf{1},\mathbf{O},\mathbf{1})$$

We can now apply the functor Σ^∞ to $\mathbf{O}$ to obtain an augmented operad in S-modules, which we shall denote by the same name. Suppose $\tilde{\mathbf{O}} \to \mathbf{O}$ is a cofibrant replacement of this augmented operad in S-modules.

We then have a homotopy commutative diagram of symmetric sequences

$$\begin{array}{ccccc}
|\mathrm{Bar}_\bullet(\mathbf{1},\tilde{\mathbf{O}},\mathbf{1})| & \xrightarrow{\tilde{\simeq}} & |\mathrm{Bar}_\bullet(\mathbf{1},\tilde{\mathbf{O}},|\mathrm{Bar}_\bullet(\tilde{\mathbf{O}},\tilde{\mathbf{O}},\mathbf{1})|) & \longrightarrow |\mathrm{Bar}_\bullet(\mathbf{1},\tilde{\mathbf{O}},|\mathrm{Bar}_\bullet(\mathbf{1},\tilde{\mathbf{O}},\mathbf{1})|) \cong & |\mathrm{Bar}_\bullet(\mathbf{1},\tilde{\mathbf{O}},\mathbf{1})| \circ |\mathrm{Bar}_\bullet(\mathbf{1},\tilde{\mathbf{O}},\mathbf{1})| \\
\simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\
|\mathrm{Bar}_\bullet(\mathbf{1},\mathbf{O},\mathbf{1})| & \xrightarrow{\tilde{\simeq}} & |\mathrm{Bar}_\bullet(\mathbf{1},\mathbf{O},|\mathrm{Bar}_\bullet(\mathbf{O},\mathbf{O},\mathbf{1})|) & \longrightarrow |\mathrm{Bar}_\bullet(\mathbf{1},\mathbf{O},|\mathrm{Bar}_\bullet(\mathbf{1},\mathbf{O},\mathbf{1})|) \cong & |\mathrm{Bar}_\bullet(\mathbf{1},\mathbf{O},\mathbf{1})| \circ |\mathrm{Bar}_\bullet(\mathbf{1},\mathbf{O},\mathbf{1})| \\
\simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\
B(\mathbf{1},\mathbf{O},\mathbf{1}) & \xrightarrow{\beta} & B(\mathbf{1},\mathbf{O},\mathbf{O},\mathbf{O},\mathbf{1}) & \xrightarrow{\delta} B(\mathbf{1},\mathbf{O},\mathbf{1},\mathbf{O},\mathbf{1}) \cong & B(\mathbf{1},\mathbf{O},\mathbf{1}) \circ B(\mathbf{1},\mathbf{O},\mathbf{1})
\end{array}$$

Writing $\mathcal{O} \in \mathrm{Alg}^{aug}(\mathrm{SSeq}(Sp))$ for the augmented ∞-operad induced by $\tilde{\mathbf{O}}$, our argument on p.139 shows that if X is a symmetric sequence, then the map $\mathrm{KD}(\mathcal{O}) \circ X \to \mathrm{KD}(\mathcal{O}) \circ \mathrm{KD}(\mathcal{O}) \circ X$ is equivalent to

$$B(\mathbf{1},\mathbf{O},\mathbf{1}) \circ_h X \xrightarrow{(\delta\circ\beta)\circ_h \mathrm{id}} (B(\mathbf{1},\mathbf{O},\mathbf{1}) \circ B(\mathbf{1},\mathbf{O},\mathbf{1})) \circ_h X$$

Here $\circ_h$ denotes a version of $\circ$ which uses homotopy orbits instead of strict orbits. We will review this and other related constructions in our Section "Variants of the Composition Product" on p.146.

Ching's Comultiplication

Ching's thesis endows the Spanier-Whitehead dual of the bar construction of any reduced operad $\mathbf{O}$ in pointed spaces with a new operad structure. For $X \in \mathbf{Sp}$ an S-module, we write $\mathbb{D}(X) = F(X, S)$ for the mapping spectrum to the (non-cofibrant) sphere spectrum. The functor $\mathbb{D}$ gives a contravariant endofunctor of $\mathbf{Sp}$ called Spanier-Whitehead duality and it sends colimits to limits.

There is a symmetric monoidal colimit-preserving functor $\Sigma^\infty : \mathbf{Top}_* \to \mathbf{Sp}$ constructed in Section II.1.1 of [EKMM97] (*not* preserving cofibrant objects). We will usually suppress Σ^∞ from our notation. We will now recall Ching's construction of the Koszul dual of a reduced operad $\mathbf{O} \in \mathrm{Alg}^{red}(\mathbf{SSeq}(\mathbf{Top}_*))$:

Definition 5.4.11. The Koszul dual operad $K(\mathbf{O}) = \mathbb{D}(B(\mathbf{1}, \mathbf{O}, \mathbf{1})) \in \mathrm{Alg}^{red}(\mathbf{SSeq}(\mathbf{Sp}))$ is a reduced operad in spectra whose underlying symmetric sequence is given by $K(\mathbf{O})_A = \mathbb{D}(B(\mathbf{1}, \mathbf{O}, \mathbf{1})_A)$. The structure map $K(\mathbf{O}) \circ K(\mathbf{O}) \longrightarrow K(\mathbf{O})$ is defined as follows:

1. For each partition $A = A_1 \coprod \ldots \coprod A_r$ of a set A into nonempty subsets A_i, we define a map of spaces $B(\mathbf{1}, \mathbf{O}, \mathbf{1})_A \to B(\mathbf{1}, \mathbf{O}, \mathbf{1})_{\underline{r}} \wedge B(\mathbf{1}, \mathbf{O}, \mathbf{1})_{A_1} \wedge \ldots \wedge B(\mathbf{1}, \mathbf{O}, \mathbf{1})_{A_r}$ by the following rule. Let $T \in B(\mathbf{1}, \mathbf{O}, \mathbf{1})_A$ be an A-labelled weighted decorated tree.

   - Assume that there is a tree $S \in \mathrm{Tree}(\{1, \ldots, r\})$ whose $\{1, \ldots, r\}$-labeling is bijective and trees $U_j \in \mathrm{Tree}(A_j)$ whose roots have only one incoming edge such that T is obtained from $S, U_1, \ldots, U_r$ by identifying the root edge of U_i with the leaf edge of S labelled by i for each $i = 1, \ldots, r$.

