NON-POSITIVE CURVATURE IN GROUP THEORY

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ABSTRACT: This article is an edited account of the four lectures that I gave at the Groups St Andrews meeting in Bath during the summer of 1997. The aim of these lectures was to introduce an audience of group theorists, with varying backgrounds, to the role that non-positive curvature plays in the theory of discrete groups. A few new results are included, but basically this is an expository article aimed as non-experts. These notes do not constitute a comprehensive survey of curvature in group theory. Nevertheless, I hope that they give the reader a substantial and enticing taste of this active area of research.

Introduction

In the last fifteen years the close connection between geometry and combinatorial group theory, which was at the heart of the pioneering work of Dehn [52], has re-emerged as a central theme in the study of infinite groups. Thus combinatorial group theory has been joined by (and to a large extent has merged with) what has become known as *geometric group theory* — this is the broad context of these lectures. My basic goal will be to illustrate how various notions of non-positive curvature can be used to illuminate and solve a range of group theoretic problems. I shall also attempt to illustrate how group theory can serve as a potent tool for exploring the geometry of non-positively curved spaces. I wish to promote the idea that manifestations of non-positive curvature are deeply inherent in combinatorial group theory, often in contexts where there is little reason, *prima facie*, to suppose that the problems concerned have any geometric content whatsoever.

This last point is particularly emphasized in the first lecture, which begins with Dehn's formulation of the basic decision problems at the heart of combinatorial group theory: the word problem, the conjugacy problem and the isomorphism problem. Following a general discussion of the foundations of geometric group theory, I shall explain how Dehn's approach to the word problem leads us naturally to the theory of hyperbolic groups à la Gromov [68]. I shall briefly outline some elements of this theory. Through hyperbolicity we enter the world of non-positive curvature and Lecture 1 concludes with some illustrations of the way in which a qualitative understanding of the geometry of non-positively curved manifolds can lead us to striking group theoretic results.

Gromov's notion of hyperbolicity, which emerges in Lecture 1 from core considerations in combinatorial group theory, encapsulates many of the features of the *global* geometry of simply-connected spaces of negative curvature, whereas classically one thinks of curvature as a local or infinitesimal concept. In Lecture 2 we will examine the geometry of spaces which are non-positively curved in a strict, local, sense first identified by A. D. Alexandrov (see [2]). Much of the importance of this

[1]The author is supported by an EPSRC Advanced Fellowship. His work on this subject has also been supported by grants from the NSF (USA) and the FNRS (Switzerland).

more local approach to curvature rests on the existence of a local-to-global theorem (the Generalized Cartan-Hadamard Theorem): if a complete geodesic space X is non-positively curved (in a local sense), then its universal cover $\widetilde{X}$ is (a strong non-positive curvature condition that implies much about the large-scale geometry and topology of the space).

A central theme in Lecture 2 is the closeness of the relationship between the global geometry of CAT(0) spaces and the structure of the groups which act properly by isometries on them. Moreover, as the Generalized Cartan-Hadamard Theorem indicates, the existence of a metric of non-positive curvature on a space tells one a great deal about the topology of the space. Thus we are presented with an environment in which group theory, geometry and topology are deeply interconnected. Such an environment is obviously an exciting one, but before one can get too excited one needs to know that there is a substantial range of examples. Thus in Lecture 3 we turn our attention to the construction and identification of non-positively curved spaces, with particular emphasis on examples that are of group-theoretic interest.

Our point of departure in Lecture 1 will be Dehn's original formulation of the basic decision problems of combinatorial group theory. (This was also the point of departure for much of twentieth century combinatorial group theory.) In Lecture 4 I shall return to the study of these basic problems in the context of non-positively curved spaces and groups, and I shall describe the state of the art in this active area.

The origins of combinatorial group theory are deeply intertwined with those of low-dimensional topology. In both subjects geometry, in particular non-positive curvature, plays a central role that is not immediately evident[2]. For most of this century, though, curvature has played little role in the study of either subject. Indeed, in the case of combinatorial group theory, the whole geometric vein in the subject has lacked prominence. Its return to prominence in recent years is due largely to the deep insights of Mikhael Gromov [67, 68, 69], and I should make it clear that although his name and that of Max Dehn do not appear on every page of these notes, their influence runs throughout.

Acknowledgements

A more comprehensive account of much of the material in these lectures can be found in my book with André Haefliger [33]. Exploring the geometry of CAT(0) spaces is an adventure that Haefliger and I have shared over the last five years and I am deeply grateful to him for sharing his many insights. I am equally grateful to both him and his wife, Minouche, for welcoming me so warmly into their home during this time.

Some of the material in Lecture 4 is based on joint work with Gilbert Baumslag, Chuck Miller and Hamish Short; I would like to take this opportunity to thank them for many hours of good-humoured conversation. Finally, I would like to thank the organizers of Groups St Andrews in Bath. I particularly wish to thank Geoff Smith

[2]In the case of 3-manifolds this insight is largely due to Thurston [111, 112].

for accommodating my disorganization both in Bath and in the process of getting this article finished within a (strenuous) stone's throw of the deadline.

The tone of the lectures

The level and tone of the lectures reflect both the background of the audience and my own tastes and prejudices. First the audience: each of them might acquiesce in the title "group theorist", but within that category one would find many whose daily thoughts revolve entirely around finite groups, while others might consider themselves to be topologists. From the point of view of these lectures, the importance of this diversity is that it would be unreasonable to assume that a majority of the audience were well-acquainted with ideas from topology and geometry (beyond such basic concepts as the fundamental group of a space). I shall assume no knowledge of differential geometry. Various comments in the text will only be transparent to readers who are familiar with the rudiments of hyperbolic geometry, but for the most part such a familiarity is not required. At some points in lectures 3 and 4 I shall need some low dimensional topology and the reader who finds this to be a problem may wish to omit the sections concerned.

With regard to my own tastes and prejudices: I came to group theory, more precisely combinatorial group theory, through problems in geometry and topology. Moreover, I am by nature inclined to seek geometry underlying any given piece of mathematics. I hope that admitting this subjectivity will not weaken my contention that geometry is inherent (though not always immediately apparent) in many aspects of group theory and that to ignore it can be debilitating.

Lecture 1: In the beginning there was Max Dehn

I shall begin be explaining what I mean by combinatorial and geometric group theory, and by framing some of the central issues in each subject. I particularly wish to emphasize the common origins of these subjects and the natural interdependence of them.

First, combinatorial group theory: roughly speaking, *combinatorial group theory* is the study of groups given by generators and defining relations. Thus a group Γ is described to us by means of a set $\mathcal{A}$ and a subset $\mathcal{R}$ of the free group F on $\mathcal{A}$; to say that $\langle \mathcal{A} \mid \mathcal{R} \rangle$ is a presentation of Γ means that Γ is isomorphic to the quotient of F by the smallest normal subgroup containing $\mathcal{R}$. This method of describing groups emerged from work on discrete groups of isometries of hyperbolic space, and was first articulated by von Dyck in [56], which is generally regarded as the first paper in combinatorial group theory. For a comprehensive history of the subject up to 1980, see [46].

1.1 Decision problems

It was some years after von Dyck's original paper that Max Dehn framed the central issues of combinatorial group theory that have occupied researchers in the field for most of this century. Building on work of Poincaré [97, 98], Dehn was working on the basic problems of recognition and classification for low-dimensional manifolds (see [52]). In that setting, one finds that for many purposes the key invariant is the fundamental group of the space at hand. Moreover, when one is presented with the space in a concrete way, the fundamental group often emerges in the form of a presentation. In the course of his attempts to recover knowledge about fundamental groups (and hence manifolds) from such presentations, Dehn came to realize that the problems that he was wrestling with were manifestations of fundamental problems in the theory of groups. Let us recall the manner in which Dehn originally formulated the three basic problems of this kind.

1.2 "Über unendliche diskontinuierliche Gruppen", (Dehn, 1912)

"The general discontinuous group is given by n generators and m relations between them, as defined by Dyck (Math. Ann., 20 and 22). The results of those works, however, relate essentially to finite groups. The general theory of groups defined in this way at present appears very undeveloped in the infinite case. Here there are above all three fundamental problems whose solution is very difficult and which will not be possible without a penetrating study of the subject.

1. The identity [word] problem: *An element of the group is given as a product of generators. One is required to give a method whereby it may be decided in a finite number of steps whether this element is the identity or not.*

2. The transformation [conjugacy] problem: *Any two elements S and T of the group are given. A method is sought for deciding the question whether S and T can be transformed into each other, i.e. whether there is an element U of the group satisfying the relation $S = UTU^{-1}$.*

3. The isomorphism problem: *Given two groups, one is to decide whether they are isomorphic or not (and further, whether a given correspondence between the generators of one group and elements of the other is an isomorphism or not).*

These three problems have very different degrees of difficulty. [...] One is already led to them by necessity with work in topology. Each knotted space curve, in order to be completely understood, demands the solution of the three above problems in a special case."

I particularly wish to emphasize Dehn's last two remarks: the degree of difficulty of these three problems varies considerably, and each arises naturally in low-dimensional topology. I would also like to add something to the second remark: Dehn was studying surface groups and knot groups, and surfaces of positive genus and knot-complements[3] all support metrics of non-positive curvature. Thus the context in which Dehn was working was deeply imbued with non-positive curvature.

[3]viewed as compact manifolds with boundary

1.3 Dehn's algorithm for solving the word problem

This is perhaps the most direct approach that one can hope for whereby the information in a finite presentation is used directly to solve the word problem in the group presented. We shall see in subsequent sections that this direct algebraic approach to the word problem provides a remarkable bridge into the world of negative curvature (1.21).

Given a finite set of generators $\mathcal{A}$ for a group Γ, one would have a particularly efficient algorithm for solving the word problem if one could construct a finite list of words $u_1, v_1, u_2, v_2, \ldots, u_n, v_n$, with $u_i = v_i$ as elements of Γ, with lengths $|v_i| < |u_i|$, and with the property that if a word w represents the identity in Γ then at least one of the u_i is a subword of w.

If such a list of words exists, then given an arbitrary word w one looks for a subword of the form u_i; if there is no such subword, one stops and declares that w does not represent the identity; if u_i occurs as a subword of w, then one replaces it with v_i and repeats the search for subwords of the new (shorter) word — this new word represents the same element of the group as w. After at most $|w|$ steps one will either have reduced w to the empty word or else verified that w does not represent the identity.

This is the algorithm that Dehn used to solve the word problem for surface groups [53].

Definition 1.4 *A finite presentation $\langle \mathcal{A} \mid \mathcal{R} \rangle$ of a group Γ is called a Dehn presentation if $\mathcal{R} = \{u_1 v_1^{-1}, \ldots, u_n v_n^{-1}\}$, where the words $u_1, v_1, \ldots, u_n, v_n$ satisfy the conditions of Dehn's algorithm.*

Readers should ask themselves how one might decide whether or not $\mathbb{Z}^2$ has a (possibly obscure) Dehn presentation. (In fact it does not.) As further food for thought, let me mention some consequences of having a Dehn presentation.

Theorem 1.5 *If a group Γ has a Dehn presentation, then: numdepth ¿3 oodeep*

1. Γ *does not contain* $\mathbb{Z}^2$*;*
2. Γ *has a solvable conjugacy problem;*
3. Γ *has only finitely many conjugacy classes of finite subgroups;*
4. $H^*(\Gamma, \mathbb{Q})$ *is finitely generated and finite dimensional.*

I have chosen to mention these properties because in each case the hypothesis and conclusion are purely algebraic, whereas the proof of each assertion goes via the global geometry of the group, as we shall see later.

In order to tease out the geometry inherent in Dehn's algorithm, we need the basic vocabulary of geometric group theory.

Geometric group theory

This is a subject close to combinatorial group theory whose distinct identity has been forged in the last ten years. It is based to a large extent on two strands of thought.

The first is that by getting a group to act on a space (preferably by isometries) one can often use knowledge of the space to elucidate the nature of the group, or else use knowledge of the group to elucidate the nature of the space.

The second idea begins with the observation (due to Cayley (1878) and Dehn (1910)) that one can regard any finitely generated group as a metric space. As the result of efforts inspired largely by the writings of Gromov, we now know that one can glean a remarkable amount of algebraic information by studying the asymptotic geometry of groups regarded as metric spaces. Gromov outlined this idea in his article *"Infinite groups as geometric objects"* for the proceedings of the ICM in Warsaw [67], and he explored it in much greater detail in his two subsequent essays *"Hyperbolic groups"* [68] and *"Asymptotic invariants of infinite groups"* [69].

1.6 The word metric

Given a group Γ with finite generating set $\mathcal{A}$, one realizes the group as a geometric object by endowing it with the left-invariant metric:

$$d(g,h) = \inf\{|w| \mid w \in F(\mathcal{A}),\ w = g^{-1}h \text{ in } \Gamma\},$$

where $|w|$ is the length of the word w.

In many contexts it is easier to work with geodesic metric spaces[4] rather than simple metric spaces. With this in mind, one often replaces the metric space (Γ, d) with its *Cayley graph.*

Cayley graphs (Gruppenbilder)

Arthur Cayley introduced these graphs in 1878 *"to study the quasi-geometry"* of (finite) groups. Dehn, who was aware of Cayley's work, made extensive use of them in his work on Fuchsian groups twenty years later.

Definition 1.7 *The Cayley graph $\mathcal{C}_{\mathcal{A}}(\Gamma)$ of a group Γ with respect to a generating set $\mathcal{A}$ is the metric graph whose vertices are in 1-1 correspondence with Γ and which has an edge (labelled a) of length one joining γ to γa for each $\gamma \in \Gamma$ and $a \in \mathcal{A}$.*

Example 1.8 *The Cayley graph of $\mathbb{Z}^2$ with respect to the standard generators is (isometric to) the 1-skeleton of the tiling of the plane by unit squares, endowed with the ℓ_1-metric.*

Remark The word metric $d_{\mathcal{A}}$ on Γ is the restriction to Γ (the vertex set) of the metric on the Cayley graph $\mathcal{C}_{\mathcal{A}}(\Gamma)$.

[4] A metric space X is called a geodesic space if every pair of points $x, y \in X$ can be joined by a geodesic, i.e. a map $c : [0, D] \to X$ such that $c(0) = x$, $c(D) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, D]$.

Note that the action of Γ on itself by left multiplication gives an embedding $\Gamma \to Isom(\Gamma, d_{\mathcal{A}})$, and this action extends to an action of Γ by isometries on the Cayley graph. (The action of $\gamma_0 \in \Gamma$ by right multiplication $\gamma \mapsto \gamma\gamma_0$ is an isometry only if γ_0 lies in the centre of Γ.)

