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Bridson, Martin R. (4-OX); **Haefliger, André** (CH-GENV-SM)

★**Metric spaces of non-positive curvature.**

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FEATURED REVIEW.

This book constitutes an excellent introduction to the theory of metric spaces of non-positive curvature. The bases of the theory as well as the most important recent developments are treated. All the results are proved in full generality with complete details, and the most interesting examples are given.

The basic notion involved is that of a geodesic metric space with curvature bounded above. A. D. Aleksandrov, who laid the foundations of the theory, gave several equivalent definitions of this property, and we recall here one of these definitions. We note that the modern terminology used here has been popularized by Gromov, whose work has recently revived the subject. Let X be a metric space. A geodesic segment in X is a distance-preserving map from a compact interval of $\mathbf{R}$ into X. The space X is said to be geodesic if the distance between any two points is equal to the length of a geodesic segment joining them. A geodesic triangle in X is a triple of points $x, y, z \in X$ together with three geodesic segments joining them pairwise. We shall denote these segments by $[x, y]$, $[y, z]$ and $[z, x]$ and we shall call them the sides of Δ. (Note that there is an abuse of notation here, since a geodesic segment $[x, y]$ is not uniquely determined by its endpoints x and y.) Let $\Delta = [x, y] \cup [y, z] \cup [z, x]$ be a geodesic triangle in X. For each real number κ, let M^2_κ be the unique complete simply connected 2-dimensional Riemannian manifold of constant curvature κ. (Thus, for $\kappa = 0$, M^2_κ is the Euclidean plane, for $\kappa < 0$, M^2_κ is the hyperbolic plane with its metric scaled by $1/\sqrt{-\kappa}$, and for $\kappa > 0$, M^2_κ is the unit sphere with its canonical metric scaled by $1/\sqrt{\kappa}$.) A comparison triangle for Δ in M^2_κ is a geodesic triangle $\overline{\Delta}$ in M^2_κ together with a map f from the disjoint union of the sides of Δ onto the union of the sides of $\overline{\Delta}$, which is consistent on the vertices of Δ and which is distance-preserving on each side of Δ. The map f is called a comparison map for Δ. For each $\kappa \leq 0$, such a pair $(\overline{\Delta}, f)$ exists for any triangle Δ in X, and for $\kappa > 0$, it exists provided the diameter of the set $\{x, y, z\}$ in X is not too large. Whenever it exists, the pair $(\overline{\Delta}, f)$ is unique up to post-composition by an isometry of M^2_κ. The basic definition is now the following. A metric space X is said to satisfy the

CAT(κ) inequality (or, more briefly, X is a CAT(κ) space) if X is a geodesic metric space such that for any geodesic triangle Δ in X and for any comparison triangle $\overline{\Delta}$ and comparison map f, if p and q are two arbitrary points on the disjoint union of the sides of Δ, we have $\operatorname{dist}(p,q) \leq \operatorname{dist}(f(p), f(q))$, where in the left-hand side one measures the distance between the images of p and q in X. (Note that the consideration of p and q being on the disjoint union of the sides of Δ and not in X is made necessary by the fact that there may be identifications between the images of the geodesic segments in X.) We note that the terminology CAT is due to Gromov, who introduced it in honor of Cartan, Aleksandrov and Toponogov.

The work of Aleksandrov goes back to the 1950s. He defined a metric space X to have curvature bounded by κ if X is a local CAT(κ) space, that is, if each point in X has a neighborhood which, equipped with the induced metric, is a CAT(κ) space. A complete Riemannian manifold with all its sectional curvatures bounded by κ is an example of a CAT(κ) space, but the theory has a much broader interest than the study of Riemannian manifolds. Some of the main non-Riemannian examples are graphs, polyhedral complexes, Bruhat-Tits buildings, and there are many others.

Let us note that Aleksandrov wrote about 150 papers and books on the subject (with several co-authors). A selection of his papers is being translated and edited, and the first volume appeared as Vol. 4 of the series Classics of Soviet Mathematics [Selected works. Part I, Translated from the Russian by P. S. V. Naidu, Gordon and Breach, Amsterdam, 1996; MR 2000a:01035].

We note by the way that H. Busemann worked independently on a slightly different theory of curvature in metric spaces, based on the convexity of the distance function, and in fact his first paper on the subject is prior to the work of Aleksandrov [see H. Busemann, Acta Math. 80 (1948), 259–310; MR 10, 623g].

More recently, M. Gromov has revived the interest in Aleksandrov's theory, in particular in his lectures on manifolds of non-negative curvature, published in 1985 [W. Ballmann, M. L. Gromov and V. Schroeder, Manifolds of nonpositive curvature, Progr. Math., 61, Birkhauser Boston, Boston, MA, 1985; MR 87h:53050], which concerns mainly Riemannian manifolds but where everything is based on the comparison inequalities, and in his work "Hyperbolic groups" [in Essays in group theory, 75–263, Springer, New York, 1987; MR 89e:20070].