     In this case, we define a weighting on S by first restricting the weighting from T and then adjusting the weight of the leaf edges of S so that the distance from root to all leaves of S is exactly 1. For each i, we define a weighting on U_i by first restricting the weights from T and then rescaling all weights by a common factor λ_i to make the root-leaf distance on U_i equal to 1. Care must be taken of degenerate cases, and we refer the interested reader to Definition 4.16 in [Chi05] for details. We decorate the vertices of S and $U_1, \ldots, U_r$ by restricting the decorations from T. The leaves of S and the root of each U_i are decorated by the non-basepoint of S^0. The resulting tuple of labelled weighted decorated trees $(S, U_1, \ldots, U_r)$ is then the image of T.
   - If T can not be built by trees in the way described above, we send T to the base point.

2. Applying $\mathbb{D}(\Sigma^\infty -)$ and summing over all decompositions of A, we obtain a Σ_r-equivariant map

$$\bigoplus_{A = A_1 \coprod \ldots \coprod A_r} K(\mathbf{O})_{\underline{r}} \otimes K(\mathbf{O})_{A_1} \otimes \ldots \otimes K(\mathbf{O})_{A_r} \longrightarrow K(\mathbf{O})_A$$

3. Passing to orbits and summing over all r gives the desired structure map.

Variants of the Composition Product

We follow [Chi12] and introduce variants of the *composition product* from p.80. Given $n \geq 2$ and a finite set J, write $\mathrm{Fin}[n]_{J/}$ for the category of sequences $(J \xrightarrow{f_{n-1}} I_{n-1} \xrightarrow{f_{n-2}} \dots \xrightarrow{f_1} I_1)$ of finite sets with diagrams

$$\begin{array}{ccccccc} & & I_{n-1} & \xrightarrow{f_{n-2}} & \dots & \xrightarrow{f_1} & I_1 \\ & \nearrow & \simeq\downarrow & & & & \simeq\downarrow \\ J & \longrightarrow & I'_{n-1} & \xrightarrow{f_{n-2}} & \dots & \xrightarrow{f_1} & I'_1 \end{array}$$

as morphisms. Here the vertical maps are bijections. Given symmetric sequences $M^1, \dots, M^n$ and a finite set J, we define a functor $(M^1, \dots, M^n) : \mathrm{Fin}[n]_{J/} \to \mathbf{Sp}$ by sending the chain $(J \xrightarrow{f_{n-1}} I_{n-1} \xrightarrow{f_{n-2}} \dots \xrightarrow{f_1} I_1)$ to $M^1(I_1) \wedge \bigwedge_{i \in I_1} M^2(f_1^{-1}(i)) \wedge \dots \wedge \bigwedge_{i \in I_{n-1}} M^{n-1}(f_{n-1}^{-1}(i))$. We define:

$$(M^1 \circ \dots \circ M^n)(J) = \operatorname*{colim}_{f \in \mathrm{Fin}[n]_{J/}} (M^1, \dots, M^n)(f) \simeq \coprod_{r_1, \dots, r_{n-1}} \left(\coprod M^1_{\underline{r}_1} \wedge \bigwedge_{i=1}^{r_1} M^2_{J_{1,i}} \wedge \dots \wedge \bigwedge_{i=1}^{r_{n-1}} M^n_{J_{n,i}} \right)_{\Sigma_{r_1} \times \dots \times \Sigma_{r_{n-1}}}$$

$$(M^1 \circ_h \dots \circ_h M^n)(J) = \operatorname*{hocolim}_{f \in \mathrm{Fin}[n]_{J/}} (M^1, \dots, M^n)(f) \simeq \coprod_{r_1, \dots, r_{n-1}} \left(\coprod M^1_{\underline{r}_1} \wedge \bigwedge_{i=1}^{r_1} M^2_{J_{1,i}} \wedge \dots \wedge \bigwedge_{i=1}^{r_{n-1}} M^n_{J_{n,i}} \right)_{h\Sigma_{r_1} \times \dots \times \Sigma_{r_{n-1}}}$$

$$(M^1 \,\hat{\circ}\, \dots \,\hat{\circ}\, M^n)(J) = \lim_{f \in \mathrm{Fin}[n]_{J/}} (M^1, \dots, M^n)(f) \simeq \prod_{r_1, \dots, r_{n-1}} \left(\prod M^1_{\underline{r}_1} \wedge \bigwedge_{i=1}^{r_1} M^2_{J_{1,i}} \wedge \dots \wedge \bigwedge_{i=1}^{r_{n-1}} M^n_{J_{n,i}} \right)^{\Sigma_{r_1} \times \dots \times \Sigma_{r_{n-1}}}$$

$$(M^1 \,\hat{\circ}^h \dots \,\hat{\circ}^h M^n)(J) = \operatorname*{holim}_{f \in \mathrm{Fin}[n]_{J/}} (M^1, \dots, M^n)(f) \simeq \prod_{r_1, \dots, r_{n-1}} \left(\prod M^1_{\underline{r}_1} \wedge \bigwedge_{i=1}^{r_1} M^2_{J_{1,i}} \wedge \dots \wedge \bigwedge_{i=1}^{r_{n-1}} M^n_{J_{n,i}} \right)^{h\Sigma_{r_1} \times \dots \times \Sigma_{r_{n-1}}}$$

The sums and products corresponding to $r_1, \dots, r_{n-1}$ range over all chains of ordered partitions of length $n-1$ of the set J into (possibly empty) sets, where the classes of the k^{th} partition are labelled as $J_{k,1}, \dots, J_{k,r_k}$. Relying on the norm in the context of S-modules (cf. [Kle01], [Rog05]), we obtain natural transformations

$$(M^1 \circ \dots \circ M^n) \longleftarrow (M^1 \circ_h \dots \circ_h M^n) \longrightarrow (M^1 \,\hat{\circ}^h \dots \,\hat{\circ}^h M^n) \longleftarrow (M^1 \,\hat{\circ}\, \dots \,\hat{\circ}\, M^n).$$

Proposition 5.4.12. *If $M^1, \dots, M^n$ are Σ-cofibrant, then the following morphisms are weak equivalences:*

$$(M^1 \circ \dots \circ M^n) \longleftarrow (M^1 \circ_h \dots \circ_h M^n) \qquad (\mathbb{D}M^1 \hat{\circ} \, \dots \, \hat{\circ}\, \mathbb{D}M^n) \longrightarrow (\mathbb{D}M^1 \hat{\circ}^h \dots \, \hat{\circ}^h \mathbb{D}M^n)$$