The point of introducing word metrics and Cayley graphs is to realize finitely generated groups as geometric objects. Since we are interested in groups Γ rather than pairs $(\Gamma, \mathcal{A})$, where $\mathcal{A}$ is a choice of generators, the geometry that we introduce ought to be, as far as possible, independent of the choice of generators. The key observation in this regard is that the word metrics associated to different choices of finite generating sets are Lipschitz equivalent, and the corresponding Cayley graphs are quasi-isometric in the following sense.

Definition 1.9 (quasi-isometry) *Let (X_1, d_1) and (X_2, d_2) be metric spaces. A (not necessarily continuous) map $f : X_1 \to X_2$ is called a (λ, ε)-quasi-isometric embedding if there exist constants $\lambda \geq 1$ and $\varepsilon \geq 0$ such that for all $x, y \in X_1$*

$$\frac{1}{\lambda} d_1(x, y) - \varepsilon \leq d_2(f(x), f(y)) \leq \lambda d_1(x, y) + \varepsilon.$$

If, in addition, there exists a constant $C \geq 0$ such that every point of X_2 lies in the C–neighbourhood of the image of f, then f is called a (λ, ε)-quasi-isometry. When such a map exists, X_1 and X_2 are said to be quasi-isometric.

The following remarks show in particular that quasi-isometry is an equivalence relation on any set of metric spaces.

(1) If there exists a (λ, ε)–quasi-isometry $f : X_1 \to X_2$ then there exists a (λ', ε')–quasi-isometry $f' : X_2 \to X_1$ (for some λ' and ε') and a constant $k \geq 0$ such that $d(ff'(x'), x') \leq k$ and $d(f'f(x), x) \leq k$ for all $x \in X_1$ and all $x' \in X_2$. Such a map f' is called a *quasi-inverse* for f.

(2) The composition of a (λ, ε) quasi-isometric embedding and a (λ', ε') quasi-isometric embedding is a (μ, ν) quasi-isometric embedding with $\mu = \lambda\lambda'$ and $\nu = \lambda'\varepsilon + \varepsilon'$. The composition of two quasi-isometries is a quasi-isometry.

(3) Let X be a metric space. Say that two maps $f, g : X \to X$ are equivalent, $f \sim g$, if $\sup_x d(f(x), g(x))$ is finite. Let $[f]$ denote the $\sim$ equivalence class of f. The *quasi-isometry group of* X, denoted $\mathcal{QI}(X)$, is the set of $\sim$ equivalence classes of quasi-isometries $X \to X$. Composition of maps induces a group structure on X, and any quasi-isometry $\phi : X \to X'$ induces an isomorphism $\phi_* : \mathcal{QI}(X) \to \mathcal{QI}(X')$.

Example 1.10 *The inclusion of a subspace Y into a metric space X is a quasi-isometry if and only if Y is quasi-dense, i.e. there exists a constant $C > 0$ such that every point of X lies in the $C-$neighbourhood of Y. For example, the natural inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ is a quasi-isometry (for the metric $d(x, y) = |x - y|$). In fact, the natural inclusion $(\Gamma, d_{\mathcal{A}}) \hookrightarrow \mathcal{C}_{\mathcal{A}}(\Gamma)$ of any finitely generated group as the vertex set its Cayley graph is a $(1, \frac{1}{2})$ quasi-isometry.*

Proposition 1.11 *A homomorphism $\varphi : \Gamma_1 \to \Gamma_2$ of finitely generated groups is a quasi-isometry if and only if $\ker(\phi)$ and $\Gamma_2/\mathrm{im}(\varphi)$ are both finite.*

This proposition can be proved directly without difficulty, but it can also be viewed in terms of the natural action of Γ_2 by isometries on its Cayley graph $\mathcal{C}_{\mathcal{A}}(\Gamma_2)$: via ϕ there is an induced action of Γ_1 on $\mathcal{C}_{\mathcal{A}}(\Gamma_2)$; this induced action is proper and cocompact (in the sense defined below) if and only if $\ker(\phi)$ and $\Gamma_2/\mathrm{im}(\varphi)$ are both finite. Thus (1.11) provides us with our first illustration of the fact that quasi-isometries arise naturally from proper cocompact group actions.

A bridge between the two aspects of geometric group theory

Definition 1.12 *A metric space is called proper if all of its closed bounded subsets are compact. When a group Γ acts on a space X by isometries, one endows the quotient (set of orbits) $\Gamma\backslash X$ with the (pseudo)metric:*

$$d(\Gamma.x, \Gamma.x') = \inf\{d(y, y') \mid y \in \Gamma.x,\ y' \in \Gamma.x'\}.$$

The action of Γ is called proper if for each bounded set $B \subset X$, the set $\{\gamma \in \Gamma \mid \gamma.B \cap B \neq \emptyset\}$ is finite. If $\Gamma\backslash X$ is a compact metric space then the action is said to be cocompact.

The following result was discovered by the Russian school in the nineteen fifties (see [58],[110]). It was rediscovered by John Milnor [87] some years later in the course of his investigations into the geometry of nilpotent groups.

Proposition 1.13 (Švarc-Milnor Lemma) *Let (X, d) be a geodesic space. If Γ acts properly and cocompactly by isometries on X, then Γ is finitely generated and for any choice of basepoint $x_0 \in X$ the map $\gamma \mapsto \gamma.x_0$ is a quasi-isometry.*

Proof The idea of the proof is as follows. First we fix a point $x_0 \in X$ and a ball B of radius D centred at x_0, where D is large enough to ensure that X is the union of the translates by Γ of the ball of radius $D/3$ about x_0. (B exists because the action is cocompact.) The set $\mathcal{A} := \{a \in \Gamma \mid B \cap a.B \neq \emptyset\}$ is finite because the action of Γ is proper, and in a moment we'll show that $\mathcal{A}$ generates Γ. If two balls $\gamma.B$ and $\gamma'.B$ intersect then $\gamma' = \gamma a$ for some $a \in \mathcal{A}$, and by drawing a line labelled a from $\gamma.x_0$ to $\gamma'.x_0$ for each such γ and γ' we obtain a copy of the Cayley graph $\mathcal{C}_{\mathcal{A}}(\Gamma)$ immersed in X. We then have to check that this version of $\mathcal{C}_{\mathcal{A}}(\Gamma)$ is not too distorted. In one direction we have:

Lemma 1.14 *Let (X, d) be a metric space. Let Γ be a group with finite generating set $\mathcal{A}$ and associated word metric $d_{\mathcal{A}}$. If Γ acts by isometries on X, then for every choice of basepoint $x_0 \in X$ there exists a constant $\mu > 0$ such that $d(\gamma.x_0, \gamma'.x_0) \leq \mu\, d_{\mathcal{A}}(\gamma, \gamma')$ for all $\gamma, \gamma' \in \Gamma$.*

Proof Let $\mu = \max\{d(x_0, a.x_0) \mid a \in \mathcal{A} \cup \mathcal{A}^{-1}\}$. If $d_{\mathcal{A}}(\gamma, \gamma') = n$ then $\gamma^{-1}\gamma' = a_1 a_2 \dots a_n$ for some $a_j \in \mathcal{A} \cup \mathcal{A}^{-1}$. Let $g_i = a_1 a_2 \dots a_i$. By the triangle inequality, $d(\gamma.x_0, \gamma'.x_0) = d(x_0, \gamma^{-1}\gamma'.x_0) \leq d(x_0, g_1.x_0) + d(g_1.x_0, g_2.x_0) + \cdots + d(g_{n-1}.x_0, \gamma^{-1}\gamma'.x_0)$. And for each i we have $d(g_{i-1}.x_0, g_i.x_0) = d(x_0, g_{i-1}^{-1} g_i.x_0) = d(x_0, a_i.x_0) \leq \mu$. Thus the lemma is proved. □

To complete the proof of the proposition we must show that $\mathcal{A}$ generates Γ and we must bound $d_{\mathcal{A}}(\gamma, \gamma')$ in terms of $d(\gamma.x_0, \gamma'.x_0)$. Because both metrics are Γ-invariant, it is enough to compare $d_{\mathcal{A}}(1, \gamma)$ and $d(x_0, \gamma.x_0)$.

Given $\gamma \in \Gamma$, let $c : [0, l] \to X$ be a geodesic joining x_0 to $\gamma.x_0$. We choose a partition $0 = t_0 < t_1 < \ldots < t_n = l$ of $[0, l]$ such that $d(c(t_i), c(t_{i+1})) \leq D/3$ for all i. For each t_i we choose an element $\gamma_i \in \Gamma$ such that $d(c(t_i), \gamma_i.x_0) \leq D/3$; choose $\gamma_0 = 1$ and $\gamma_n = \gamma$. Then, for $i = 1, \ldots, n$ we have $d(\gamma_i.x_0, \gamma_{i-1}.x_0) \leq D$ and hence $a_i := \gamma_{i-1}^{-1}\gamma_i \in \mathcal{A}$. Thus $\mathcal{A}$ generates Γ:

$$\gamma = \gamma_0(\gamma_0^{-1}\gamma_1) \ldots (\gamma_{n-2}^{-1}\gamma_{n-1})(\gamma_{n-1}^{-1}\gamma_n) = a_1 \ldots a_{n-1}a_n.$$

If we take as coarse a partition $0 = t_0 < t_1 < \ldots < t_n = l$ as possible with $d(c(t_i), c(t_{i+1})) \leq D/3$, then $n \leq (3/D)d(x_0, \gamma.x_0) + 1$. Since γ can be expressed as a word of length n, we get $d_{\mathcal{A}}(1, \gamma) \leq (d(x_0, \gamma.x_0) + 1)(3/D) + 1$. □

1.15 Some consequences of the Švarc-Milnor Lemma

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$\mathbb{Z}^n$ is quasi-isometric to Euclidean n-space $\mathbb{E}^n$.

If G is a Lie group that any cocompact lattice $\Gamma \subset G$ is quasi-isometric to G.

Let T_n denote the connected metric tree in which every vertex has valence n and every edge has length 1. If $n, m \geq 3$ then T_n is quasi-isometric to T_m.

To prove (3), note that the Cayley graph of the free group F_r of rank r is T_{2r}, so the case where n and m are even follows from the fact that every finitely generated free group of rank ≥ 2 is a subgroup of finite index in F_2. Hence all such groups (and their Cayley graphs) are quasi-isometric.

In the case where n is odd, T_n is quasi-isometric to the Cayley graph of $G_{2,n} = \mathbb{Z}_2 * \mathbb{Z}_n$, which has a finitely generated free subgroup of finite index, namely the kernel of the abelianization map $G_{2,n} \to \mathbb{Z}_2 \times \mathbb{Z}_n$.

A surprising number of group-theoretical properties that one might think had nothing to do with geometry turn out to be invariants of quasi-isometry, for example finite presentability, the existence of a nilpotent subgroup of finite index [70], the solvability of the word problem [3], the existence of an abelian subgroup of finite index ([93], [32]) and the existence of representations as lattices in various semi-simple Lie groups (see [60] for references). I quote these results to exemplify the theme that geometry is lying just under the surface of much of group theory.

Classical curvature

Let us now turn our attention to the main subject of these lectures *non-positive curvature*[5].

Figure 1.4 portrays the fact that classical (sectional) curvature can be quantified in terms of the geometry of triangles. With this in mind, in seeking generalized

[5]Terminology: In English one normally uses the term "negative curvature" to mean curvature strictly less than zero and one says "non-positive curvature" when one wishes to include zero. The French term "courbure negative" should normally be translated as "non-positive curvature"

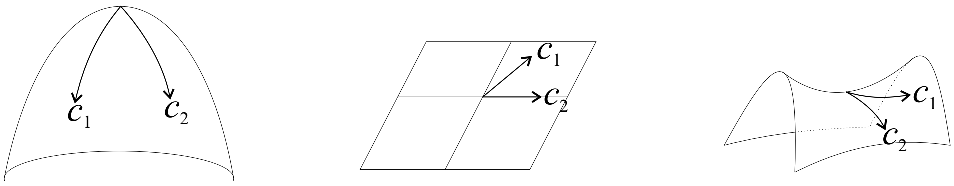

Figure 1.1: Sectional curvature κ

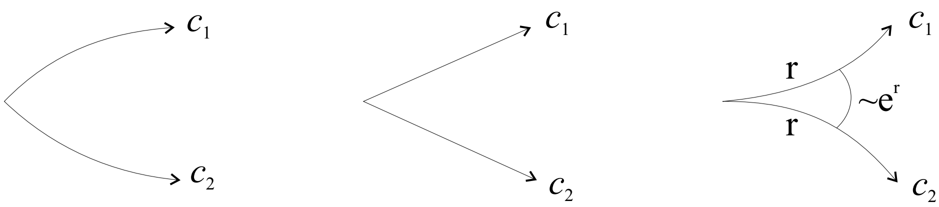

Figure 1.2: The divergence of geodesics in each case

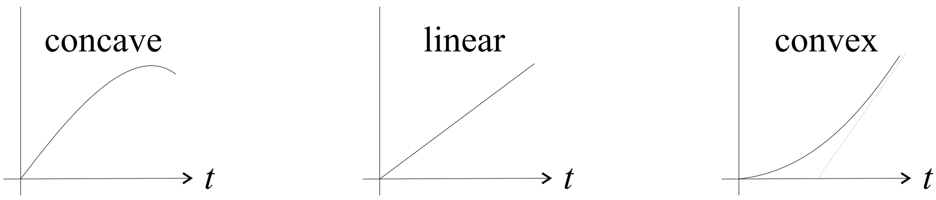

Figure 1.3: The function $t \mapsto d(c_1(t), c_2(t))$ in each case

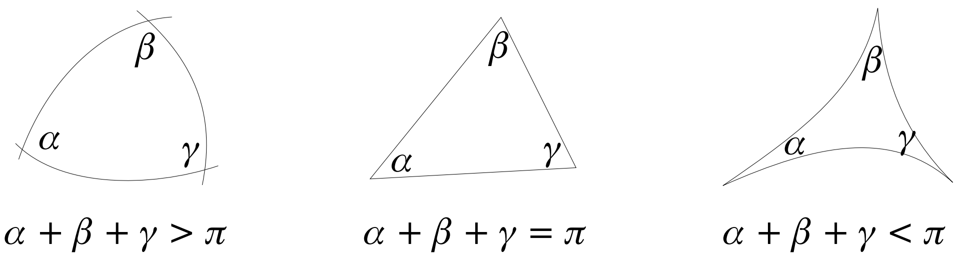

Figure 1.4: Triangles in each case

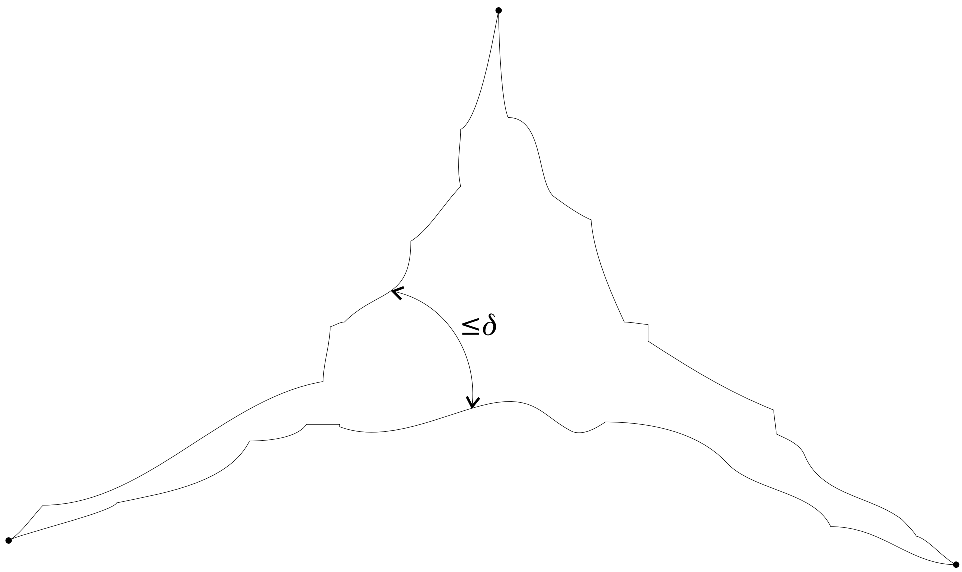

Figure 1.5: A triangle in a hyperbolic space

notions of non-positive curvature we concentrate our attention on the geometry of triangles. By definition, a *(geodesic) triangle* $\Delta = \Delta(p, q, r)$ in a metric space X consists of three points $p, q, r \in X$, its *vertices*, and a *choice* of three geodesic segments joining them, its *sides*.

