Non-positive curvature and complexity for finitely presented groups

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ICM Madrid, 24 August 2006.

NPC and complexity for f.p. groups

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Finitely Presented Groups

geometry

The 2 Strands of

The universe of finite

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Non-Positiv Curvature

local curvature, CAT(0) NPC groups

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Dehn functions Isoperimetric spectra

Drawing in and Reaching out

Outline

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Finitely presented groups

$$\Gamma \cong \langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle \equiv \mathcal{P}$$

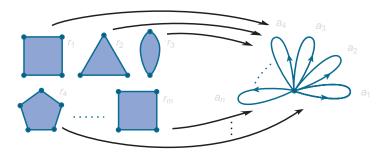


Figure: The standard 2-complex K(P)

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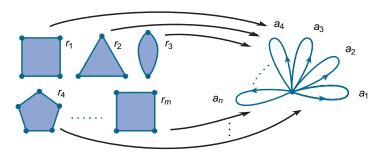


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The group springing into action

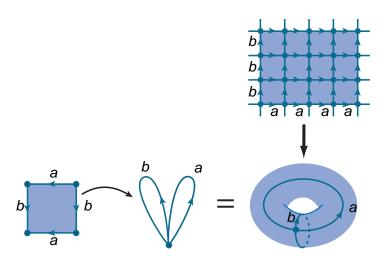


Figure: The 2-complex and Cayley graph for $\langle a, b \mid [a, b] \rangle$

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Subdirect products of hyperbolic groups Solution of Grothendieck's

Strand 1: Study (and manufacture) group actions on spaces in order to elucidate the structure of both the groups and the spaces.

One may prefer discrete cocompact actions by isometries but ...

sometimes weaken admission criteria to obtain a more diverse class of groups,

sometimes demand more structure to narrow the focus on groups and spaces of exceptional character.

The first strand in Geometric Group Theory

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Drawing in and Reaching out

Study finitely generated groups as geometric objects in their own right, via their intrinsic geometry.

$$\Gamma = \langle a_1, \ldots, a_n \mid r_1, r_2, \ldots \rangle$$

Word Metric:

$$d(\gamma_1, \gamma_2) = \min\{|w| : w \in F(\mathcal{A}), \ w \stackrel{\Gamma}{=} \gamma_1^{-1} \gamma_2\}.$$

Cayley Graph (1878) =
$$\widetilde{K(\mathcal{P})}^{(1)}$$

- Word metric and Cayley graph are independent o generating set, up to quasi-isometry.
- Thus one is particularly interested in properties of groups and spaces invariant under quasi-isometry
- Large-scale (coarse) geometry and topology

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Drawing in and Reaching out

Subdirect products of hyperbolic groups Solution of Grothendieck's

$\Gamma \cong \langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle$

"The general discontinuous group is given [as above]. There are above all three fundamental problems.

- ► The identity [word] problem
- ▶ The transformation [conjugacy] problem
- ▶ The isomorphism problem

[...] One is already led to them by necessity with work in topology. Each knotted space curve, in order to be completely understood, demands the solution of the three"

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Higman Embedding (1961): Every recursively presented group is a subgroup of a finitely presented group.

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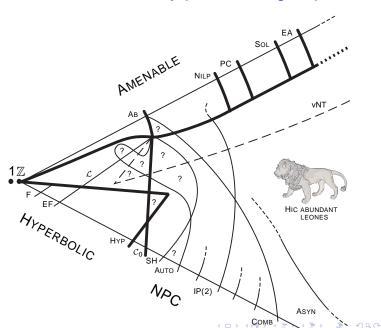
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Reaching out

Gromov's hyperbolic groups

If Γ is hyperbolic then it

- acts properly, cocompactly on a contractible complex
- has only finitely many conjugacy classes of finite subgroups and its abelian subgroups are virtually cyclic
- striking algorithmic properties: the set of geodesic words for Γ (wrt any finite gen set) is a regular language:
 ∃ finite state automaton recognising words labelling geodesics in Cayley graph; hyperbolic groups are automatic.
- Rapidly-solvable word and conjugacy problems.
- [Sela] The isomorphism problem is solvable among torsion-free hyperbolic groups.