Proof. A straightforward generalisation of the proof of Lemma 9.20 in [AC11] shows that the $\Sigma_{r_1} \times \dots \times \Sigma_{r_{n-1}}$-spectra appearing in the definition of $(M^1 \circ \dots \circ M^n)(J)$ are (projectively) cofibrant whenever some $r_i \neq 1$. Since taking coinvariants is a left Quillen functor, the first claim then follows by Theorem 24.3.1. of [MS06]. The second claim follows from the first by Spanier-Whitehead duality. □

Given n, ℓ, r, Ching [Chi12] defines maps $(M^1 \circ \ldots \circ M^n) \longrightarrow (M^1 \circ \ldots \circ M^\ell \circ (M^{\ell+1} \circ \ldots \circ M^{\ell+r}) \circ \ldots \circ M^n)$. By suitably cofibrantly replacing diagrams in Ching's construction, we also obtain a natural morphism

$$(M^1 \circ_h \ldots \circ_h M^n) \longrightarrow (M^1 \circ_h \ldots \circ_h M^\ell \circ_h (M^{\ell+1} \circ_h \ldots \circ_h M^{\ell+r}) \circ_h M^{\ell+r+1} \circ_h \ldots \circ_h M^n)$$

Dually, we also have natural arrows

$$(M^1 \mathbin{\hat{\circ}} \ldots \mathbin{\hat{\circ}} M^n) \longleftarrow (M^1 \mathbin{\hat{\circ}} \ldots \mathbin{\hat{\circ}} M^\ell \mathbin{\hat{\circ}} (M^{\ell+1} \mathbin{\hat{\circ}} \ldots \circ M^{\ell+r}) \mathbin{\hat{\circ}} M^{\ell+r+1} \circ \ldots \circ M^n)$$

$$(M^1 \mathbin{\hat{\circ}^h} \ldots \mathbin{\hat{\circ}^h} M^n) \longleftarrow (M^1 \mathbin{\hat{\circ}^h} \ldots \mathbin{\hat{\circ}^h} M^\ell \mathbin{\hat{\circ}^h} (M^{\ell+1} \mathbin{\hat{\circ}^h} \ldots \mathbin{\hat{\circ}^h} M^{\ell+r}) \mathbin{\hat{\circ}^h} M^{\ell+r+1} \mathbin{\hat{\circ}^h} \ldots \mathbin{\hat{\circ}^h} M^n)$$

Lemma 5.4.13. *We have a homotopy commutative diagram*

$$\begin{array}{ccc}
(M^1 \circ \ldots \circ M^n) & \longrightarrow & (M^1 \circ \ldots \circ M^\ell \circ (M^{\ell+1} \circ \ldots \circ M^{\ell+r}) \circ M^{\ell+r+1} \circ \ldots \circ M^n) \\
\uparrow & \textcircled{A} & \uparrow \\
(M^1 \circ_h \ldots \circ_h M^n) & \longrightarrow & (M^1 \circ_h \ldots \circ_h M^\ell \circ_h (M^{\ell+1} \circ_h \ldots \circ_h M^{\ell+r}) \circ_h M^{\ell+r+1} \circ_h \ldots \circ_h M^n) \\
\downarrow & \textcircled{B} & \downarrow \\
(M^1 \mathbin{\hat{\circ}^h} \ldots \mathbin{\hat{\circ}^h} M^n) & \longleftarrow & (M^1 \mathbin{\hat{\circ}^h} \ldots \mathbin{\hat{\circ}^h} M^\ell \mathbin{\hat{\circ}^h} (M^{\ell+1} \mathbin{\hat{\circ}^h} \ldots \mathbin{\hat{\circ}^h} M^{\ell+r}) \mathbin{\hat{\circ}^h} M^{\ell+r+1} \mathbin{\hat{\circ}^h} \ldots \mathbin{\hat{\circ}^h} M^n) \\
\uparrow & \textcircled{C} & \uparrow \\
(M^1 \mathbin{\hat{\circ}} \ldots \mathbin{\hat{\circ}^h} M^n) & \longleftarrow & (M^1 \mathbin{\hat{\circ}} \ldots \mathbin{\hat{\circ}} M^\ell \mathbin{\hat{\circ}} (M^{\ell+1} \mathbin{\hat{\circ}} \ldots \mathbin{\hat{\circ}} M^{\ell+r}) \mathbin{\hat{\circ}} M^{\ell+r+1} \mathbin{\hat{\circ}} \ldots \mathbin{\hat{\circ}} M^n)
\end{array}$$

Proof. Squares Ⓐ and Ⓒ follow from the comparison between homotopy (co)limits and ordinary (co)limits. We now focus on the middle square. In order to not get lost in overly cumbersome notation, we only prove the case necessary for our specific purposes: assume that $n = 3$, that $M^1_{\underline{0}}$ and $M^2_{\underline{0}}$ are both zero and that M^3 is given by the spectrum X concentrated in degree 0. For $\ell = 0$, the claim turns out to be obvious.

So let $\ell = 1$ and fix positive integers $j_1, \ldots, j_t$ and $k_1, \ldots, k_t$. Write $r = \sum k_i j_i$. We observe that the asserted claim is equivalent to the commutativity of the following square for all such sequences of numbers:

$$\begin{array}{ccc}
(M^1_{\underline{r}} \otimes (M^2_{\underline{j_1}} \otimes X^{\otimes j_1})^{\otimes k_1} \otimes \ldots)_{h\Sigma_{j_1} \wr \Sigma_{k_1} \times \ldots \times \Sigma_{j_t} \wr \Sigma_{k_t}} & \longrightarrow & (M^1_{\underline{r}} \underset{h\Sigma_{k_1} \times \ldots \times \Sigma_{k_t}}{\otimes} (M^2_{\underline{j_1}} \underset{h\Sigma_{j_1}}{\otimes} X^{\otimes j_1})^{\otimes k_1} \otimes \ldots) \\
\downarrow & & \downarrow \\
(M^1_{\underline{r}} \otimes (M^2_{\underline{j_1}} \otimes X^{\otimes j_1})^{\otimes k_1} \otimes \ldots)^{h\Sigma_{j_1} \wr \Sigma_{k_1} \times \ldots \times \Sigma_{j_t} \wr \Sigma_{k_t}} & \longleftarrow & (M^1_{\underline{r}} \overset{h\Sigma_{k_1} \times \ldots \times \Sigma_{k_t}}{\otimes} (M^2_{\underline{j_1}} \overset{h\Sigma_{j_1}}{\otimes} X^{\otimes j_1})^{\otimes k_1} \otimes \ldots)
\end{array}$$