Generalized notions of non-positive curvature

We shall be primarily concerned with two generalizations of non-positive curvature. The first generalization we consider is Gromov's notion of hyperbolicity, a global manifestation of strictly negative curvature. Gromov's approach is essentially to ignore the local structure of the space and to try instead to encapsulate the essence of negative curvature by requiring that triangles in the spaces under consideration behave on the large scale like geodesic triangles in real hyperbolic space $\mathbb{H}^n$.

1.16 Definition of hyperbolicity

Given $\delta > 0$, a geodesic metric space X is said to be $\delta-$hyperbolic if for every geodesic triangle $\Delta \subseteq X$, each edge of Δ is contained in the $\delta-$neighbourhood of the union of the other two sides (see Figure 1.5). X is said to be *hyperbolic* (in the sense of Gromov) if it is $\delta-$hyperbolic for some δ.

A finitely generated group is said to be hyperbolic (or "word hyperbolic") if its Cayley graph is hyperbolic in the above sense. (We shall see in (1.27) that hyperbolicity is an invariant of quasi-isometry among geodesic spaces and therefore this definition of a hyperbolic group does not depend on a choice of generators.)

I should mention that there are a number of useful variations on the definition given above. (Gromov credits Rips with the definition that we have given, "the slim triangles condition".) For an account of these variations, and a proof of their equivalence, we refer to [22], [33, III.H], [49], [64] and [106]. In these lectures I shall give only a very brief introduction to the theory of hyperbolic spaces and groups. For more details see the references listed above. Gromov's original article [68] is not easy to read, but contains an abundance of fascinating ideas beyond those examined in the above references.

Exercise 1.17 *Find δ_0 such that the hyperbolic plane $\mathbb{H}^2$ is δ_0-hyperbolic. (Hint: Because the area of a hyperbolic triangle is less than π, there is a bound on the size of semi-circles that one can inscribe in a triangle.)*

The second notion of curvature that we shall consider in these notes is local in nature. This approach is due to A.D. Alexandrov [2] (and to a lesser extent Busemann [40]); it is the subject of my book with Haefliger [33] (see also [6]). The idea is to compare triangles in the given space to triangles in the Euclidean plane $\mathbb{E}^2$. If all small triangles in the given space are no fatter than their comparison triangles in $\mathbb{E}^2$, then the space is said to be non-positively curved. More precisely:

Let X be a metric space. A comparison triangle in $\mathbb{E}^2$ for a geodesic triangle $\Delta = \Delta(p,q,r)$ in X is a triangle $\overline{\Delta} = \Delta(\overline{p},\overline{q},\overline{r})$ in $\mathbb{E}^2$ such that $d(\overline{p},\overline{q}) = d(p,q)$, $d(\overline{q},\overline{r}) = d(q,r)$ and $d(\overline{p},\overline{r}) = d(p,r)$. Note that $\overline{\Delta}$ is unique up to isometry. A point $\overline{x} \in [\overline{q},\overline{r}]$ is called *a comparison point* for $x \in [q,r]$ if $d(q,x) = d(\overline{q},\overline{x})$. Comparison points on $[\overline{p},\overline{q}]$ and $[\overline{p},\overline{r}]$ are defined in the same way.

Definition 1.18 (CAT(0) space) *Let X be a metric space. Let $\Delta \subset \mathbb{E}^2$ be a geodesic triangle in X. Let $\overline{\Delta} \subset M^2_\kappa$ be a comparison triangle for Δ. Then Δ is said to satisfy the CAT(0) inequality if for all $x, y \in \Delta$ and comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$,*

$$d(x,y) \leq d(\overline{x},\overline{y}).$$

If X is a geodesic space and all geodesic triangles in X satisfy the CAT(0) inequality, then X is said to be a CAT(0) space.[6]

A metric space X is said to be non-positively curved (in the sense of Alexandrov) if it is locally a CAT(0) space, i.e. for every $x \in X$ there exists $r_x > 0$ such that the ball $B(x,r_x)$, endowed with the induced metric, is a CAT(0) space. Henceforth, whenever I say that a space is non-positively curved, I shall mean this in the sense of Alexandrov.

Remarks (1) Standard comparison theorems in Riemannian geometry show that a complete Riemannian manifold is non-positively curved in the above sense if and only if all of its sectional curvatures are non-positive (see [48] or [33, II.1A]).

(2) By taking comparison triangles in the hyperbolic plane instead of the Euclidean plane, one obtains the notion of a CAT(-1) space and *curvature* ≤ -1. More generally, by taking comparison triangles in the plane of constant sectional curvature κ one gets the notion of a CAT(κ) space and *curvature* $\leq \kappa$.

[6]This terminology is due to Gromov. The initials are in honour of Cartan, Alexandrov and Toponogov.

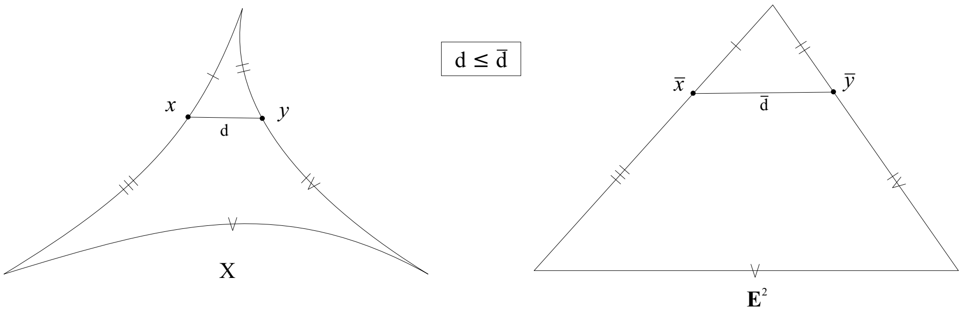

Figure 1.6: The CAT(0) inequality

By reversing the direction of the inequality in (1.18) one gets the notion of *curvature* ≥ 0. The study of spaces with lower curvature bounds is also interesting (see [38]), but has less to do with group theory.

Lemma 1.19 *CAT*(-1) *spaces are* δ_0*-hyperbolic, where* δ_0 *is as in (1.17).*

In the remainder of this lecture I shall concentrate on δ-hyperbolic spaces. First though, to whet the reader's appetite for CAT(0) spaces, let me state a group-theoretic result that I will explain later in the context of CAT(0) spaces.

Let $\mathcal{P} = \langle \mathcal{A} \mid \mathcal{R} \rangle$ be a presentation for Γ, with $\mathcal{R} \neq \emptyset$. Regard the elements of $\mathcal{R}$ as cyclic words and consider the following graph $W(\mathcal{P})$: the vertex set of $W(\mathcal{P})$ is $\{a, \overline{a} \mid a \in \mathcal{A}\}$; the edges of $W(\mathcal{P})$ are in 1–1 correspondence with the subwords of length two in the cyclic words $r \in \mathcal{R}$ — for each occurrence of $(ab^{-1})^{\pm 1}$ there is an edge connecting a to b, and for each occurrence of $(ab)^{\pm 1}$ there is an edge connecting a to $\overline{b}$.

Theorem 1.20 *If each* $r \in \mathcal{R}$ *has length at least* 4 *and* $W(\mathcal{P})$ *contains no circuits of length less than* 4*, then numdepth ¿3 oodeep*

1. Γ *is torsion-free;*
2. Γ *and all of its f.p. subgroups have solvable word and conjugacy problems;*
3. $H^n(\Gamma, \mathbb{Z}) = 0$ *for* $n \geq 3$*; and*
4. *if* Γ *contains a solvable subgroup of finite index, then* $|\mathcal{A}| = 2$, $|\mathcal{R}| = 1$ *and* $\Gamma \cong \mathbb{Z}^2$ *or* $\mathbb{Z} \rtimes \mathbb{Z}$.

Dehn's algorithm: the word problem and negative curvature

We now have a sufficient vocabulary to explain how negative curvature is hidden in the obvious solution to the word problem that one gets from a Dehn presentation.

Theorem 1.21 (Gromov, Cannon, Dehn) *A group is hyperbolic if and only if it admits a (finite) Dehn presentation.*

Max Dehn proved that Fuchsian groups admit Dehn presentations [53]. Jim Cannon extended Dehn's theorem to include the fundamental groups of all compact negatively curved manifolds [42, 43]. In the generality stated above, the theorem is due to Gromov [68]. (Cannon gave a proof in [43] and alternative proofs can be found elsewhere, e.g. [106].)

Proof [an outline] A detailed proof requires more elements of the geometry of hyperbolic spaces than we shall establish here, but we indicate the main points.

First suppose that the Cayley graph of Γ with respect to some finite generating set $\mathcal{A}$ is δ-hyperbolic. By studying the geometry of quasi-geodesics in hyperbolic spaces one sees (Corollary 1.30) that every closed loop in the Cayley graph $\mathcal{C}_{\mathcal{A}}(\Gamma)$ contains a subpath of length at most 8δ that is not a geodesic (and whose endpoints are at vertices). It follows that we obtain a Dehn presentation $\langle \mathcal{A} \mid \mathcal{R} \rangle$ for Γ by defining $\mathcal{R}$ to be the set of words $u_i v_i^{-1}$, where u_i varies over all words of length at most 8δ in the generators and their inverses and v_i is a word that is equal to u_i in Γ and whose length $|v_i|$ is less than $|u_i|$.

The proof of the converse is less direct; it goes via the linear isoperimetric inequality. □

Definition 1.22 (Isoperimetric inequalities) *A finitely presented group* $\Gamma = \langle \mathcal{A} \mid \mathcal{R} \rangle$ *is said to satisfy a linear isoperimetric inequality if there exists a constant* K *such that every word* w *in the kernel of the natural map* $F(\mathcal{A}) \to \Gamma$ *can be expressed in the free group* $F(\mathcal{A})$ *as a product of at most* $K|w|$ *conjugates of relators* $r \in \mathcal{R}^{\pm 1}$.

Quadratic, polynomial and (sub)exponential isoperimetric inequalities are all defined similarly. In the quadratic case, for example, the requirement is that w can be expressed as a product of at most $K|w|^2$ conjugates of relators.

One can show that this definition (but not the value of the constant K) is independent of the given finite presentation of Γ. Indeed the optimal type of isoperimetric inequality satisfied by a finitely presented group is an invariant of quasi-isometry [3].

It is obvious that if a finitely presented group has a Dehn presentation then the group satisfies a linear isoperimetric inequality. Thus the following result of Gromov completes the proof of (1.21).

Theorem 1.23 *If a group satisfies a linear isoperimetric inequality then it is hyperbolic.*

An elementary proof of this result, due to Hamish Short, can be found in an appendix to [63]. (See also Lysenok [82].)

Remark Gromov [67] found a useful companion for Theorem 1.23: if a group satisfies a sub-quadratic isoperimetric inequality then it satisfies a linear isoperimetric inequality. For a proof see [23], [90] or [95].

In Lecture 4 we shall discuss the conjugacy problem and the isomorphism problem in the class of hyperbolic groups.

Quasi-geodesics in δ-hyperbolic spaces

The essence of a geodesic space is encoded in the geometry of its geodesics. What do we wish to know about geodesics in hyperbolic spaces? Well, we are primarily interested in groups, and a finitely generated group is only well-defined as a geometric object up to quasi-isometry. Moreover, our main tool for relating the geometry of groups to the geometry of the spaces on which they act (1.13) also concerns only the quasi-isometry type of the group. Thus we are led to ask which properties of geodesics are preserved by quasi-isometries.

Definition 1.24 (quasi-geodesic) *A (λ, ε) quasi-geodesic in a metric space X is a (λ, ε) quasi-isometric embedding $c : I \to X$, where I is an interval of the real line (bounded or unbounded) or $\mathbb{Z}$. More explicitly,*

$$\frac{1}{\lambda}|t - t'| - \varepsilon \leq d(c(t), c(t')) \leq \lambda |t - t'| + \varepsilon$$

for all $t, t' \in I$. If $I = [a, b]$ then $c(a)$ and $c(b)$ are called the endpoints of c.

In general quasi-geodesics need not resemble geodesics.

Exercise 1.25 *Consider the spiral $[0, \infty) \to \mathbb{E}^2$ given in polar coordinates by $t \mapsto (t, \log(1 + t))$. Prove that this is a quasi-geodesic.*

In hyperbolic spaces quasi-geodesics approximate geodesics closely — this is a key difference between Euclidean and hyperbolic geometry.

Theorem 1.26 (Stability of quasi-geodesics) *For all $\delta > 0, \lambda \geq 1, \varepsilon \geq 0$ there exists a constant $R = R(\delta, \lambda, \varepsilon)$ such that the following statement holds. If X is a δ-hyperbolic geodesic space, c is a (λ, ε) quasi-geodesic in X and $[p, q]$ is a geodesic segment joining the endpoints of c, then $[p, q]$ lies in the R-neighbourhood of the image of c and vice versa.*

It follows from this theorem that a group is hyperbolic if and only if for every $\lambda, \varepsilon > 0$, the (λ, ε) quasi-geodesic triangles in its Cayley graph are uniformly slim. This property is obviously preserved under quasi-isometry, therefore:

Corollary 1.27 *Hyperbolicity is an invariant of quasi-isometry amongst geodesic spaces (in particular Cayley graphs, hence finitely generated groups). More generally, if X and Y are geodesic spaces, X is hyperbolic and $Y \hookrightarrow X$ is a quasi-isometric embedding, then Y is hyperbolic.*

The following result gives a local criterion for recognising quasi-geodesics.