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The universe of finitely

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CAT(0) and CAT(-1) conditions

A.D. Alexandrov

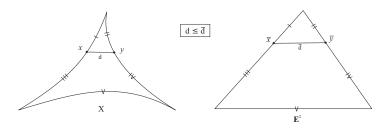


Figure: The CAT(0) inequality

then in \tilde{X} all triangles satisfy this inequality.

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local curvature, CAT(0)

CAT(0) and CAT(-1) conditions

A.D. Alexandrov

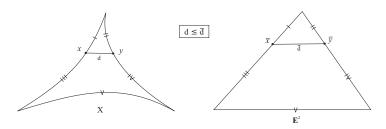


Figure: The CAT(0) inequality

Local-to-global: If X is complete and every point has a neighbourhood in which triangles satisfy this inequality, then in \tilde{X} all triangles satisfy this inequality.

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CAT(0) spaces

Metric spaces of non-positive curvature, *Bridson-Haefliger* Grund. Math. Wiss. **319**

- connections with many branches of mathematics
- local-to-global phenomena
- rigidity
- complexes of groups (à la Haefliger)
- connections with 3-dimensional geometry
- combination theorems; verifiability
- A great deal one can say about the structure of groups that act by isometries on CAT(0) and CAT(-1) spaces

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Negative curvature and hyperbolic groups

Identify key robust feature of CAT(-1) spaces X

If Γ acts geometrically on X (basepoint p), articulate wha remains of the feature when it is pulled-back via the Γ -equivariant quasi-isometry $\gamma \mapsto \gamma . p$ (fixed $p \in X$).

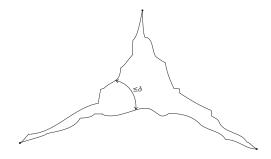


Figure: The slim triangles condition

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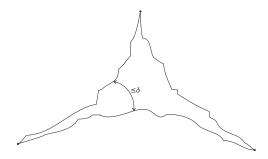


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Coarse convexity

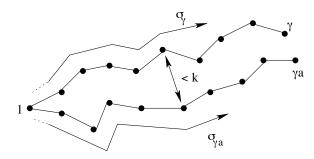


Figure: The fellow-traveller property

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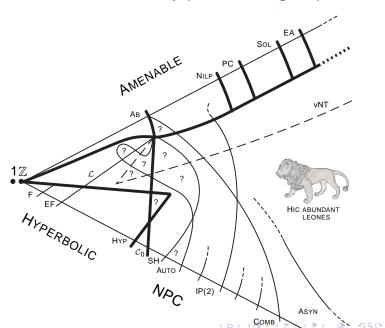
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Theorem

∃ combable groups that are not bicombable or automatic.

Hierarchy of formal languages, $Reg \subset CF \subset Ind$.

NPC groups

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Hierarchy of formal languages, $Reg \subset CF \subset Ind$.

Theorem

∃ Ind-combable groups that are not Reg-combable (automatic); some have quadratic Dehn functions, some cubic.

Theorem

∃ combable groups with unsolvable conjugacy problem.

Theorem

Isomorphism problem unsolvable for combable groups.

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Subdirect products of hyperbolic groups
Solution of Grothendieck's

Finitely presented $\Gamma = \langle \mathcal{A} \mid \mathcal{R} \rangle$, word $w \in \mathcal{F}(\mathcal{A})$ with w = 1 in Γ

Area(w):= min
$$\left\{ N : w = \prod_{j=1}^{N} u_j r_j u_j^{-1} \text{ freely, } r_j \in \mathcal{R}^{\pm 1} \right\}$$

The Dehn function $\delta(n)$ of the presentation is

$$\delta(n) = \max\{\operatorname{Area}(w)|\ w \in \ker(F(A) \to \Gamma),\ |w| \le n\}$$

where |w| denotes the length of the word w.