Since the composition $X \to X_{hG} \to X^{hG} \to X$ is simply $\sum_{g \in G} g$, it suffices to check that the composite

$$\begin{array}{ccc}
 & & \left(M^1_{\underline{r}} \otimes (M^2_{\underline{j_1}} \otimes X^{\otimes j_1})^{\otimes k_1} \otimes \ldots\right) \\
 & \swarrow & \uparrow \\
\left(M^1_{\underline{r}} \underset{h\Sigma_{k_1} \times \ldots \times \Sigma_{k_t}}{\otimes} (M^2_{\underline{j_1}} \underset{h\Sigma_{j_1}}{\otimes} X^{\otimes j_1})^{\otimes k_1} \otimes \ldots\right) & \longrightarrow & \left(M^1_{\underline{r}} \overset{h\Sigma_{k_1} \times \ldots \times \Sigma_{k_t}}{\otimes} (M^2_{\underline{j_1}} \overset{h\Sigma_{j_1}}{\otimes} X^{\otimes j_1})^{\otimes k_1} \otimes \ldots\right)
\end{array}$$

is equal to $\left(\sum_{g \in \Sigma_{j_1} \wr \Sigma_{k_1} \times \ldots \times \Sigma_{j_t} \wr \Sigma_{k_t}} g \right)$. But this follows since naturality of the norm implies that the map in question is equivalent to

$$\left(\sum_{(h_1, \ldots, h_\ell) \in \Sigma_{k_1} \times \ldots \times \Sigma_{k_t}} h_1 \otimes \ldots \otimes h_\ell \right) \circ \left(\left(\sum_{g^{1,1} \in \Sigma_{j_1}} g^{1,1} \otimes \ldots \otimes \sum_{g^{1,k_1} \in \Sigma_{j_1}} g^{1,k_1} \right) \otimes \ldots \otimes \left(\sum_{g^{\ell,1} \in \Sigma_{j_t}} g^{\ell,1} \otimes \ldots \otimes \sum_{g^{\ell,k_t} \in \Sigma_{j_t}} g^{\ell,k_t} \right) \right)$$

□

If N is a symmetric sequence with vanishing constant term, then composition products with N as a second factor simplify since we only need to consider decompositions into nonempty finite sets when computing the composition product. Given a finite set J, the group Σ_r acts freely on the *finite* set $S_r(J)$ of ordered decompositions of J into *nonempty* finite subsets. This implies:

Lemma 5.4.14. *If $N \in \mathrm{SSeq}(\mathbf{Sp})$ has vanishing constant term, then the following diagram commutes:*

$$\begin{array}{ccc}
\coprod_{r=0}^{\infty} \left(\coprod_{[J_1 \coprod \ldots \coprod J_r] \in S_r/\Sigma_r} M_{\underline{r}} \otimes N_{J_1} \otimes \ldots \otimes N_{J_r} \right) & \longrightarrow & \prod_{r=0}^{\infty} \left(\prod_{[J_1 \coprod \ldots \coprod J_r] \in S_r/\Sigma_r} M_{\underline{r}} \otimes N_{J_1} \otimes \ldots \otimes N_{J_r} \right) \\
\simeq \uparrow & & \simeq \downarrow \\
(M \circ N)(J) & \longrightarrow & (M \ \hat{\circ}\ N)(J) \\
\simeq \uparrow & & \simeq \downarrow \\
(M \circ_h N)(J) & \longrightarrow & (M \ \hat{\circ}^h N)(J)
\end{array}$$

Proof. This follows from Theorem 5.2.5 in [Rog05] which identifies the norm on freely induced G-spectra. □

Preliminary Observations on S-modules

Proposition 5.4.15. *Let $f : X \to Y$ be a weak equivalence of spectra which are either cofibrant or suspension spectra of well-pointed spaces. Then $\mathbb{D}(f) : \mathbb{D}(Y) \to \mathbb{D}(X)$ is a weak equivalence of S-modules.*

Proof. Let $S_c \to S$ be a cofibrant replacement of the sphere spectrum. Since S-modules satisfy the "very strong unit axiom" (see Example 6 in [Mur15]), we know that $S_c \otimes X \to S_c \otimes Y$ is a weak equivalence. The spectra $S_c \otimes X$ and $S_c \otimes Y$ are cofibrant. This follows either by the axioms of a monoidal model category or by combining Theorem VII.4.6. in [EKMM97] with Proposition 10.3.18 of [MS06]. Since S is fibrant, this implies that $F(S_c \otimes X, S) \to F(S_c \otimes Y, S)$ is a weak equivalence.We then consider the following diagram:

$$\begin{array}{ccc}
F(X, S) & \longrightarrow & F(Y, S) \\
\simeq \downarrow & & \simeq \downarrow \\
F(S_c, F(X, S)) & \longrightarrow & F(S_c, F(Y, S)) \\
\cong \downarrow & & \cong \downarrow \\
F(S_c \otimes X, S) & \xrightarrow{\simeq} & F(S_c \otimes Y, S)
\end{array}$$

The top vertical arrows are weak equivalences by Lemma 4.2.7 of [Hov99]. This implies the claim. $\square$

Proposition 5.4.16. *Let $X_1, \ldots, X_n$ be well-pointed spaces. Assume we are given a weak equivalence $Y_i \to X_i$ with Y_i a cofibrant S-module for all i (we denote the suspension spectrum of X_i by the same name). Then the natural map $Y_1 \otimes \ldots \otimes Y_n \to X_1 \otimes \ldots \otimes X_n$ is a weak equivalence.*

Proof. For each i, let $\widetilde{X}_i \to X_i$ be a trivial fibration with cofibrant domain. Since **Sp** satisfies the very strong monoid axiom in the sense of [Mur15], the map $(S_c \otimes X_1) \otimes \ldots \otimes (S_c \otimes X_n) \to X_1 \otimes \ldots \otimes X_n$ is a weak equivalence. For each i, the spectrum $S_c \otimes X_i$ is cofibrant by Theorem VII.4.6. in [EKMM97] and Proposition 10.3.18 of [MS06]. By Corollary 2.12 in [GS07], there are homotopy-unique weak equivalences $S_c \otimes X_i \xrightarrow{\simeq} \widetilde{X}_i \xleftarrow{\simeq} Y_i$ over X_i for all i. We therefore obtain a diagram

$$\begin{array}{ccc}
(S_c \otimes X_1) \otimes \ldots \otimes (S_c \otimes X_n) & \xrightarrow{\simeq} & \\
\simeq \downarrow & & \\
\widetilde{X}_1 \otimes \ldots \otimes \widetilde{X}_n & \longrightarrow & X_1 \otimes \ldots \otimes X_n \\
\simeq \uparrow & & \\
Y_1 \otimes \ldots \otimes Y_n & \nearrow &
\end{array}$$