Definition 1.28 *Let X be a metric space and fix $k > 0$. A path $c : [a, b] \to X$ is said to be a k-local geodesic if $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [a, b]$ with $|t - t'| \leq k$.*

Theorem 1.29 (8δ-local geodesics are q.g.) *Let X be a δ-hyperbolic geodesic space and let $c : [a, b] \to X$ be a 8δ-local geodesic. Then c is a (λ, ε)-quasi-geodesic, where λ depends only on δ and $\varepsilon < 8\delta$.*

For a proof see [49], [64], [106] or [33, III.H].

Corollary 1.30 *If X is a δ-hyperbolic geodesic space, then no path $c : [a, b] \to X$ with $a \neq b$ and $c(a) = c(b)$ is an 8δ-local geodesic.*

In Lecture 4 we shall discuss the conjugacy problem and the isomorphism problem in the class of hyperbolic groups.

Quasification and finite subgroups of hyperbolic groups

I wish to illustrate how profitable it can be to transport classical arguments from the theory of non-positively curved manifolds into the world of hyperbolic and related groups. The key to all such adaptations is that one must find an appropriate way to "quasify" the key role that non-positive (or negative) curvature is playing in the classical setting; one then attempts to encapsulate a robust form of the salient feature of curvature in the more relaxed world of hyperbolic spaces. The first example of this is the very definition of δ-hyperbolic space: instead of insisting that geodesic triangles satisfy the CAT(-1) inequality, which imposes tight control on the local structure of the space, one observes that many important aspects of the global geometry of CAT(-1) spaces rely only on the fact that the triangles are uniformly slim – thus one arrives at a notion of negative curvature (namely δ-hyperbolicity) that is much more robust, indeed quasi-isometry invariant.

To illustrate how this general philosophy of quasification can be useful, we prove the following:

Theorem 1.31 *If a finitely generated group Γ is hyperbolic, then it contains only finitely many conjugacy classes of finite subgroups.*

The proof is adapted from the following classical argument of Elie Cartan [44]. Suppose that we have a finite (more generally, compact) group of isometries G of a complete Riemannian manifold X of non-positive curvature. Fix $x \in X$ and consider $G.x \subset X$. This is a bounded set, and in a complete Riemannian manifold of non-positive curvature there are various intrinsic notions of "centre" that one can associate to bounded sets. We focus on the *circumcentre* (this is not the notion of centre used by Cartan, but it was used by Bruhat and Tits [37] in a later adaptation of Cartan's argument) — the circumcentre of a bounded set is the centre of the unique smallest ball that contains that set. (See [33, II.2] for a proof that there is indeed a unique such ball.)

Since $G.x$ is G-invariant, so is its centre c, and hence G fixes a point of X. If G is a subgroup of a group Γ that is acting cocompactly on X, say $X = \Gamma.K$ where K is compact, then $\gamma.c \in K$ for some $\gamma \in \Gamma$ and hence G is conjugate in Γ to a subgroup of the stabilizer of some point of K. If the action of Γ is proper, then the union of such stabilizers is finite and therefore Γ has only finitely many conjugacy classes of finite subgroups.

The existence of a circumcentre for bounded sets was the key feature of the above argument. By quasifying the usual proof of the existence of such centres in non-positively curved spaces we get:

Lemma 1.32 (quasi-centres) *Let X be a δ-hyperbolic geodesic space. Let $Y \subset X$ be a non-empty bounded subspace. Let $r_Y = \inf\{\rho \geq 0 \mid Y \subset B(x, \rho),$ some $x \in X\}$. For all $\varepsilon > 0$, the set $C_\varepsilon(Y) = \{x \in X \mid Y \subseteq B(x, r_Y + \varepsilon)\}$ has diameter less than $(4\delta + 2\varepsilon)$.*

Proof Given $x, x' \in C_\varepsilon(Y)$, let m be the midpoint of a geodesic segment $[x, x']$. For each $y \in Y$ we consider a geodesic triangle with vertices x, x', y and with $[x, x']$ as one of the sides. As X is δ-hyperbolic, m is within a distance δ of some $p \in [x, y] \cup [x', y]$; suppose that $p \in [x, y]$. Then, since $d(x, m) = d(x, x')/2$ and $d(p, x) \geq d(x, m) - \delta$, we have $d(y, p) = d(y, x) - d(p, x) \leq d(y, x) + \delta - d(x, x')/2$, and hence

$$d(y, m) \leq r_Y + \varepsilon + 2\delta - \frac{1}{2} d(x, x').$$

But $d(y, m) \geq r_Y$ for some $y \in Y$, hence $\varepsilon + 2\delta - \frac{1}{2} d(x, x') \geq 0$. □

Proof[of Theorem 1.31] Let Γ be a hyperbolic group and consider the natural action of Γ on its Cayley graph. Let $H \subset \Gamma$ be a finite subgroup, and let $C_1(H)$ be as in the lemma. $C_1(H)$ contains at least one vertex and the action of H leaves $C_1(H)$, and hence its vertex set, set-wise invariant. If x is one of the vertices of $C_1(H)$, then $x^{-1}Hx$ leaves $x^{-1}C_1(H)$ invariant. But $x^{-1}C_1(H)$ is a set of diameter less than $(4\delta + 2)$ containing the identity 1, and it contains $x^{-1}Hx = (x^{-1}Hx).1$. Thus every finite subgroup of Γ is conjugate to a subset of the ball of radius $(4\delta+2)$ about the identity. □

The proof given above does not appear in the literature but I believe that it is known to a number of researchers in the field, in particular Brian Bowditch and Ilya Kapovich. Alternative proofs were given by Ol'shanskii and Bogopolskii and Gerasimov [21].

Quasiconvexity and abelian subgroups of hyperbolic groups

A subspace C of a geodesic space X is said to be *convex* if for all $x, y \in C$, each geodesic joining x to y is contained in C. Following Gromov [68], one can quasify this notion: a subspace C of a geodesic metric space X is said to be *quasiconvex* if there exists a constant $k > 0$ such that for all $x, y \in C$, some geodesic joining x to y is contained in the k-neighbourhood of C.

Exercise 1.33 *Let G be a group with finite generating set $\mathcal{A}$. Let $H \subset G$ be a subgroup. Prove that if H is a quasiconvex subset of the Cayley graph $\mathcal{C}_\mathcal{A}(G)$ then H is finitely generated, and $H \hookrightarrow G$ is a quasi-isometric embedding (with respect to any choice of word metrics).*

In the circumstances of the above exercise one says that H is a *quasiconvex subgroup* of G. In the light of (1.27) we have:

Proposition 1.34 *The quasiconvex subgroups of hyperbolic groups are hyperbolic.*

The following results are due to Gromov [68]. The ideas underlying them have been used by other authors, particularly Hamish Short, to obtain similar results in wider contexts (see [59], [63], [4], [107]).

Proposition 1.35 *Let Γ be a hyperbolic group. numdepth ¿3 oodeep*

1. *The centralizer $C(\gamma)$ of every $\gamma \in \Gamma$ is a quasiconvex subgroup.*
2. *If the subgroups $H_1, H_2 \subset \Gamma$ are quasiconvex then so is $H_1 \cap H_2$.*

Corollary 1.36 *Suppose that Γ is hyperbolic and that $\gamma \in \Gamma$ has infinite order. Then: numdepth ¿3 oodeep*

1. *The map $\mathbb{Z} \to \Gamma$ given by $n \mapsto \gamma^n$ is a quasi-geodesic.*
2. *$\langle\gamma\rangle$ has finite index in $C(\gamma)$. In particular Γ does not contain $\mathbb{Z}^2$.*

Proof $C(\gamma)$ is quasiconvex, hence finitely generated and hyperbolic. By intersecting the centralizers of a generating set for $C(\gamma)$, we see that the centre $Z(C(\gamma))$ is also finitely generated and hyperbolic. It is easy to see that a finitely generated abelian group is hyperbolic if and only if it contains a cyclic subgroup of finite index. Hence $Z(C(\gamma))$ contains $\langle\gamma\rangle$ as a subgroup of finite index, and since $Z(C(\gamma)) \hookrightarrow C(\gamma)$ and $C(\gamma) \hookrightarrow \Gamma$ are quasi-isometric embeddings (1.33), so is $\langle\gamma\rangle \hookrightarrow \Gamma$. This proves (1).

Fix a finite generating set with respect to which Γ is δ-hyperbolic. Let d be the associated word metric. Define the *translation number* $\tau(\gamma)$ to be $\lim_{n\to\infty} d(1, \gamma^n)/n$. (This limit exists because the function $n \mapsto d(1, \gamma^n)$ is sub-additive.) It follows from (1) that $\tau(\gamma) > 0$.

$\tau(\gamma^r) = r\,\tau(\gamma)$ and $\tau(x^{-1}\gamma x) = \tau(\gamma)$ for all $x \in \Gamma$. Thus, replacing γ by a suitable power if necessary, we may assume that γ is not conjugate to any element a distance 4δ or less from the identity. I claim that it follows from this that if $[\gamma, g] = 1$ then g lies within a distance $K := d(1, \gamma) + 4\delta$ of $\langle\gamma\rangle$. For suppose not, choose g so that $d(g, \langle\gamma\rangle) = d(g, 1) > K$, and consider a geodesic quadrilateral Q in the Cayley graph of Γ with vertices $\{1, \gamma, g, g\gamma = \gamma g\}$. The side of Q joining γ to γg is chosen to be the translate by γ of the side joining 1 to g; we write g_t to denote the point a distance t from 1 on this side.

δ-hyperbolicity (1.16) implies that g_t lies within a distance 2δ of a point on one of the other three sides of Q. If $d(1, g) - d(1, \gamma) - 2\delta > t > 2\delta$, then this other point must be on the side joining γ to γg; say it is $\gamma g_{t'}$. Since $t' = d(\gamma g_{t'}, \langle\gamma\rangle)$, we have $t' \leq t + 2\delta$. Similarly $t \leq t' + 2\delta$, and hence $d(g_t, \gamma g_t) \leq 4\delta$. But this implies that $d(1, g_t^{-1}\gamma g_t) \leq 4\delta$, contrary to our assumption on γ. This proves (2). □

1.37 Translation numbers are rational

A remarkable result of Gromov [68], an elegant proof of which has been given by Delzant [55], states that associated to each word metric on a hyperbolic group Γ there is an integer M such that each translation number $\tau(\gamma)$, $\gamma \in \Gamma$ (as defined in the preceding proof) is a rational number whose denominator divides M.

1.38 Semihyperbolic groups

One obvious omission from these lectures is the theory of semihyperbolic groups as laid out by myself and Alonso in [4] (following the influence of [68], [59] and [63]). In this theory one seeks to encapsulate the global essence of non-positive curvature (as opposed to strictly negative curvature) by a suitable analogue of the δ-hyperbolic condition. In the theory of semihyperbolic groups quasi-convexity again plays a key role.

Lecture 2: The geometry of spaces

In this lecture I shall present those elements of the basic theory of CAT(0) spaces that are most useful from the point of view of combinatorial and geometric group theory. All of this material is taken from my book with André Haefliger [33]. My main aim is to illustrate the close connection between the geometry of CAT(0) spaces and the structure of the groups which act on them.

A key feature in this theory is the existence of a local-to-global theorem for non-positively curved[7] spaces (Theorem 2.2). This is important from the point of view of group theory because in many cases it provides a remedy to the problems caused by the fact that the rather dull local structure of Cayley graphs precludes any local-to-global analysis of their geometry: if one can realise a group Γ as the fundamental group of a compact non-positively curved space X, then one can analyze Γ via its action by deck-transformations on the universal covering $\widetilde{X}$, and we have the option of studying Γ laid out as the set of translates of a basepoint in $\widetilde{X}$ rather than as the set of vertices of the Cayley graph. Theorem 2.2 tells us that $\widetilde{X}$ is a CAT(0) space. The action of Γ on $\widetilde{X}$ is proper, cocompact and by isometries. Thus if by a study of CAT(0) spaces we can deduce facts about groups which act on them in this manner, then we will have an approach to understanding a (hopefully wide) class of groups.

In order to motivate such a study let me list some of the benefits of getting groups to act on CAT(0) spaces.

Theorem 2.1 (A summary of results) *Let Γ be a group that acts properly and cocompactly by isometries on a CAT(0) space X. Then: numdepth ¿3 oodeep*

1. Γ *is finitely presented.*
2. Γ *satisfies a quadratic isoperimetric inequality (and hence has a solvable word problem).*

[7]In all that follows "non-positive curvature" will be meant in the sense of (1.18).

3. Γ *has a solvable conjugacy problem.*
4. Γ *has only finitely many conjugacy classes of finite subgroups.*
5. *Every solvable subgroup of* Γ *is virtually abelian.*
6. *Every abelian subgroup of* Γ *is finitely generated.*
7. *If* Γ *is torsion-free, then it is the fundamental group of a compact cell complex whose universal cover is contractible.*
8. Γ *does not contain subgroups of the form* $\langle x, y \mid x^{-1}y^p x = y^q \rangle$ *with* $|p| \neq |q|$.
9. *If* $A \cong \mathbb{Z}^n$ *is central in* Γ *then there exists a subgroup of finite index in* Γ *that contains* A *as a direct factor.*

The class of groups which act properly and cocompactly by isometries on $CAT(0)$ *spaces is closed under the following operations: numdepth ¿3 oodeep*

(a) *direct products,*

(b) *free products with amalgamation along virtually cyclic subgroups.*

All of these results are proved in [33]. I will give a sketch of the proof of parts (1) to (9) in the course this lecture and lecture 4. Part (a) is an easy exercise and (b) is proved in (3.11).

Local-to-global

The following theorem is a variation on a result of M. Gromov [68], following an idea of A. D. Alexandrov. A detailed proof was given by W. Ballmann in the locally compact case ([64], Chap.10). S. Alexander and R. L. Bishop [1] proved the stronger result stated below under the additional hypothesis that $\widetilde{X}$ is a geodesic space. The form of the result stated here is proved in [33, II.4].

The metric d on a geodesic space X is said to be *convex* if given any pair of geodesic paths $c, c' : [0,1] \to X$ parametrized proportional to the arc length, the following inequality holds for all $t \in [0,1]$:

$$d(c(t), c'(t)) \leq (1-t)\, d(c(0), c'(0)) + t\, d(c(1), c'(1)).$$

The metric is said to be *locally convex* if every point has a neighbourhood on which the induced metric is convex.