The book under review is divided into three parts.

In Part 1, the authors present the basic geometric notions associated to geodesic metric spaces. In particular, they develop the notions of length of a curve, angle and length space. They study in detail the model spaces M^2_κ and more generally the n-dimensional spaces M^n_κ, with their isometry groups. They also consider infinite-dimensional examples. They explain some basic operations on spaces such as products, κ-cones, gluings, joins and limits. These operations had already been studied by Aleksandrov and his school. The authors apply these results to study properties of spaces constructed as polyhedral M^n_κ-complexes. These spaces had already been studied by the first author in his thesis. In the last chapter of Part 1, the authors consider group actions on metric and topological spaces. A properly discontinuous cocompact action of a group by a group of isometries on a simply connected length space provides naturally a finite presentation of the group. The group itself is then regarded as a metric space, and the questions considered include quasi-isometric invariants, growth and rigidity.

Part 2 concerns CAT(κ) spaces. The authors investigate in particular the question of how these spaces behave with respect to the operations studied in Part 1. They consider then the question of the existence of flat polygons and flat strips in CAT(0) spaces. This is based again on ideas of Aleksandrov, who noticed that a nontrivial equality in the definition of a geodesic triangle in a CAT(0) space implies that the triangle spans an isometrically embedded Euclidean triangle in the space.

The most important features of non-positively curved metric spaces are probably the local implies global properties. A famous example is the generalized Cartan-Hadamard theorem. One version of this theorem says the following. Let X be a complete connected metric space. Then, (1) if the metric on X is locally convex (that is, if the distance function locally satisfies Busemann's convexity condition), then the universal cover, $\tilde{X}$, equipped with the induced metric, is globally convex (which implies in particular that there is, up to parametrization, a unique geodesic segment joining each pair of points), and (2) if X is of curvature $\leq \kappa$, with $\kappa \leq 0$, then $\tilde{X}$ (equipped with the induced length metric) is a CAT(κ) space.

The result had been stated (in a slightly different form) by Gromov. The outline of the proof which is given in the book under review is due to S. Alexander and R. L. Bishop, who gave a proof of the result under the additional hypothesis that $\tilde{X}$ is a geodesic metric space.

The authors then study M^n_κ-polyhedral complexes of bounded curvature. They establish a result due to Gromov which gives a necessary and sufficient condition on the existence of a CAT(0) metric on a polyhedral complex in terms of the structure of the link of vertices. An important ingredient is a theorem of Berestovskii, which states that if Y is a geodesic metric space, then the Euclidean cone C_0Y is a CAT(0) space if and only if Y is a CAT(1) space. The authors treat in detail the case of 2-dimensional complexes, which is by far the most interesting case.

The authors proceed in Part 2 with the study of groups acting isometrically on CAT(0) spaces. In this context, they prove results which generalize theorems of Gromoll-Wolf and Lawson-Yau on fundamental groups of compact non-positively curved Riemannian manifolds. The authors then study the geometry of the boundary at infinity of a CAT(0) space X, in particular the Tits metric on the boundary, an object which encapsulates the geometry of subspaces of X. By studying the boundary equipped with this metric, one can determine how X splits as a product of spaces. The discussion contains a special section on symmetric spaces.

In Part 3, the authors consider several more specialized topics related to spaces of non-positive curvature. They start with a chapter on Gromov δ-hyperbolic spaces, including the basic theory of quasi-geodesics, divergence of geodesics, area and isoperimetric inequalities, the boundary at infinity and the visual metrics on that boundary. Part 3 also contains a chapter on the various decision problems for isometry groups of CAT(0) spaces and for hyperbolic groups (word, conjugacy and isomorphism problems). In relation with these problems, the authors study cone-types and growth questions for finitely-generated groups. Then they consider quasi-convexity, translation length, free subgroups, the Rips complex for hyperbolic groups, the theory of semi-hyperbolic groups, the structure of subgroups, amalgamation, HNN extensions and residual finiteness.

Part 3 contains finally a detailed exposition of the theory of complexes of groups, a theory which had been developed by the second author and which is a generalization of the Bass-Serre theory of graphs of groups. Complexes of groups are natural and useful tools for the study of group actions on simply-connected polyhedral complexes in terms of local data on the quotient spaces. The authors prove a basic result stating that a non-positively curved complex of groups is developable. This result is treated in the broad context of groupoids of isometries.

In conclusion, the book under review assembles several most interesting results in a very active subject which otherwise were spread over various articles and books. This book is written very carefully, and it should be useful at different levels as a textbook for a course on geometry or on geometric group theory, with no special background needed. At the same time, this book constitutes an invaluable reference for actual research in the field.

Athanase Papadopoulos (F-STRAS-IM)