The vertical maps are weak quivalence as they are obtained by smashing weak equivalences between cofibrant objects. The claim now follows by the "2-out-of-3"-property. $\square$

We will now refine Proposition 8.5. in [AC11] to the situation relevant to us:

Proposition 5.4.17. *Let $\mathbf{O}$ be a reduced operad in pointed spaces and assume $\widetilde{\mathbf{O}} \to \mathbf{O}$ is a cofibrant replacement of the corresponding reduced operad in spectra (denoted by the same symbol).*

Then the map $|B_\bullet(\mathbf{1}, \widetilde{\mathbf{O}}, \mathbf{1})| \to |B_\bullet(\mathbf{1}, \mathbf{O}, \mathbf{1})|$ is a weak equivalence of symmetric sequences.

Proof. For any n, k, the map $B_k(\mathbf{1}, \widetilde{\mathbf{O}}, \mathbf{1})(n) \to B_k(\mathbf{1}, \mathbf{O}, \mathbf{1})(n)$ is the coproduct of maps of the form $\widetilde{\mathbf{O}}_{k_1} \otimes \ldots \otimes \widetilde{\mathbf{O}}_{k_n} \to \mathbf{O}_{k_1} \otimes \ldots \otimes \mathbf{O}_{k_n}$. These maps are equivalences by Lemma 5.4.16 since every term of a cofibrant operad is either a cofibrant spectrum or equal to S^0. Theorem 24.3.1. of [MS06] then implies that the morphism $B_\bullet(\mathbf{1}, \widetilde{\mathbf{O}}, \mathbf{1})(n) \to B_\bullet(\mathbf{1}, \mathbf{O}, \mathbf{1})(n)$ is a levelwise weak equivalence of simplicial S-modules. For any nonnegative integer t, the latching map $\operatorname{colim}_{t \twoheadrightarrow s} \ (B_s(\mathbf{1}, \mathbf{O}, \mathbf{1})(n)) \to B_t(\mathbf{1}, \mathbf{O}, \mathbf{1})(n)$ can be written as the inclusion of a spectrum X into a coproduct $X \coprod Y$. Such an inclusion satisfies the homotopy extension property and we therefore deduce that $B_\bullet(\mathbf{1}, \mathbf{O}, \mathbf{1})(n)$ is a proper simplicial S-module in the sense of Definition X.2.1 in [EKMM97]. A similar argument establishes that $B_\bullet(\mathbf{1}, \widetilde{\mathbf{O}}, \mathbf{1})$ is a proper simplicial S-module. By Theorem X.2.4 (ii) of [EKMM97], this implies that the induced map on realisations is a weak equivalence. $\square$

Remark 5.4.18. We warn the reader that there is a clash of notation: For us, the terms "fibrations" and "cofibrations" refer to the respective distinguished classes of maps in the model category of S-modules. In [EKMM97] and [MS06], these maps are called "q-fibrations" and "q-cofibrations".

The Link

We can now articulate the compatibility between Ching's comultiplication map defined on p.145 and the comultiplication map on p.142 coming from Lurie's Koszul duality .

Proposition 5.4.19. *Let* $\mathbf{O}$ *be a reduced operad in* $\mathrm{Alg}^{aug}(\mathbf{SSeq}(\mathbf{Top}_*))$*. The following diagram of symmetric sequences in* $\mathbf{Sp}$ *commutes up to homotopy:*

$$\begin{array}{ccccccccc} K(\mathbf{O}) \circ K(\mathbf{O}) & & \longrightarrow & & & & & & K(\mathbf{O}) \\ \downarrow \simeq & & & & & & & & \| \\ \mathbb{D}(B(\mathbf{1},\mathbf{O},\mathbf{1})) \,\hat{\circ}\, \mathbb{D}(B(\mathbf{1},\mathbf{O},\mathbf{1})) & \xrightarrow{\cong} & \mathbb{D}(B(\mathbf{1},\mathbf{O},\mathbf{1},\mathbf{O},\mathbf{1})) & \xrightarrow{\mathbb{D}(\delta)} & \mathbb{D}(B(\mathbf{1},\mathbf{O},\mathbf{O},\mathbf{O},\mathbf{1})) & \xrightarrow[\simeq]{\mathbb{D}(\beta)} & \mathbb{D}(B(\mathbf{1},\mathbf{O},\mathbf{1})) \end{array}$$

The upper horizontal map is Ching's map from p.145. The maps β *and* δ *are defined in the section "Explicit Comonadic Comultiplication" starting on p. 142.*

Proof. Fix a finite set A and an ordered decomposition $(A_1, \dots, A_r) \in S_r$ into nonempty subsets. Consider

$$B(\mathbf{1},\mathbf{O},\mathbf{1})_A \xrightarrow{\delta\circ\beta} B(\mathbf{1},\mathbf{O},\mathbf{1},\mathbf{O},\mathbf{1})_A \xrightarrow{\epsilon_{(A_1,\dots,A_r)}} B(\mathbf{1},\mathbf{O},\mathbf{1})_{\underline{r}} \wedge B(\mathbf{1},\mathbf{O},\mathbf{1})_{A_1} \wedge \dots \wedge B(\mathbf{1},\mathbf{O},\mathbf{1})_{A_r}$$