Theorem 2.2 (Generalized Cartan-Hadamard Theorem) *Let* X *be a complete connected metric space. numdepth ¿3 oodeep*

1. *If the metric on* X *is locally convex, then the induced length metric on the universal covering* $\widetilde{X}$ *is (globally) convex. (In particular there is a unique geodesic segment joining each pair of points in* $\widetilde{X}$ *and geodesic segments in* $\widetilde{X}$ *vary continuously with their endpoints.)*
2. *If* X *is of curvature* $\leq \kappa$*, where* $\kappa \leq 0$*, then* $\widetilde{X}$ *(with the induced length metric) is a* $CAT(\kappa)$ *space.*

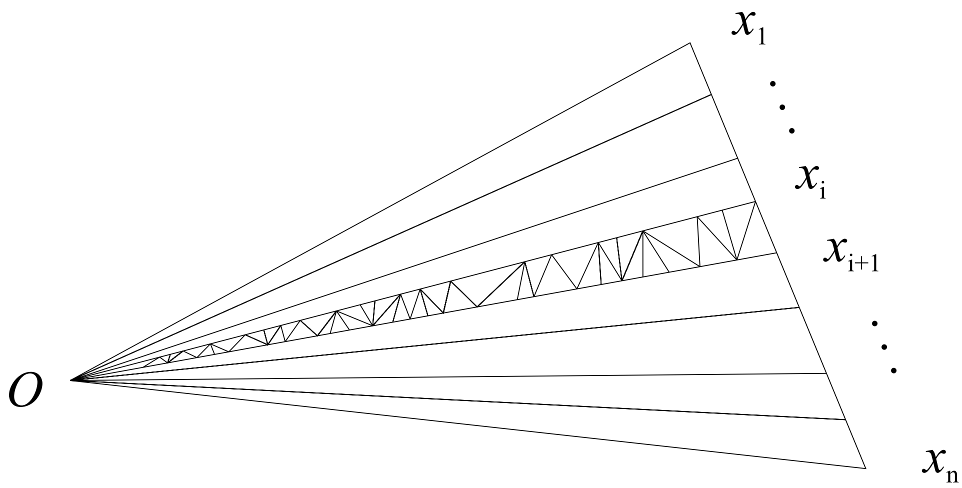

Figure 2.1: Alexandrov's Patchwork

The second part of Theorem 2.2 is deduced from the first by a 'patchwork' process indicated in the following picture. Following Alexandrov [2], one proves a gluing lemma: given a geodesic triangle $\Delta(p, q, r)$ and a point m on the side $[p, r]$, if the triangles $\Delta(p, m, r)$ and $\Delta(q, m, r)$ both satisfy the CAT(κ) inequality, then so does $\Delta(p, q, r)$. One applies the gluing lemma repeatedly, first to fill out the narrow subtriangles in Figure 2.1, such as $\Delta_i = \Delta(0, x_i, x_{i+1})$, then to assemble the Δ_i into the given (large) triangle.

We note three consequences of Theorem 2.2 that are important from the point of view of group theory.

Corollary 2.3 *If X is a compact non-positively curved space, then: numdepth ¿3 oodeep*

1. *The universal cover of X is contractible.*
2. *$\Gamma = \pi_1 X$ has a finite $K(\Gamma, 1)$.*
3. *If Y is a compact length space and $f : Y \to X$ is a local isometry, then $f_* : \pi_1 Y \to \pi_1 X$ is an injection.*

(1) is proved in (2.5). In order to deduce (2) from (1) one has to show that X has the homotopy type of a finite simplicial complex — see [33, II.5]. (3) is proved in [33, II.4].

Convexity and its consequences

One of the most basic properties determining the nature of CAT(0) spaces is the convexity of the metric.

Proposition 2.4 *If X is a CAT(0) space, then the metric on X is convex (in the sense defined above).*

Proof First assume that we are given geodesics $c, c' : [0,1] \to X$ with $c(0) = c'(0)$. Consider a comparison triangle $\overline{\Delta} \subset \mathbb{E}^2$ for $\Delta(c(0), c(1), c'(1))$. Given $t \in [0,1]$, elementary Euclidean geometry tells us that $d(\overline{c(t)}, \overline{c'(t)}) = t\, d(\overline{c(1)}, \overline{c'(1)}) = t\, d(c(1), c'(1))$. And by the CAT(0) inequality, $d(c(t), c'(t)) \leq d(\overline{c(t)}, \overline{c'(t)})$. Hence we obtain $d(c(t), c'(t)) \leq t\, d(c(1), c'(1))$.

In the general case, we introduce the linearly reparametrized geodesic $c'' : [0,1] \to X$ with $c''(0) = c(0)$ and $c''(1) = c'(1)$. By applying the preceding special case, first to c and c'' and then to c' and c'' with reversed orientation, we obtain: $d(c(t), c''(t)) \leq t\, d(c(1), c''(1))$ and $d(c''(t), c'(t)) \leq (1-t)\, d(c''(0), c'(0))$. Hence,

$$d(c(t), c'(t)) \leq d(c(t), c''(t)) + d(c''(t), c'(t)) \leq t\, d(c(1), c'(1)) + (1-t)\, d(c(0), c'(0)),$$

as required. □

Remark 2.5 *Let X be a CAT(0) space, fix $x_0 \in X$ and define a homotopy $H : X \times [0,1] \to X$ by $H(x,t) = c_x(t)$, where $c_x : [0,1] \to X$ is a constant-speed parametrization of the geodesic connecting x_0 to x. It follows from the above proposition that H is a continuous map. Thus X is contractible.*

I shall not include many proofs in this lecture (all relevant proofs can be found in Part II of [33]). The above proof is given so as to illustrate the nature of elementary arguments using comparison triangles. One of the hallmarks of the theory of CAT(0) spaces is that by concatenating many such elementary arguments one arrives at non-trivial results remarkably quickly. This remark applies in particular to the following results.

Recall that a subset C of a CAT(0) spaces X is said to be *convex* if the geodesic in X joining each pair of points $x, y \in C$ lies entirely in C.

Proposition 2.6 *Let X be a complete space. numdepth ¿3 oodeep*

1. *If $C \subseteq X$ is closed and convex, then for every $x \in X$ there exists a unique point $\pi(x) \in C$ such that $d(x, \pi(x)) = d(x, C) := \inf_{y \in C} d(x,y)$. Moreover the map π does not increase distances.*
2. *Every bounded subset of X has a unique circumcentre.*

Following the discussion in (1.31), from 2.6(2) we deduce:

Corollary 2.7 *numdepth ¿3 oodeep*

1. *If a group Γ acts by isometries on a complete CAT(0) space X, then every finite subgroup of Γ fixes a point of X.*
2. *If the action of Γ is proper and cocompact, then Γ contains only finitely many conjugacy classes of finite subgroups.*
3. *If Y is a compact non-positively curved space, then $\pi_1 Y$ is torsion-free.*

The action of the fundamental group of any space on the universal cover is free, so part (3) of the above corollary follows from part (2) and Theorem 2.2.

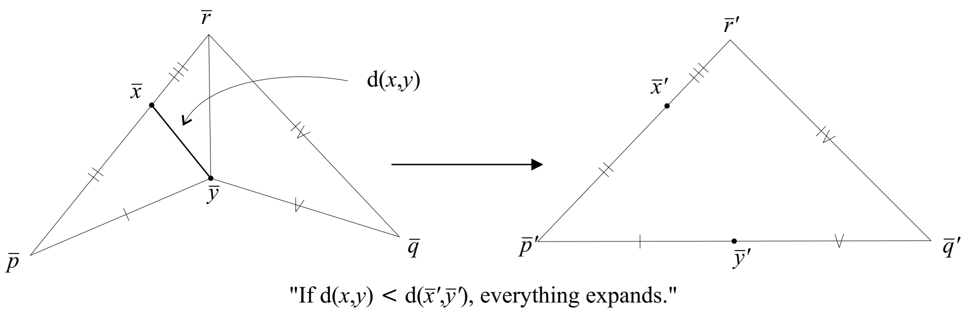

Figure 2.2: The idea behind the Flat triangle lemma

Flat subspaces of CAT(0) spaces

The two most classical examples of CAT(0) spaces are Euclidean space $\mathbb{E}^n$ and hyperbolic space $\mathbb{H}^n$. As every undergraduate knows, the flavours of the geometry enjoyed by these two spaces are quite distinct. Given an arbitrary CAT(0) space, then, one might wonder whether it is essentially hyperbolic in nature, or more Euclidean. More precisely, one might ask whether it contains significant flat (Euclidean) subspaces, and if it does not then can one make precise the idea that it is morally speaking negatively curved? We shall see that in the case of cocompact CAT(0) spaces such a dichotomy does indeed exist (2.10).

Our search begins with a remarkable observation of Alexandrov: given a geodesic triangle Δ in a CAT(0) space X, compare Δ with its comparison triangle $\overline{\Delta} \subset \mathbb{E}^2$ as in (1.18); if there is *any* non-trivial equality $d(x, y) = d(\overline{x}, \overline{y})$, then one can fill Δ to get an isometrically embedded triangular Euclidean disc in X.

To state this more precisely, we define the *convex hull* of a subset A of a geodesic space X to be the intersection of all of the convex subspaces of X containing A. (Despite what our Euclidean intuition may suggest, the convex hull of a geodesic triangle in a CAT(0) space may not be 2-dimensional. For example, in the complex hyperbolic plane $\mathbb{C}\mathrm{H}^2$ the convex hull of three points in general position is 4-dimensional — see [33, II.10].)

Proposition 2.8 (Flat triangle lemma) *Let $\Delta(p, q, r)$ be a geodesic triangle in a CAT(0) space, and let $\Delta(\overline{p}, \overline{q}, \overline{r}) \subset \mathbb{E}^2$ be its comparison triangle. If there exist points $\overline{x}$ in the interior of $[\overline{p}, \overline{r}]$ and $\overline{y} \in [\overline{p}, \overline{q}]$ with $\overline{y} \neq \overline{p}$ such that $d(x, y) = d(\overline{x}, \overline{y})$, then the triangle $\Delta(p, q, r)$ is flat; more precisely, the convex hull of Δ in X is isometric to the convex hull of $\overline{\Delta}$ in $\mathbb{E}^2$.*

A *geodesic line* in a metric space X is, by definition, a map $c : \mathbb{R} \to X$ such that $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in \mathbb{R}$. Two geodesic lines c, c' in X are said to be *asymptotic* if there exists a constant K such that $d(c(t), c'(t)) \leq K$ for all $t \in \mathbb{R}$. We remind the reader that if a function $\mathbb{R} \to \mathbb{R}$ is convex and bounded, then it is constant.

Theorem 2.9 (The flat strip theorem) *Let X be a CAT(0) space, and let c :*

$\mathbb{R} \to X$ *and* $c' : \mathbb{R} \to X$ *be geodesic lines in* X*. If* c *and* c' *are asymptotic, then the convex hull of* $c(\mathbb{R}) \cup c'(\mathbb{R})$ *is isometric to a flat strip* $\mathbb{R} \times [0, D] \subset \mathbb{E}^2$*.*

Proof [33] Let π be the projection of X to the closed convex subspace $c(\mathbb{R})$. By reparametrizing if necessary, we may assume that $c(0) = \pi(c'(0))$.

The function $t \mapsto d(c(t), c'(t))$ is convex, non-negative and bounded, hence constant, equal to D say. Similarly, for all $a \in \mathbb{R}$, the function $t \mapsto d(c(t+a), c'(t))$ is constant. In particular $d(c(a+t), c'(t)) = d(c(a), c'(0)) \geq d(c(0), c'(0))$, and hence $\pi(c'(t)) = c(t)$ for all t.

Given $t < t'$, we consider the quadrilateral in X which is the union of the geodesic segments $[c(t), c(t')]$, $[c(t'), c'(t')]$, $[c'(t'), c'(t)]$ and $[c'(t), c(t)]$. We divide this into two geodesic triangles by introducing the diagonal $[c(t), c'(t')]$. Let $t_m = (t + t')/2$. By the triangle inequality, $d(c(t_m), c'(t_m)) \leq d(c(t_m), m) + d(m, c'(t_m))$, where m is the midpoint of $[c(t), c'(t')]$. And by hypothesis, $d(c(t_m), c'(t_m)) = D$.

Consider the parallelogram in the Euclidean plane obtained by joining comparison triangles $\overline{\Delta}(\overline{c(t)}, \overline{c'(t')}, \overline{c'(t)})$ and $\overline{\Delta}(\overline{c(t)}, \overline{c'(t')}, \overline{c(t')})$ along the side $[\overline{c(t)}, \overline{c'(t')}]$. We have $d(\overline{c(t_m)}, \overline{c'(t'_m)}) = d(\overline{c(t_m)}, \overline{m}) + d(\overline{m}, \overline{c'(t'_m)}) = D$. But by the CAT(0) inequality, $d(\overline{c(t_m)}, \overline{m}) \geq d(c(t_m), m)$ and $d(\overline{m}, \overline{c'(t'_m)}) \geq d(m, c'(t'_m))$. Thus we must have equality throughout, and by the flat triangle lemma, the convex hull of $\{c(t), c'(t), c(t'), c'(t')\}$ is isometric to a Euclidean rectangle.

Sending $t \to -\infty$ and $t' \to \infty$, it follows that the map $\mathbb{R} \times [0, D] \to X$ which sends (t, s) to the point a distance s from $c(t)$ on the geodesic segment $[c(t), c'(t)]$ is an isometry onto the convex hull of $c(\mathbb{R}) \cup c'(\mathbb{R})$. □

By means of a (somewhat subtle) limiting and compactness argument based on the Flat triangle lemma, one can prove the following result. This result was proved in the setting of smooth manifolds by Eberlein [57]. Gromov pointed out that it should work in arbitrary CAT(0) spaces and I provided a proof of Gromov's suggestion [27]. Bowditch [24] provided an alternative proof that works in even greater generality.

Theorem 2.10 (Flat plane theorem) *Let* X *be a proper* $CAT(0)$ *space that admits a cocompact action by a group of isometries. Then* X *is* δ*-hyperbolic for some* $\delta > 0$ *if and only if it does not contain a subspace isometric to the Euclidean plane.*

Product decompositions and isometries

In view of the Flat strip theorem, the terms *parallel* and *asymptotic* are synonymous when applied to geodesic lines in a CAT(0) space. From the Flat strip theorem we deduce:

Theorem 2.11 (A product decomposition theorem) *Let* X *be a* $CAT(0)$ *space and let* $c : \mathbb{R} \to X$ *be a geodesic line. numdepth ¿3 oodeep*

1. *The union of the images of all geodesic lines* $c' : \mathbb{R} \to X$ *parallel to* c *is a convex subspace* X_c *of* X*.*

2. *Let p be the restriction to X_c of the projection on the complete convex subspace $c(\mathbb{R})$. Let $X_c^0 = p^{-1}(c(0))$. Then, X_c^0 is convex (in particular it is a CAT(0) space) and X_c is canonically isometric to the product $X_c^0 \times \mathbb{R}$.*

Proof [33] Given two points $x_1, x_2 \in X_c$ we fix geodesic lines c_1 and c_2 parallel to c, such that x_1 lies in the image of c_1 and x_2 lies in the image of c_2. Because c_1 is parallel to c_2, we can apply the Flat strip theorem and deduce that the convex hull of $c_1(\mathbb{R}) \cup c_2(\mathbb{R})$ is isometric to a flat strip; in particular it is the union of images of geodesic lines parallel to c. This proves that X_c is convex.