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The Filling Theorem

[Gromov]

W smooth, complete, Riemannian manifold.

• $c: S^1 \rightarrow W$ null-homotopic, rectifiable loop

$$\overline{\mathsf{FArea}(c)} = \inf \{ \operatorname{Area}(F) \mid F : \mathbb{D}^2 \to W, \ F|_{\partial \mathbb{D}^2} = c \}.$$

 \bullet The 2-diml isoperimetric function $[0,\infty) \to [0,\infty)$ is

$$\operatorname{Fill}_0^{\mathbf{M}}(r) := \sup \{ \operatorname{FArea}(c) \mid c : S^1 \to \widetilde{M}, \ I(c) \leq r \}$$

• Filling Theorem: $\forall M$ closed, $\mathrm{Fill}_0^M(x) \simeq \delta_{\pi_1 M}(x)$.

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The isoperimetric spectrum

$$\mathbf{IP} = \{ \alpha \mid \mathbf{n}^{\alpha} \simeq \ \mathsf{Dehn} \ \mathsf{function} \} \subseteq [1, \infty)$$

- Gromov (Bowditch, Papasoglu, Olshanskii,...): $(1,2) \cap IP = \emptyset$ and $\alpha = 1$ are the *hyperbolic groups*.
- At $\alpha = 2$ one has a rich and diverse class of groups.
- N ⊆ IP [Baumslag-Miller-Short, B-Pittet, Gromov]
- Bridson ('94): ∃ infinitely many non-integers in IP.
- Brady-B ('98): $\forall p \geq q$ integers, $2 \log_2(2p/q) \in \mathbb{P}$.
- Sapir-Birget-Rips ('98): Deeper analysis $\delta(x) \succeq x^4$.
- B,B,Forester,Shankar ('04) $\mathbb{Q} \cap [2,\infty) \subseteq IP$.

Theorem (B-Brady '98)

$$\overline{\mathbf{IP}} = \{1\} \cup [2, \infty).$$

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Subdirect products of hyperbolic groups

$G = \langle a_1, a_2, c, s_1, s_2 | a_1 a_2 = c = a_2 a_1, s_i^{-1} a_i^r s_i = c \rangle$

Figure: Half of snowflake diagram and dual tree

If *k* is the radius of the dual tree,

- Growth of the base of central Δ is r^k
- Area of central \triangle is roughly square of the base.
- Length of boundary is roughly 2^k.
- Area $\geq (r^k)^2 = (2^k)^{2\log_2(r)} = |\partial|^{2\log_2(r)}$

Basic snowflake groups [Brady-B]

$$G = \langle a_1, a_2, c, s_1, s_2 | a_1 a_2 = c = a_2 a_1, s_i^{-1} a_i^r s_i = c \rangle$$

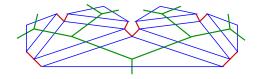


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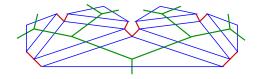


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General BBFS construction based on Perron-Frobenius theory

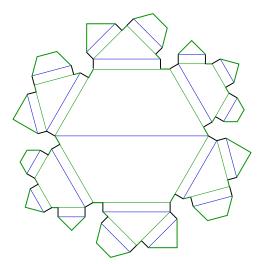


Figure: A snowflake disk based on the matrix $P = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$

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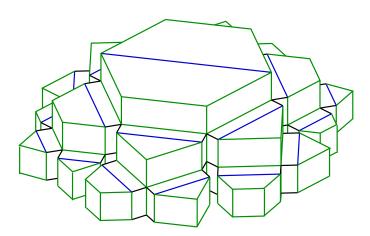


Figure: Snowflake spheres

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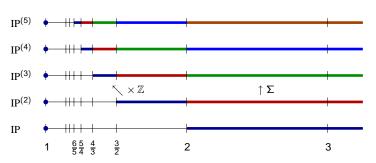


Figure: Complete picture of IP, partial for $IP^{(k)}$.