Here $\delta \circ \beta$ sends an A-labelled weighted decorated tree T to the weighted decorated 2-stage tree obtained by first scaling the weights of T by 2, then introducing an additional vertex with distance 1 from the root on every edge which "crosses the middle line", decorating the "new" vertices "on the middle line" by the non-basepoint in S^0, and finally decorating the "old" vertices v on the middle line by applying $\mathbf{O}(i(v)) \to \mathbf{1}(i(v))$. The second map $\epsilon_{(A_1,\dots,A_r)}$ identifies the quotient $(\coprod_{\sigma\in\Sigma_r} B(\mathbf{1},\mathbf{O},\mathbf{1})_{\underline{r}}\wedge B(\mathbf{1},\mathbf{O},\mathbf{1})_{A_{\sigma(1)}}\wedge\dots\wedge B(\mathbf{1},\mathbf{O},\mathbf{1})_{A_{\sigma(r)}})_{\Sigma_r}$ sitting inside $B(\mathbf{1},\mathbf{O},\mathbf{1},\mathbf{O},\mathbf{1}) \cong B(\mathbf{1},\mathbf{O},\mathbf{1}) \circ B(\mathbf{1},\mathbf{O},\mathbf{1})$ with the right hand side and projects the rest off to zero. The composite map thus sends an A-labelled weighted decorated tree T to the basepoint unless the "middle line" crosses precisely r edges, no vertices, and the partition of A obtained by identifying points if they lie over the same "crossed edge" agrees with $A = A_1 \coprod \dots \coprod A_r$. If this happens, then the composite map "cuts the tree in the middle", multiplies the weights of the resulting $(r + 1)$ trees by 2, and thereby obtains a point in $B(\mathbf{1},\mathbf{O},\mathbf{1})_{\underline{r}} \wedge B(\mathbf{1},\mathbf{O},\mathbf{1})_{A_1} \wedge \dots \wedge B(\mathbf{1},\mathbf{O},\mathbf{1})_{A_r}$.
The 'down-right" map in the above square can be constructed by applying $\mathbb{D}(\Sigma^\infty-)$ to $\epsilon_{(A_1,\dots,A_r)} \circ \delta \circ \beta$, summing over all $(A_1, \dots, A_r) \in S_r$, dividing out by Σ_r, and finally summing over all r.

We also have another natural map $\gamma : B(\mathbf{1},\mathbf{O},\mathbf{1})_A \xrightarrow{\gamma} B(\mathbf{1},\mathbf{O},\mathbf{1})_{\underline{r}} \wedge B(\mathbf{1},\mathbf{O},\mathbf{1})_{A_1} \wedge \ldots \wedge B(\mathbf{1},\mathbf{O},\mathbf{1})_{A_r}$ given by Ching's construction. The map γ sends an A-labelled weighted decorated tree T on the left to the basepoint unless it is grafted along the partition $A = A_1 \coprod \ldots \coprod A_r$. If this is indeed the case, the map γ first ungrafts T into an $\{1,\ldots,r\}$-labelled tree S and A_i-labelled trees U_i with induced decorations and then weights S by rescaling the leaf weights and each U_i by rescaling all weights by a constant factor λ_i. If we apply $\mathbb{D}(\Sigma^\infty -)$ to this map, sum over all $(A_1,\ldots,A_r) \in S_r$, divide out by Σ_r, and finally sum over all $r \geq 0$, we obtain precisely the structure map $K(\mathbf{O}) \circ K(\mathbf{O}) \to K(\mathbf{O})$ given by Ching's construction.

The Homotopy. It suffices to construct a homotopy H from $\epsilon_{(A_1,\ldots,A_r)} \circ \delta \circ \beta$ to γ for each $(A_1,\ldots,A_r) \in S_r$ which interacts well with the Σ_r-action. Assume that $T \in \mathrm{Tree}(A)$ is an A-labelled tree which can be obtained by grafting along the given partition $A = A_1 \coprod \ldots \coprod A_r$. Given a weight w on T, we write a_i for the vertex of T corresponding to A_i and define $d_i = d_i(w)$ to be the distance from the root r of T to a_i and $s_i = s_i(w)$ to be the length of the unique edge *under* a_i. Let $S \subset w(T)$ be the collection of weights w with $d_i(w) = 1$ and $s_i(w) = 0$ for some i. For $t \in [0,1]$, we define a function $\kappa_t = \kappa_t(T) : (w(T) - S) \to w(T)$ as follows:

- If $(u < v)$ is an edge with $v < a_i$ for all i, we define $\kappa_t(w)(u < v) = \max\left(1 - t, \frac{1}{2}\right) \cdot w(u < v)$.Coming
- If $(u < v)$ is an edge with $a_i \leq u$, we set $\kappa_t(w)(u < v) = \left(\frac{1-(d_i - s_i)(1-t) - \max(t - \frac{1}{2}, 0)}{1 - d_i + s_i}\right) \cdot w(u < v)$.
- For an edge $(u \leq a_i)$, we set $\kappa_t(w)(u < a_i) = \begin{cases} \left(\frac{1-(d_i-s_i)(1-t)}{1-d_i+s_i}\right) \cdot w(u < v) & \text{if } t \leq \frac{1}{2} \\ \frac{s_i + d_i}{2} + (1 - d_i)\left(t - \frac{1}{2(1-d_i+s_i)}\right) & \text{if } t \geq \frac{1}{2} \end{cases}$

We observe that $\kappa_t(w)$ is indeed a valid weight on T and depends continuously on t.

At a first glance, the function κ_t seems to suffer from serious defects: It is undefined for weights in S. Even when it is defined, it does not necessarily send trees for which some edge e has length zero to trees with the same property (the case $|A| = 2$ illustrated above is instructive).

Nonetheless, we can use the map κ_t to continuously modify the map $F := \epsilon_{(A_1,\ldots,A_r)} \circ \delta \circ \beta$ from above. We begin by considering the following composite map $\overline{H}_t$:

$$(w(T) - S)_+ \wedge (\mathbf{1},\mathbf{O},\mathbf{1})_A \xrightarrow{\kappa_t \wedge \mathrm{id}} w(T)_+ \wedge (\mathbf{1},\mathbf{O},\mathbf{1})_A \xrightarrow{F} B(\mathbf{1},\mathbf{O},\mathbf{1})_{\underline{r}} \wedge B(\mathbf{1},\mathbf{O},\mathbf{1})_{A_1} \wedge \ldots \wedge B(\mathbf{1},\mathbf{O},\mathbf{1})_{A_r}$$

Write $E(T) \subset w(T)$ for the subspace of $w(T)$ consisting of all weights such that "the horizontal line" of distance $\frac{1}{2}$ from the root cuts through precisely r edges (and no vertices) whose upper points $a_1, \ldots a_r$

partition A as $A_1 \coprod \ldots \coprod A_r$. Let $D(T)$ be the complement of $E(T)$. Given a weight w in $w(T)$, we write

$$d(w, D(T)) = \min_{w' \in D(T)} \left(\max_{(u<v) \text{ an edge in T}} |w(u<v) - w'(u<v)| \right)$$

We now consider the projection $p : w(T) \to \overline{w(T)} := w(T)/D(T)$. If $w_1, w_2, \ldots$ is any sequence in $w(T)$ such that $d(w_i, D(T)) \to 0$, then $p(w_1), p(w_2), \ldots$ converges to the collapsed point in the quotient topology.