Given $x \in X_c$, let c_x be the unique geodesic line in X_c with $p(c_x(0)) = c(0)$.

Let $j : X_c^0 \times \mathbb{R} \to X_c$ be the bijection defined by $j(x,t) = c_x(t)$. Using the Flat strip theorem, it is clear that this map is an isometry provided $d(x,x') = d(c_x(\mathbb{R}), c_{x'}(\mathbb{R}))$ for all $x, x' \in X_c^0$. To complete the proof one has to check that, given three geodesic lines $c_i : \mathbb{R} \to X$, $i = 1,2,3$, if the union of each pair of these lines is isometric to the union of two parallel lines in $\mathbb{E}^2$, and if $p_{i,j}$ is the map that assigns to each point of $c_j(\mathbb{R})$ the unique closest point on $c_i(\mathbb{R})$, then $p_{1,3} \circ p_{3,2} \circ p_{2,1} = p_{1,1}$, the identity of $c_1(\mathbb{R})$. (See [33, II.2.15].) □

The above splitting forms the basis of a number of important results concerning the structure of groups which act by isometries on CAT(0) spaces.

Definition 2.12 *Given an isometry γ of a metric space X, the translation length of γ is defined to be $|\gamma| := \inf\{d(x, \gamma.x) \mid x \in X\}$. We also define $Min(\gamma) := \{x \in X \mid d(x,\gamma.x) = |\gamma|\}$. If $Min(\gamma) \neq \emptyset$, then γ is called semisimple. If $|\gamma| > 0$ and $Min(\gamma) \neq \emptyset$, then γ is called a hyperbolic isometry.*

Proposition 2.13 *If a group Γ acts properly and cocompactly by isometries on a CAT(0) space X, then every element $\gamma \in \Gamma$ of infinite order acts as a hyperbolic isometry. If γ is a hyperbolic isometry of X then: numdepth ¿3 oodeep*

1. *$Min(\gamma)$ is closed and convex.*
2. *$Min(\gamma)$ is a union of parallel geodesic lines each of which is γ-invariant. (These lines are called axes for γ.)*
3. *$Min(\gamma)$ is isometric to a product $Y \times \mathbb{R}$, and the restriction of γ to $Min(\gamma)$ is of the form $(y,t) \mapsto (y, t + |\gamma|)$, where $y \in Y, t \in \mathbb{R}$.*
4. *Every isometry α which commutes with γ leaves $Min(\gamma) = Y \times \mathbb{R}$ invariant, and its restriction to $Y \times \mathbb{R}$ is of the form (α', α''), where α' is an isometry of Y and α'' is a translation of $\mathbb{R}$.*

Exercise 2.14 Prove Theorem 2.1(8).

The following result places severe restrictions on the way in which central extensions can act by isometries on CAT(0) spaces. Notice that the conditions on the action are very weak here — in particular no discreteness is assumed.

Theorem 2.15 *Let X be a CAT(0) space and let Γ be a finitely generated group acting by isometries on X. If Γ contains a central subgroup $A \cong \mathbb{Z}^n$ that acts faithfully by hyperbolic isometries (apart from the identity element), then there exists a subgroup of finite index $H \subset \Gamma$ which contains A as a direct factor.*

Proof [33] Fix $\alpha \in A$. According to 2.13(4), the action of Γ leaves $Min(\alpha) = Y \times \mathbb{R}$ invariant, and the restriction of each $\gamma \in \Gamma$ to $Y \times \mathbb{R}$ is of the form (γ', γ''), where γ' is an isometry of Y and γ'' is a translation of $\mathbb{R}$. The map $\gamma \mapsto \gamma''$ defines a homomorphism from Γ to a finitely generated group of translations of $\mathbb{R}$. Such a group of translations is isomorphic to $\mathbb{Z}^m$, for some m, so we have a surjective homomorphism $\psi : \Gamma \to \mathbb{Z}^m$. The image under ψ of A is non-trivial, because it contains α.

We compose ψ with the projection of $\mathbb{Z}^m$ onto a suitable direct summand so as to obtain a homomorphism $\phi : \Gamma \to \mathbb{Z}$ such that $\phi(A)$ is non-trivial. We then choose $a \in A$ so that $\phi(a)$ generates $\phi(A)$. Let $H_0 = \phi^{-1}(\phi(A))$ and note that H_0 has finite index in Γ. The map $\phi(a) \mapsto a$ splits $\phi|_{H_0}$, giving $H_0 = \ker\phi \times \langle a \rangle$ (since a is central) and $A = A' \times \langle a \rangle$, where $A' = A \cap \ker\phi$. By induction on m (the rank of A) we may assume that A' is a direct factor of a subgroup of finite index $H' \subset \ker\phi$. Let $H = H' \times \langle a \rangle$. □

To give a taste of how one might use a result such as this, let me mention two applications. The first shows that (2.15) obstructs a suggested approach to the construction of faithful linear representations of mapping class groups. An unpublished result of Geoff Mess (see [33, II.7]) shows that centralizers in the mapping class group $\mathcal{M}$ of closed surfaces of genus at least three are not virtually direct products. Thus (2.15) restricts the ways in which $\mathcal{M}$ can act on CAT(0) spaces (cf. [78]).

Recall that a Coxeter group ("generalized reflection group") is a group of the form $\langle s_1, \dots, s_n \mid s_i^2 = (s_i s_j)^{m_{ij}} = 1,\ i,j = 1,\dots,n \rangle$. Gabor Moussong [89] shows that every Coxeter group acts properly and cocompactly by isometries on a CAT(0) space, thus:

Corollary 2.16 *The mapping class groups of closed surfaces of genus at least three do not admit faithful representations into Coxeter groups.*

Combining Corollary 2.16 with some standard (but non-trivial) results in 3-manifold topology, in [33, II.7] we prove:

Theorem 2.17 *If a closed 3-manifold M admits a metric that has non-positive curvature (in the sense of Alexandrov), and if the centre of $\pi_1 M$ is non-trivial, then M has a finite-sheeted covering $\widehat{M} = \Sigma \times S^1$, where Σ is a closed surface.*

Flat torus theorem

We have seen (Flat plane theorem) that the existence of flat planes is the only obstruction that can prevent a cocompact CAT(0) space from being hyperbolic. But how might such planes arise? Here we find another beautiful connection between geometry and group theory, first discovered in the setting of smooth manifolds by Gromoll-Wolf [66] and Lawson-Yau [79].

Theorem 2.18 (Flat torus theorem) *Let A be a free abelian group of rank n acting properly by semi-simple isometries on a $CAT(0)$ space X. Then: numdepth ¿3 oodeep*

1. $Min(A) = \bigcap_{\alpha \in A} Min(\alpha)$ *is non-empty and splits as a product* $Y \times \mathbb{E}^n$.
2. *Every element* $\alpha \in A$ *leaves* $Min(A)$ *invariant and respects the product decomposition;* α *acts as the identity on the first factor* Y *and as a translation on the second factor* $\mathbb{E}^n$.
3. *The quotient of each* n*-flat* $\{y\} \times \mathbb{E}^n$ *by the action of* A *is an* n*-torus.*
4. *If an isometry of* X *normalizes* A*, then it leaves* $Min(A)$ *invariant and preserves the product decomposition.*
5. *If a subgroup* $\Gamma \subset Isom(X)$ *normalizes* A*, then a subgroup of finite index in* Γ *centralizes* A*. Moreover, if* Γ *is finitely generated, then* Γ *has a subgroup of finite index that contains* A *as a direct factor.*

The proof proceeds by induction on the rank of A; the base step is contain in (2.13). In the inductive step one looks at the action of A on $Min(a_1)$ where a_1 is a primitive element of A. See [33, II.7] for a complete proof, generalizations, and proofs of the following consequences.

Corollary 2.19 *If* Γ *is the fundamental group of a compact geodesic space of non-positive curvature, then numdepth ¿3 oodeep*

1. *every abelian subgroup of* Γ *is finitely generated,*
2. *virtually abelian subgroups of* Γ *satisfy the ascending chain condition,*
3. *every solvable subgroup of* Γ *is virtually abelian.*

The ideas used to prove the Product Decomposition Theorem and the Flat torus theorem can also be used to prove the following theorem, which was first proved for smooth manifolds in [66] and [79]. In the setting of CAT(0) spaces this result is due to Claire Baribaud [11]. A more general result is proved in [33, II.6.22].

Theorem 2.20 (Splitting theorem) *Let* Y *be a compact geodesic space of non-positive curvature in which every local geodesic can be extended to a locally isometric embedding of the real line. If the fundamental group of* Y *splits as a product* $\Gamma = \Gamma_1 \times \Gamma_2$ *and* Γ *has trivial centre, then* Y *splits as a product* $Y = Y_1 \times Y_2$*, isometrically, where* $\pi_1 Y_i = \Gamma_i$ *for* $i = 1, 2$.

A fundamental dichotomy in group theory?

Let me end this lecture by returning, in the light of the Flat torus theorem, to the dichotomy exposed by the Flat plane theorem.

Question 2.21 *Suppose that* Γ *is the fundamental group of a compact non-positively curved space. If* Γ *is not hyperbolic, must it contain a free abelian subgroup of rank two ?*

I suspect that the answer to this question is no, but in restricted settings the answer is yes: Bangert and Schroeder [10] proved that the answer is yes for real-analytic manifolds; Lee Mosher and I [34] proved the same for closed topological 3-manifolds (following earlier work of Mosher [88], Schroeder [101] and Buyalo [41]); and the answer is also yes in the case where Γ is a cocompact lattice in a semi-simple Lie group [83].

An intriguing possibility is that the dichotomy suggested in (2.21) may have little to do with non-positive curvature, *per se*, but instead it might be a special case of a much more general phenomenon.

Question 2.22 *Let* Γ *be a finitely presented group. Suppose that* Γ *admits an Eilenberg-Maclane space* $K(\Gamma, 1)$ *that has only finitely many cells in each dimension (equivalently, suppose that the cohomology of* Γ *is finitely generated in each dimension). If* Γ *is not hyperbolic, must it contain a subgroup of the form* $\langle x, y \mid y^{-1}x^p y = y^q\rangle$ *?*

Question 2.21 is a special case of Question 2.22. Another interesting special case of Question 2.22 is where Γ is a subgroup of a hyperbolic group.

The dichotomy suggested in Theorem 2.2 does not hold for finitely presented groups in general. Thus far two types of counterexamples are known: Noel Brady [25] showed that there exist hyperbolic groups G and finitely presented subgroups $H \subset G$ that are not hyperbolic (cf. (1.36)), and Slava Grigorchuk [65] has constructed finitely presented HNN extensions of the form $H = B*_\phi$ where B is an infinite, finitely generated, torsion group and $\phi : B \to B$ is an epimorphism.

In each of these examples, the third homology group of H is not finitely generated and therefore H does not have an Eilenberg-Maclane complex $K(H, 1)$ with finitely many 3-cells.

Lecture 3: Building CAT(0) spaces of interest in group theory

In Lecture 1 we saw how the basic decision problems of combinatorial group theory lead to the study of non-positive curvature. Lecture 2 was about the basic structure of CAT(0) spaces with emphasis on the close connections between geometry, topology and group theory in the presence of non-positive curvature. The purpose of Lecture 3 is to put some meat on the attractive theoretical skeleton that has emerged: I shall explain various examples and general methods for constructing

CAT(0) spaces, and I shall try to illustrate how such constructions can give rise to interesting new groups and new insights into well-known groups. Let me begin by mentioning one result of the latter kind.

Theorem 3.1 *Every Coxeter group has a solvable conjugacy problem.*

Surprisingly, no algebraic proof of this result is known. The only known proof relies on the theorem of Moussong that I mentioned earlier: every finitely generated Coxeter group W acts properly and cocompactly by isometries on a piecewise-Euclidean complex. In (4.5) I shall explain why groups which act in this way have a solvable conjugacy problem.

Example 3.2 *(1) A simply-connected, complete, Riemannian manifold is a* $CAT(0)$ *space if and only if all of its sectional curvatures are non-positive (see [48], [33, II.1A]). Examples include* $\mathbb{E}^n$, $\mathbb{H}^n$*, and any symmetric space of non-compact type.*

(2) Metric graphs. A metric graph (1-complex) X *consists of a set of vertices and a set of edges joining them; each edge is given a metric that makes it locally isometric to a closed interval of the real line; a (pseudo)metric is defined on* X *by setting the distance between two points equal to the length of the shortest path between them, where length is measured in the given (local) metrics on the edges.*

I encourage you to consider the many pathologies that can arise if one does not impose any restrictions on the lengths of the edges (particularly if there are vertices of infinite valence) cf. [33, I.1.9]. I also encourage you to check that if there are only finitely many different edge lengths then the metric defined above makes X *a complete geodesic space of non-positive curvature.*

Truncated hyperbolic space. Given a geodesic space X *with metric* d *and an arc-connected subspace* $Y \subset X$*, there are two metrics with which one might endow* Y*: one might simply define the distance between* $y, y' \in Y$ *to be* $d(y, y')$*, or one might define the distance to be the infimum of the lengths of paths that join* y *to* y' *in* Y *(where length is measured using* d*). The second construction is called the induced path metric on* Y*. In order to see the difference between these metrics, consider the case where* Y *is the unit circle in the Euclidean plane.*

If one removes a disjoint collection of open horoballs from real hyperbolic space $\mathbb{H}^n$ *then the induced path metric is a* $CAT(0)$ *space. This is proved in [33, II.11]. It follows that non-uniform lattices in* $SO(n, 1)$ *act properly and cocompactly by isometries on* $CAT(0)$ *spaces (namely these truncated hyperbolic spaces). The presence of nilpotent subgroups that are not virtually abelian show that non-uniform lattices in other rank 1 Lie groups do not admit such actions (by 2.1(5)) and hence the corresponding truncated symmetric spaces are not non-positively curved.*

Polyhedral complexes

The simplest examples of geodesic metric spaces that are not manifolds are provided by metric graphs. A wider and much more interesting class of spaces is provided by the higher dimensional analogues of graphs, *metric polyhedral complexes.* Roughly speaking, to construct a piecewise-Euclidean complex K one takes the disjoint union of a family of convex polyhedra from Euclidean space $\mathbb{E}^n$ and one identifies them along isometric faces. As with graphs, a (pseudo)metric on K is defined by setting the distance between each pair of points equal to the length of paths joining them, where length is measured in the given metrics on the individual cells. In my thesis [26] I proved that if the set of isometry types of the cells, denoted $Shapes(K)$, is finite, then every pair of points in K can be joined by a shortest path, in other words K is a geodesic space.