 $\mathbf{IP}^{(k)} = \{ \alpha \mid n^{\alpha} \simeq k \text{-diml Dehn function} \}$

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Drawing in and Reaching out

Subdirect products of hyperbolic groups

Tame: Finitely presented subdirect products of free groups, surface groups and limit groups are remarkably tame, for example:

Theorem (B, Howie, Miller, Short) If the Γ_i are limit groups and $S \subset \Gamma_1 \times \cdots \times \Gamma_n$ is of type

 FP_n , then S is virtually a direct product of limit groups.

Wild: Finitely presented subgroups of more general hyperbolic groups can be utterly wild.

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Solution of Grothendieck's Problems

 \exists algorithm with input a finite aspherical presentation $\mathcal Q$

and output a FINITE presentation of $P \subset H \times H$, with H hyperbolic.

$$P := \{(\gamma_1, \gamma_2) \mid p(\gamma_1) = p(\gamma_2)\} \subset H \times H$$

is the fibre-product associated to s.e.s.

$$1 \rightarrow N \rightarrow H \xrightarrow{p} Q \rightarrow 1$$

with N fin gen, H 2-diml hyperbolic, Q = |Q|

"1-2-3 Thm" refers to fact that N, H and Q are of type $\mathcal{F}_1, \mathcal{F}_2$ and \mathcal{F}_3 respectively. [Baumslag, B, Miller, Short]

Refinements (B-Haefliger, Wise) place more stringent conditions on H, e.g. locally CAT(-1) or residually finite

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 $A \neq 0$ a commutative ring, Γ a finitely generated group, $\operatorname{Rep}_{\Delta}(\Gamma)$ the category of Γ -actions on fin. pres. Δ -modules. Any homomorphism $u: \Gamma_1 \to \Gamma_2$ of groups induces

$$u_A^* : \operatorname{Rep}_A(\Gamma_2) \to \operatorname{Rep}_A(\Gamma_1).$$

Question (G, 1970): If Γ_1 and Γ_2 are finitely presented

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If $u: \Gamma_1 \to \Gamma_2$ is a homomorphism of finitely generated groups, u_{Δ}^* is an equivalence of categories if and only if $\hat{u}:\hat{\Gamma}_1\to\hat{\Gamma}_2$ is an isomorphism of profinite groups.

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Solution of Grothendieck's Problem

Qu: If Γ_1 and Γ_2 are finitely presented and residually finite, must $u: \Gamma_1 \to \Gamma_2$ be an isomorphism if $\hat{u}: \hat{\Gamma}_1 \to \hat{\Gamma}_2$ is an isomorphism?

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Grothendieck proved that the answer is yes in many cases, e.g. arithmetic groups. Platonov-Taygen (later Bass-Lubotzky, Pyber) proved answer no for finitely generated groups in general.

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Theorem (B-Grunewald, 2003)

 \exists residually finite, hyperbolic groups H and finitely presented subgroups $P \hookrightarrow \Gamma := H \times H$ of infinite index, such that P is not abstractly isomorphic to Γ , but the inclusion $u : P \hookrightarrow \Gamma$ induces an isomorphism $\hat{u} : \hat{P} \to \hat{\Gamma}$.

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Outline of the proof

Suppose Q has no finite quotients.

$$1 \rightarrow N \rightarrow H \rightarrow Q \rightarrow 1$$

Lemma.
$$H_2(Q, \mathbb{Z}) = 0 \implies \hat{N} \rightarrow \hat{H}$$
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- ▶ Build Q with aspherical presentation, no finite quotients and H₂(Q Z) = 0
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- ▶ Apply the embracing algorithm: $P \subset H \times H$
- ▶ Extend the lemma to deal with fibre products

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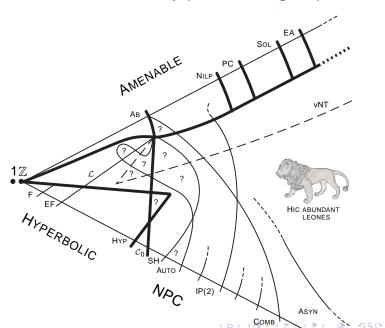
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