We now observe that the map $w(T)_+ \wedge (\mathbf{1},\mathbf{O},\mathbf{1})_A \to B(\mathbf{1},\mathbf{O},\mathbf{1})_{\underline{r}} \wedge B(\mathbf{1},\mathbf{O},\mathbf{1})_{A_1} \wedge \ldots \wedge B(\mathbf{1},\mathbf{O},\mathbf{1})_{A_r}$ sends $D(T) \wedge (\mathbf{1},\mathbf{O},\mathbf{1})_A$ to the basepoint.

We therefore obtain a factorisation of $\overline{H}_t(T)$ as

$$\begin{array}{ccc}
(w(T) - S)_+ \wedge (\mathbf{1},\mathbf{O},\mathbf{1})_A & & \\
\downarrow{\scriptstyle \kappa_t \wedge \mathrm{id}} & \searrow & \\
\overline{w(T)}_+ \wedge (\mathbf{1},\mathbf{O},\mathbf{1})_A & \longrightarrow & B(\mathbf{1},\mathbf{O},\mathbf{1})_{\underline{r}} \wedge B(\mathbf{1},\mathbf{O},\mathbf{1})_{A_1} \wedge \ldots \wedge B(\mathbf{1},\mathbf{O},\mathbf{1})_{A_r}
\end{array}$$

We can extend $\kappa_t : (w(T) - S) \longrightarrow \overline{w(T)}$ to $w(T)$ by setting $\kappa_t(p)$ equal to the collapsed point for all $p \in S$. This extended map is continuous as if w_n is any sequence of weights in $w(T) - S$ converging to a point w in S, then $d(w_n, D(T))$ is eventually zero. We denote the extension of $\overline{H}_t(T)$ to $w(T)_+ \wedge (\mathbf{1},\mathbf{O},\mathbf{1})_A$ by $H_t(T)$.

We can now define a map $\coprod_{T \in \mathrm{Tree}(A)} w(T)_+ \wedge (\mathbf{1},\mathbf{O},\mathbf{1})_A(T) \to B(\mathbf{1},\mathbf{O},\mathbf{1})_{\underline{r}} \wedge B(\mathbf{1},\mathbf{O},\mathbf{1})_{A_1} \wedge \ldots \wedge B(\mathbf{1},\mathbf{O},\mathbf{1})_{A_r}$ by first collapsing the summands with T not grafted along the partition $A = A_1 \coprod \ldots \coprod A_r$ to the basepoint and then using H_t on the summands corresponding to suitably grafted trees T.

We claim that this map indeed descends to the coend $B(\mathbf{1},\mathbf{O},\mathbf{1})_A$. For this, suppose T is obtained by grafting along the partition $A_1 \coprod \ldots \coprod A_r$ and write $a_1, \ldots, a_r$ for the points in T corresponding to the various sets. Assume $T/e \leq T$ is obtained from T by collapsing an edge $e = (u < v)$ in T. Any weight w on T/e gives rise to a weight $\tilde{w}$ on T with $\tilde{w}(e) = 0$. We need to show that the following square commutes:

$$\begin{array}{ccc}
w(T/e)_+ \wedge (\mathbf{1},\mathbf{O},\mathbf{1})_A(T) & \longrightarrow & w(T)_+ \wedge (\mathbf{1},\mathbf{O},\mathbf{1})_A(T) \\
\downarrow & & \downarrow \\
w(T/e)_+ \wedge (\mathbf{1},\mathbf{O},\mathbf{1})_A(T/e) & \longrightarrow & B(\mathbf{1},\mathbf{O},\mathbf{1})_{\underline{r}} \wedge B(\mathbf{1},\mathbf{O},\mathbf{1})_{A_1} \wedge \ldots \wedge B(\mathbf{1},\mathbf{O},\mathbf{1})_{A_r}
\end{array}$$

- If $a_i \leq u$ or $v < a_i$ for some i, then we observe that $\kappa_t(T)(\tilde{w})$ and $\kappa_t(T/e)(w)$ agree on all edges since the values of d_i and s_i do not change when we collapse e. Hence the square commutes.

- If our collapsed edge e ends at a point $v = a_i$, then for each $t \in [0,1]$, either the weight $\kappa_t(T)(\tilde{w})$ on e is zero or the distance between the root r and a_i in the weight $\kappa_t(T)(\tilde{w})$ is at most $\frac{1}{2}$. This implies that H_t maps $(T, \tilde{w})$ to the basepoint of the space on the bottom right of the above square. The tree T/e is no longer obtained by grafting along the partition $A_1 \coprod \ldots \coprod A_r$ and thus also maps to the basepoint.

A similar argument shows that H_t interacts well with bud collapses and covers the remaining cases. We have thus defined the desired homotopy from $H_0 = \epsilon_{(A_1,\ldots,A_r)} \circ \delta \circ \beta$ to $H_1 = \gamma$. $\square$

This argument is the non-formal input to the following result:

Corollary 5.4.20. *Let* $\mathbf{O}$ *be a reduced operad in* $\mathbf{Top}_*$ *and* $\widetilde{\mathbf{O}} \to \mathbf{O}$ *a cofibrant replacement of the corresponding reduced operad in* $\mathbf{Sp}$*. Let* $\widetilde{K(\mathbf{O})} \to K(\mathbf{O})$ *be a cofibrant replacement of Ching's reduced operad* $K(\mathbf{O})$*. Write* $\mathcal{O}, K(\mathcal{O}) \in \mathrm{Alg}^{aug}(\mathrm{SSeq}(Sp))$ *for the operads induced by* $\widetilde{\mathbf{O}}$ *and* $\widetilde{K(\mathbf{O})}$*, respectively.*

Then there is a natural transformation $\nu_X : K(\mathcal{O}) \circ \mathbb{D}X \to \mathbb{D}(\mathrm{KD}(\mathcal{O}) \circ X)$ *of endofunctors of* Sp *and the following diagram commutes up to homotopy:*