An entirely analogous construction gives piecewise hyperbolic complexes (obtained by identifying faces among a disjoint union of convex polyhedra from hyperbolic space $\mathbb{H}^n$), or piecewise spherical complexes (obtained by identifying faces among a disjoint union of convex polyhedra from the n-sphere $\mathbb{S}^n$). More generally, one can take convex polyhedra from any simply-connected manifold of constant sectional curvature, and this gives the general notion of a metric polyhedral complex (see [33, I.7]).

There are various ways of characterizing complexes whose curvature is bounded above (see [33, II.5]); we mention two of them.

Theorem 3.3 *Let K be a piecewise Euclidean or piecewise hyperbolic complex with $Shapes(K)$ finite. The following conditions are equivalent. numdepth ¿3 oodeep*

1. *K is non-positively curved.*
2. *Every $x \in K$ has a neighbourhood in which there is a unique geodesic joining each pair of points.*
3. *K satisfies the link condition.*

The *link condition* can be described as follows. First think of a vertex v of a convex polyhedron P (e.g. a solid cube) in Euclidean space of some dimension and imagine how a small (radius ε) sphere centred at v intersects P. The intersection is a small spherical polyhedron (a spherical triangle if P is a cube). Now rescale your mental picture so that this spherical polyhedron is based on a sphere of radius one rather than radius ε. It is then called the link of v in P.

If K is as in Theorem 3.3, then the *link* of a vertex $v \in K$, denoted $Lk(v)$, is obtained by taking the link of v in each of the individual cells incident at v and assembling them according to the way in which the cells intersect. Thus $Lk(v)$ is a piecewise spherical complex. It is the analogue in metric complexes of the unit tangent space in differential geometry: the points of $Lk(v)$ are the directions of geodesics issuing from v, and the distance between each pairs of points should be thought of as the angle at v between the geodesics issuing in these directions. One says that K *satisfies the link condition* if for all vertices $v \in K$ and all $x, y \in Lk(v)$, if $d(x, y) < \pi$ then there is a unique geodesic joining x to y in $Lk(v)$.

2-dimensional complexes

I shall describe a number of constructions of non-positively curved 2-dimensional complexes; further constructions are given in [33, II.5]. It is more difficult to con-

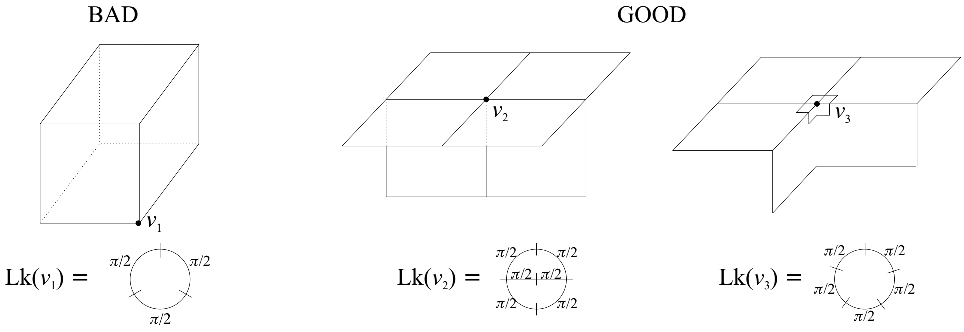

Figure 3.1: $Lk(v)$

struct examples in higher dimensions, but examples do exist, for instance Bruhat-Tits buildings of affine type (see [37] and [36]). One of the advantages of working with 2-dimensional complexes[8] is that it is easy to check the link condition.

Example 3.4 (The link condition for 2-complexes) *Let K be a piecewise Euclidean or piecewise hyperbolic 2-complex and let $v \in K$ be a vertex. Then $Lk(v)$, as defined above, is a metric graph and K is non-positively curved if and only if there does not exist a vertex such that $Lk(v)$ contains an embedded circuit of length strictly less than 2π.*

Figure 3.1 should be compared with Figure 1.1.

Squared 2-complexes

Squared 2-complexes are, by definition, piecewise Euclidean 2-complexes in which all of the 2-cells are squares of side length 1. In this simple setting the link condition reduces to the statement that the link of each vertex contains no embedded circuits of combinatorial length less than four.

Exercise Check that you thoroughly understand the following assertion: the product of any two metric graphs with edge lengths 1 is a squared 2-complex of non-positive curvature. (The links of vertices in this case are complete bipartite graphs.)

In what follows we shall use the term *tree* to mean a simply connected metric graph all of whose edges have length 1.

Theorem 3.5 *Let K be a compact, connected, non-positively curved squared complex. If the link of each vertex in K is isomorphic to a complete bi-partite graph, then the universal cover of K is the product of two trees.*

This result seems to have been discovered independently by a number of authors [115], [7], [39]. A detailed proof can be found in [35].

[8]hereafter abbreviated to "2-complexes"

This theorem relates the study of squared 2-complexes to the study of the automorphism groups of products of trees. This brings us into a very rich world, because trees are combinatorial analogues of symmetric spaces of non-compact type, and in keeping with this analogy one finds that many of the groups which act on products of trees enjoy remarkable geometric and arithmetic properties. The most striking result in this direction is the following construction due to Marc Burger and Shahar Mozes [39].

Theorem 3.6 (Burger-Mozes) *There exist finitely presented simple groups that act freely and cocompactly by isometries on a product of two trees. Moreover, one can construct such groups by amalgamating two finitely generated free groups along subgroups of finite index.*

But the number of generators and relations is rather large. Prior to the work of Burger and Mozes, my student Dani Wise had (by hand) produced many interesting examples of non-positively curved squared complexes including some whose fundamental groups have no non-trivial finite quotients [115].

Notice that by (1.13) any group Γ which acts properly and cocompactly on a product of trees in quasi-isometric to a product of finitely generated free groups, but (3.6) shows that Γ may have utterly different residual properties to products of free groups.

Ian Acheson and Dani Wise [115] (independently) noticed that work of Weinbaum [114], originally phrased in the language of small cancellation theory, essentially proves the following result.

Theorem 3.7 *The fundamental group of the complement of every tame, prime, alternating link in* $\mathbb{S}^3$ *is the fundamental group of a compact non-positively curved squared complex.*

Remark It follows from (3.12) that one can remove the hypothesis that the link is prime at the expense of replacing the conclusion "squared" by "polyhedral". The complex in (3.7) has two vertices, one edge for each region in the plane of projection and one square for each crossing. The idea of the construction is this: one regards the alternating projection as lying on a plane in $\mathbb{R}^3$ and one places one vertex a above the plane and one b below it. For each complementary region x_i of the projection one introduces an edge labelled x_i oriented from a to b. Then for each crossing in the projection of the link one attaches a square (2-cell) by gluing the boundary circuit to the path that traces out the edges $x_{i_1}, x^*_{i_2}, x_{i_3}, x^*_{i_4}$ in order, where the asterisk denotes reversed orientation and the regions $x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}$ are those that provide the corners at the crossing; they are read from above the plane in clockwise order and the strand of the link that is between x_{i_1} and x_{i_2} passes over the other strand. The natural presentation of the fundamental group of this complex is essentially Dehn's presentation of the link group [81]. (See [115] or [33, II.7] for details.)

Exercise The Hopf link consists of two circles that cannot be separated in 3-dimensional space and are such that the link has a projection with only two crossings. Show that the complex that the above construction ascribes to this link is a torus, but that this torus is not a metric product.

In [35] the geometry of non-positively curved squared complexes is used to investigate subgroups of the direct product of two free groups. In particular we give a geometric explanation of the following theorem of Baumslag and Roseblade [15].

Theorem 3.8 ([15]) *If F_1 and F_2 are free groups and H is a finitely presented subgroup of $F_1 \times F_2$, then either H is free or else it contains a subgroup of finite index $H_1 \times H_2$, where $H_1 \subset F_1$ and $H_2 \subset F_2$.*

The modified Rips construction

The following algorithmic construction gives rise to a host of interesting examples of non-positively curved complexes and allows one to translate the pathologies of arbitrary group presentations into pairs of groups $N \subset \Gamma$, where Γ is hyperbolic and N is finitely generated. In Lecture 4 I shall explain how one can exploit this transfer of pathology to expose the diverse nature of decision problems for subgroups of groups which act properly and cocompactly on CAT(0) spaces.

Theorem 3.9 *There is an algorithm that associates to every finite presentation a short exact sequence*

$$1 \to N \to \Gamma \to G \to 1,$$

where G is the group given by the presentation, N is a finitely generated group and Γ is the fundamental group of a compact, negatively curved, piecewise hyperbolic 2-complex K. Moreover one can remetrize K as a piecewise Euclidean squared complex of non-positive curvature.

This construction is based closely on an idea of E. Rips. In [99] he proved a version of this theorem in which instead of forcing Γ to be a fundamental group of the above type, he arranged for it to satisfy an arbitrarily strict small cancellation condition. The details of the adaptation given above were worked out by Wise in [116] and a slightly different account is given in [33, II.5].

In outline, Rips's idea is that given $G = \langle \mathcal{X} \mid \mathcal{R} \rangle$, one should introduce new generators $\underline{a} = \{a_1, \dots, a_n\}$, "unwrap" the old relations, and add a new set of relations U to make the subgroup generated by $\underline{a}$ normal. To "unwrap" $r \in \mathcal{R}$ we replace $r = 1$ by a relation of the form $r = w_r(\underline{a})$; let $\tilde{\mathcal{R}}$ denote the set of these new relations. To construct U, for each $x \in \mathcal{X}$ and $a_i \in \underline{a}$ we choose words $u_{i,x}$ and $u'_{i,x}$ and define $x^{-1}a_i x = u_{i,x}(\underline{a})$ and $x a_i x^{-1} = u'_{i,x}(\underline{a})$. The key to the construction is that the words $w_r, u_{i,x}$ and $u'_{i,x}$ must be long enough and sufficiently independent to ensure that

$$\Gamma := \langle \mathcal{X}, \underline{a} \mid \tilde{\mathcal{R}}, U \rangle$$

satisfies a small cancellation condition. (See [81] for background on small cancellation theory.)

In Rips's original construction one only needs two new generators, but in (3.9) one needs more in order to promote the small cancellation condition to the (more restrictive) link condition for an associated polyhedral complex.

Example 3.10 (Non-hopfian groups) *Following earlier work of Baumslag and Solitar [16] and Meier [85], in [116] Dani Wise considered the following groups:*

$$T(n) = \langle a, b, t_a, t_b \mid [a, b] = 1,\ t_a^{-1} a t_a = (ab)^n,\ t_b^{-1} b t_b = (ab)^n \rangle.$$

If $n \geq 2$, certain non-trivial commutators, for example $g_0 = [t_a(ab)t_a^{-1}, b]$, lie in the kernel of the epimorphism $T(n) \twoheadrightarrow T(n)$ given by $a \mapsto a^n, b \mapsto b^n, t_a \mapsto t_a, t_b \mapsto t_b$. Groups which admit such self-surjections with kernel are called non-Hopfian.

$T(n)$ is the fundamental group of the non-positively curved 2-complex $X(n)$ that one constructs as follows: take the (skew) torus formed by identifying opposite sides of a rhombus with sides of length n and small diagonal of length 1; the loops formed by the images of the sides of the rhombus are labelled a and b respectively; to this torus attach two tubes $S \times [0, 1]$, where S is a circle of length n; one end of the first tube is attached to the loop labelled a and one end of the second tube is attached to the loop labelled b; in each case the other end of the tube wraps n times around the image of the small diagonal of the rhombus. One checks easily that this complex satisfies the link condition.

New spaces from old

A natural way to construct interesting new metric spaces is to take a disjoint collection of known metric spaces and glue them together along subspaces. Intuitively speaking, this means that we glue the spaces by means of an equivalence relation and then define the distance between points in the resulting space by taking the infimum of the lengths (measured in the original metrics) of paths joining the points, as we did in the construction of polyhedral spaces. In order to make this description precise we need two technical devices. First, given two metric spaces X_1, X_2, we extend the given metrics to a metric on their disjoint union $X_1 \coprod X_2$ by defining $d(x, y) = \infty$ if $x \in X_1$ and $y \in X_2$. Secondly, given a metric space X and an equivalence relation $\sim$ on X, *the quotient pseudometric* $\overline{d}$ on the set of equivalence classes $X/\sim$ is given by the formula:

$$\overline{d}(\overline{x}, \overline{y}) = \inf \sum_{i=1}^{n} d(x_i, y_i),$$

where the infimum is taken over all sequences $C = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$ of points of X such that $x_1 \in \overline{x}$, $y_n \in \overline{y}$, and $y_i \sim x_{i+1}$ for $i = 1, \ldots, n-1$. ($\overline{d}$ may fail to be a metric because the distance between distinct classes $\overline{x}$ and $\overline{y}$ is zero.)

Theorem 3.11 ([33, II.11]) *Let X and A be metric spaces of non-positive curvature and suppose that A is compact. Let $\phi_1, \phi_2 : A \to X$ be local isometries and let $\overline{X} = X \coprod (A \times [0, 1])/\sim$, where $\sim$ is generated by $(a, 0) \sim \phi_1(a)$ and $(a, 1) \sim \phi_2(a)$. Then the quotient pseudometric on $\overline{X}$ is actually a metric and $\overline{X}$ is non-positively curved.*

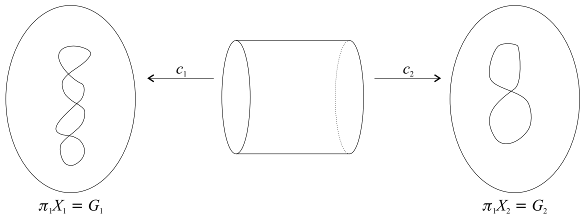

Figure 3.2: Proof of Corollary 3.12

We give one application to indicate how this result can be used to mimic group theoretic constructions in the world of non-positive curvature.

Corollary 3.12 *If G_1 and G_2 act properly and cocompactly by isometries on some $CAT(0)$ space, then so too does any amalgamated free product of the form $G_1 *_C G_2$, where C contains a cyclic subgroup of finite index.*

Proof We sketch the proof in the torsion-free case with $C = \mathbb{Z}$; see [33, II.11] for the general case. For $i = 1, 2$ let X_i be a compact non-positively curved space with $\pi_1 X_i = G_i$. The idea of the proof is shown in Figure 3.2.