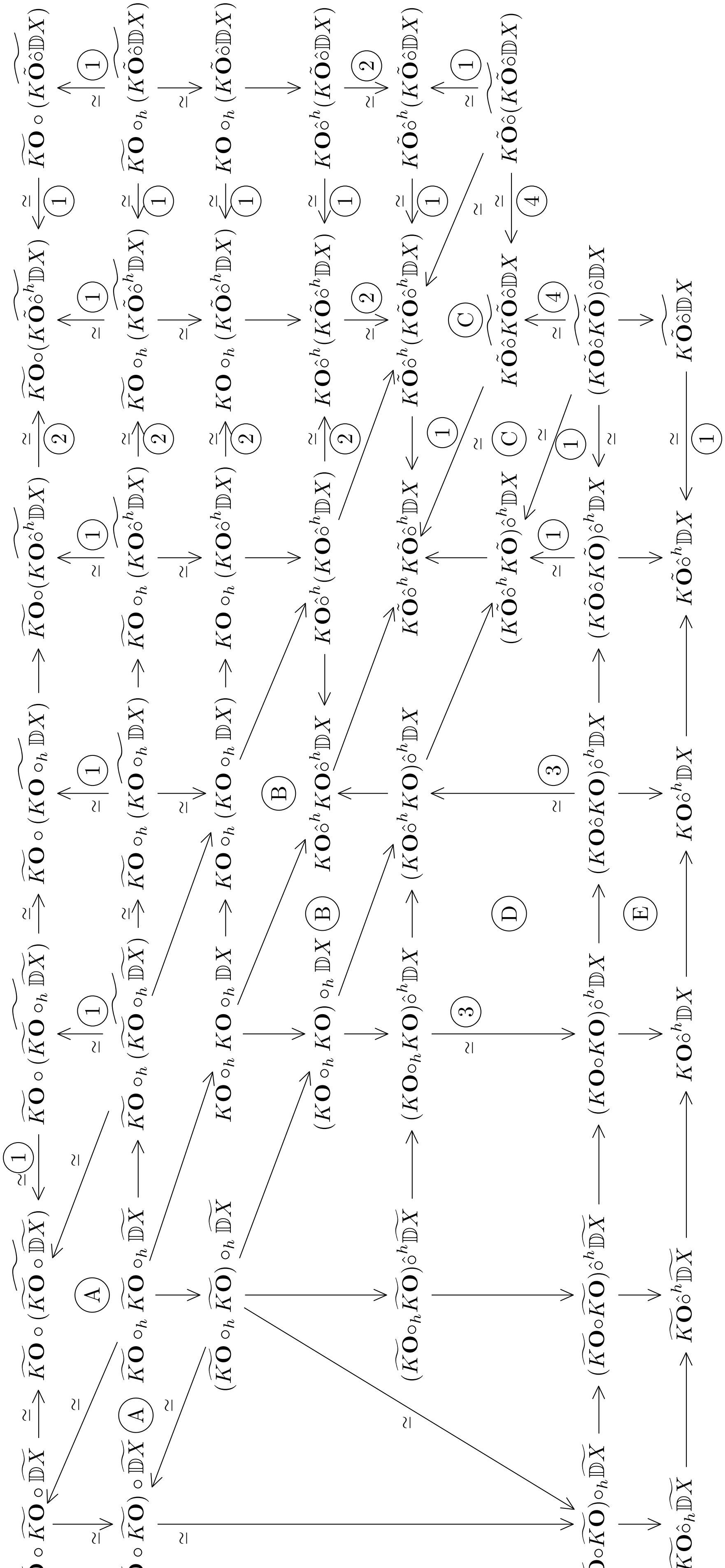
$$\begin{array}{ccc}
K(\mathcal{O}) \circ K(\mathcal{O}) \circ \mathbb{D}X & \longrightarrow & K(\mathcal{O}) \circ \mathbb{D}X \\
\downarrow{\scriptstyle K(\mathcal{O})\circ\nu_X} & & \\
K(\mathcal{O}) \circ \mathbb{D}(\mathrm{KD}(\mathcal{O}) \circ X) & & \downarrow{\scriptstyle \nu_X} \\
\downarrow{\scriptstyle \nu_{\mathrm{KD}(\mathcal{O})\circ X}} & & \\
\mathbb{D}(\mathrm{KD}(\mathcal{O}) \circ \mathrm{KD}(\mathcal{O}) \circ X) & \longrightarrow & \mathbb{D}(\mathrm{KD}(\mathcal{O}) \circ X)
\end{array}$$

Proof. We write $\widetilde{(-)}$ for the cofibrant replacement functor on symmetric sequences. In this proof, the symbol $K(\mathbf{O})$ denotes the symmetric sequence $\mathbb{D}(B(\mathbf{1}, \mathbf{O}, \mathbf{1}))$. It can be endowed with Ching's operad structure $K(\mathbf{O}) \circ K(\mathbf{O}) \to K(\mathbf{O})$ from p.145 and with the morphism $K(\mathbf{O}) \hat{\circ} K(\mathbf{O}) \to K(\mathbf{O})$ obtained by applying $\mathbb{D}$ to the map $B(\mathbf{1}, \mathbf{O}, \mathbf{1}) \xleftarrow{\simeq} B(\mathbf{1}, \mathbf{O}, B(\mathbf{O}, \mathbf{O}, \mathbf{1})) \to B(\mathbf{1}, \mathbf{O}, \mathbf{1}) \circ B(\mathbf{1}, \mathbf{O}, \mathbf{1})$ from p.142. For any cofibrant S-module X (thought of as a symmetric sequence), there is a morphism

$$\widetilde{K(\mathbf{O})} \circ \widetilde{\mathbb{D}(X)} \xleftarrow{\simeq} \widetilde{K(\mathbf{O})} \circ_h \widetilde{\mathbb{D}(X)} \xrightarrow{\simeq} K(\mathbf{O}) \circ_h \mathbb{D}(X) \to K(\mathbf{O}) \hat{\circ}^h \mathbb{D}(X) \xrightarrow{\simeq} K(\tilde{\mathbf{O}}) \hat{\circ}^h \mathbb{D}(X) \xleftarrow{\simeq} \widetilde{K(\tilde{\mathbf{O}}) \circ X}$$

The claim follows from the diagram on the following p.154. The squares Ⓐ, Ⓑ, and Ⓒ commute by Lemma 5.4.13. The square Ⓔ commutes by Proposition 5.4.19 – this is the non-formal component of this proof. Square Ⓓ commutes by Lemma 5.4.14, and all other square commute for obvious reasons.

We argue that the arrows labelled with ‘ $\simeq$ ‘ are indeed weak equivalences. We use that for any Σ-cofibrant symmetric sequence A, the functor $A \circ (-)$ preserves weak equivalences between cofibrant symmetric sequences and that the functors $(-) \circ_h (-)$, $(-) \hat{\circ}^h (-)$ send weak equivalences between pairs to weak equivalences. The arrows labelled by ① are weak equivalences by Proposition 5.4.12. The arrows ② are weak equivalences by Proposition 5.4.15 and Proposition 5.4.17, and the corresponding claim for the arrows decorated by ③ follows by Lemma 5.4.14. The arrows ④ are equivalences because $\circ$ is associative. $\square$



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