The generators of the cyclic subgroups identified by the amalgamated free product can be represented by closed geodesics (i.e. locally isometric embeddings of circles), $c_1 : S_1 \to X_1$ and $c_2 : S_2 \to X_2$. By scaling the metric on X_1, we may assume that these geodesics have the same length. Then we can apply the theorem with $X = X_1 \coprod X_2$ and $A = S_1 = S_2$ and $\phi_i = c_i$ for $i = 1, 2$. The Seifert - van Kampen theorem implies that $\pi_1 \overline{X}$ is the given amalgamated free product $G_1 *_{\mathbb{Z}} G_2$. $\square$

Corollary 3.12 does not extend (without further hypotheses) to amalgamated free products along finitely generated free groups (or free abelian) groups of higher rank. The question of when these more general amalgamations do act nicely on CAT(0) spaces is discussed at some length in [33, III]. Note too that in the light of 2.1(8), we know that $\mathbb{Z} *_{\mathbb{Z}} = \langle x, t \mid t^{-1} x t = x^2 \rangle$ provides a counterexample to the most naive HNN analogue of (3.12).

An embedding theorem

The following is one a number of embedding theorems of a similar ilk proved in [31].

Theorem 3.13 *Every compact, connected, non-positively curved space X admits an isometric embedding into a compact, connected, non-positively curved space $\overline{X}$ such that $\overline{X}$ has no non-trivial finite-sheeted coverings (equivalently $\pi_1 \overline{X}$ has no non-trivial finite quotients). If X is a polyhedral complex of dimension $n \geq 2$, then one can arrange for $\overline{X}$ to be a complex of the same dimension.*

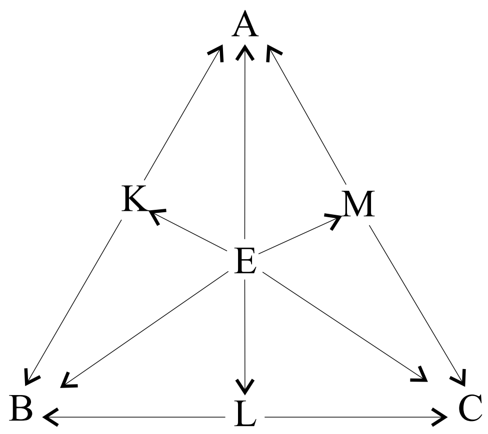

Figure 3.3: A triangle of groups

Given an arbitrary group Γ_0 with finite generating set $\{a_1, \ldots, a_n\}$, in order to embed Γ_0 in a group with no non-trivial finite quotients one can proceed as follows. First, to reduce to the case where the a_i have infinite order, replace Γ_0 by $\Gamma = \Gamma_0 * \langle t \rangle$ and $\{a_1, \ldots, a_n\}$ by $\{t, ta_1, \ldots, ta_n\}$. Let G be a torsion-free group (such as that in (3.6)) that has no finite quotients. Form $\Gamma *_{\mathbb{Z}} G$ by identifying a_1 with an element of infinite order in G, then identify the resulting group with another copy of G, this time identifying a_2 with an element of infinite order in this second copy of G. Repeat a total of n times. In the group that one obtains, Γ amalgamated with n copies of G, each along a copy of $\mathbb{Z}$, all of the obvious generators have been identified with something that must have trivial image in every finite quotient.

In the light of (3.6) and (3.12), the above operations can be carried out within the class of fundamental groups of compact non-positively curved spaces. Moreover, by making the gluing tubes sufficiently long one can ensure that the natural embedding of X into the final space is isometric. This proves (3.13). An alternative proof, which uses the elementary construction from (3.10) instead of (3.6), is given in [31].

3.14 Complexes of groups

I want to mention one other general technique for constructing group actions on CAT(0) spaces: complexes of groups in the sense of Haefliger. The foundations of the subject are rather technical, so I am afraid that I can only give a rather vague outline of the theory — see [71, 72] and [33] (II.12) and (III) for details.

To illustrate the construction, let's consider *triangles of groups*. A triangle of groups is a diagram of groups and monomorphisms as shown is Figure 3.3.

One way in which such diagrams are obtained is by recording the stabilizers of vertices, edges and the 2-cell in a strict fundamental domain for the simplicial action of a group Γ on a simply connected simplicial 2-complex X; these stabilizers are called the *local groups* of the complex. If the triangle of groups arises in this way, then by taking the pushout of the diagram in the category of groups one recovers

Γ, and hence it is reasonable to call Γ the *fundamental group* of the triangle of groups. Moreover, by taking the geometric realization of the poset of the cosets of the local groups in Γ, one can recover the complex X and the action of Γ on it (see [71], [108], [26]).

A similar process of recovery works for more general group actions on complexes where there is a strict fundamental domain, and is explained in [72] and [33, II.12]. In this more general setting, the stabilizers are arranged not into a triangle of groups but into a complex of groups in the sense of Haefliger. A more sophisticated notion is needed in order to handle actions which do not have a strict fundamental domain (see [72] or [33, III.C]).

The theory of complexes of group generalizes the Bass-Serre theory of graphs of groups [105]. In contrast to the 1-dimensional case (graphs), complexes of groups do not in general arise as the pattern of stabilizers of a group action on a simply connected space, i.e. they may not be "developable". However, if the complex of groups is non-positively curved in a suitable sense (see [71, 72], [33, III.C], [108]), then it is indeed developable, and its fundamental group will act on a CAT(0) space in a manner dictated by the complex of groups.

3-manifolds of non-positive curvature

Let me finish this lecture by turning from complexes to manifolds. In this section I shall assume more knowledge of topology than is required elsewhere in this series.

Theorem 3.15 *Let M be a compact 3-manifold with non-empty boundary. If every 2-sphere in M bounds a ball, then M supports a metric of non-positive curvature.*

Remarks (1) Throughout this article, we use the term *non-positive curvature* in the sense of A.D. Alexandrov. In the setting of the above theorem one can always arrange for the metrics considered to be smooth Riemannian metrics on the interior of the manifold, but the boundary will not in general be locally convex — indeed if it were then it would perforce be π_1-injective (2.3) which cannot happen if $\pi_1 M$ is free.

(2) The case where the boundary of M has zero Euler characteristic was dealt with by Leeb in his thesis [80]. The argument sketched below is worked out in detail in [29].

The first step in the proof involves cutting the given manifold up according to the JSJ decomposition [75], [76], [112]. This means that one cuts along a maximal non-parallel family of π_1-embedded discs, tori and Klein bottles. Thurston's Geometrization Theorem for Haken manifolds states that the pieces each have a geometric structure. It is easy to show that each piece supports a metric of non-positive curvature such that the boundary tori and Klein bottles are convex. In order to reassemble the original manifold we must perturb the metrics on the pieces so that the pairs of tori and Klein bottles to be glued are isometric.

On the geometrically finite hyperbolic pieces, the tori and Klein bottles correspond to cusps truncated by horoballs (cf. 3.2(3)), and by truncating further down the cusp one can shrink the area of each such boundary component without changing the metric away from the cusp. One can also adjust the conformal type of each such boundary component using a standard warping technique, which is explained nicely by Viktor Schroeder in [102]. (Schroeder also tells me that there is a more detailed account by Buyalo in an appendix to the Russian edition of [118].)

Having organized the tori and Klein bottle boundary components on the geometric pieces appropriately so that they can be glued (the Seifert fibred pieces do not cause any serious problems and are dealt with in [80]), the only problem that we face in trying to reassemble the original manifold is that we must be able to perform the boundary sums (the inverse operation of cutting along discs) in a manner that preserves non-positive curvature. For this one uses gluing tubes of different types:

3.16 Gluing tubes

Suppose that we have two 3-manifolds with flat boundary components and we wish to glue them along isometric discs in these components. To achieve this we take a tube of *Euclidean type* (the product of an appropriately metrized topological disc D with $[0,1]$) and we attach the ends $D \times \{0\}$ and $D \times \{1\}$ to the discs on the boundaries of the manifolds. The resulting manifold is the boundary sum of the pieces, and if the pieces were non-positively curved, so is the result, by (3.11).

If the boundary of the pieces were not flat but instead contained an isometrically embedded hyperbolic disc (of small radius)[9], then instead of using a product cylinder one would use a tube of *hyperbolic type* (the intersection of a tubular neighbourhood of a geodesic c in $\mathbb{H}^3$ with the complement of two disjoint open half-spaces whose bounding hyperplanes are orthogonal to c).

The use of suitable tubes also allows one to combine pieces with hyperbolic patches on their boundary with pieces that have Euclidean boundary. For this one uses a tube of *mixed type*: such a tube is a compact subset of $\mathbb{H}^3$ obtained by intersecting a tubular neighbourhood of a geodesic c, with the complement of an open horoball about one endpoint of c, *and* a closed half-space (containing the open horoball) that is bounded by a hyperplane orthogonal to c. In the induced path metric, this tube is non-positively curved; both ends of the tube are convex in this metric; the end lying on the bounding hyperplane is isometric to a disc in the hyperbolic plane, and the end on the horosphere is isometric to a Euclidean disc.

The full process of reassembling the 3-manifold also requires that one combine certain scaled copies of the tubes of mixed type. See [29] for details.

Lecture 4: Decision problems

In the first half of this lecture we return to our point of departure in Lecture 1: the basic decision problems of combinatorial group theory as framed as Dehn. I shall describe how one solves the word and conjugacy problems for groups which

[9]This is what happens on boundary components of higher genus associated to geometrically finite hyperbolic pieces.

act properly and cocompactly by isometries on CAT(0) spaces and I shall describe a solution to the conjugacy problem for hyperbolic groups. I shall also discuss the isomorphism problem briefly (without proofs) and explain how it is connected to classification problems in geometry and topology, following [33, III].

In the second half of this lecture I shall present a number of recent results that relate to the following question: *Let X be a compact non-positively curved space and let $\Gamma \leq \pi_1 X$ be a subgroup. Under what circumstances can one deduce that Γ is the fundamental group of a compact non-positively curved space?*

Non-positive curvature and the word and conjugacy problems

We discussed the word problem for hyperbolic groups in Lecture 1, indeed it was the bridge that led us to into our discussion of negative curvature. Recall:

Theorem 4.1 *The following statements are equivalent for f.p. groups: numdepth ¿3 oodeep*

1. Γ *is a hyperbolic group.*
2. Γ *has a finite Dehn presentation.*
3. Γ *satisfies a linear isoperimetric inequality.*
4. Γ *satisfies a sub-quadratic isoperimetric inequality.*

There exist non-hyperbolic groups that act properly and cocompactly by isometries on CAT(0) spaces, e.g. $\mathbb{Z}^2$, so point (3) of the above theorem shows the quadratic bound in the following result is sharp.

Theorem 4.2 *If Γ acts properly and cocompactly by isometries on a CAT(0) space X, then it is finitely presented and satisfies a quadratic isoperimetric inequality.*

Proof I'll sketch the proof, following [33, III]. Fix $x_0 \in X$ and $D > 0$ such that X is the union of the balls $\gamma.B(x_0, D/3)$. General considerations of group actions (cf. 1.13) yield:

$\mathcal{A} = \{a \in \Gamma \mid d(a.x_0, x_0) \leq 2D{+}1\}$ generates Γ. Given $\gamma \in \Gamma$, if $d(x_0, \gamma.x_0) \leq 4D{+}1$ then $\gamma = a_1 \dots a_l$ for some $a_i \in \mathcal{A}$ and $l \leq 4$. Let $\mathcal{R}$ be the set of words in the free group $F(\mathcal{A})$ that have length at most 10 and represent the identity in Γ. Then $\langle \mathcal{A} \mid \mathcal{R} \rangle$ is a finite presentation for Γ.

We shall see that if a word $w = b_1 \dots b_m$, with $b_j \in \mathcal{A}^{\pm 1}$, represents $1 \in \Gamma$ then w can be expressed in the free group $F(\mathcal{A})$ as a product of at most $(D+1)m^2$ conjugates of relators $r \in \mathcal{R}$. The proof will also show that the length of the conjugating elements is bounded by a linear function of m.

To each $\gamma \in \Gamma$ we associate a word σ_γ in the generators $\mathcal{A}$ as follows. Let c_γ be the unique geodesic joining x_0 to $\gamma.x_0$ in X. For each integer $i < d(x_0, \gamma.x_0) + 1$ let $\sigma_\gamma(i) \in \Gamma$ be such that $d(c_\gamma(i), \sigma_\gamma(i).x_0) \leq D$, with $\sigma_\gamma(0) = 1$ and $\sigma_\gamma(i) = \gamma$ when i is large enough. Note that $a_i = \sigma_\gamma(i)^{-1}\sigma_\gamma(i+1) \in \mathcal{A}$. Define σ_γ to be $a_0 \dots a_n$, where n is the least integer greater than $d(x_0, \gamma.x_0)$. Note that $\gamma = \sigma_\gamma$ in Γ. We choose σ_1 to be the empty word.

We fix $\gamma \in \Gamma$ and $b \in \mathcal{A}$ and compare σ_γ with $\sigma_{\gamma'}$ where $\gamma' = \gamma b$. By appending letters a_i that represent the identity if necessary, we may write $\sigma_\gamma = a_0 \dots a_n$ and $\sigma_{\gamma'} = a'_0 \dots a'_n$, where $n = n(\gamma, \gamma') = \max\{|\sigma_\gamma|, |\sigma_{\gamma'}|\}$. Now $d(\gamma.x_0, \gamma'.x_0) \leq 2D + 1$ (because $b \in \mathcal{A}$), so from the convexity of the metric on X (2.4) we have $d(c_\gamma(i), c_{\gamma'}(i)) < 2D + 1$ for all i and hence $d(\sigma_\gamma(i).x_0, \sigma_{\gamma'}(i).x_0) < 4D + 1$. If for each i we choose a word $\alpha(i)$ of length at most 4 that is equal to $\sigma_\gamma(i)^{-1}\sigma_{\gamma'}(i)$, then $\alpha(i)^{-1}a_{i+1}\alpha(i+1){a'_{i+1}}^{-1} \in \mathcal{R}$. We choose $\alpha(n)$ to be b (the difference between γ and γ'), and α_0 to be the empty word.

If we write $p_\gamma(i)$ for the word $a_0 \dots a_i$ (the i-th prefix of σ_γ), and define $p_{\gamma'}(i)$ similarly, then we have the following equality in the free group on $\mathcal{A}$:

$$\sigma_\gamma b \sigma_{\gamma'}^{-1} = \prod_{i=1}^{n(\gamma,\gamma')} p_{\gamma'}(i-1) \left[\alpha(i-1)^{-1} a_i \alpha(i) {a'_i}^{-1}\right] p_{\gamma'}(i-1)^{-1}.$$

Each of the words in square brackets belongs to $\mathcal{R}$.

Finally, we consider the given word $w = b_1 \dots b_m$ that represents the identity in Γ. Let $\gamma_0 = 1$ and let $\gamma_j \in \Gamma$ be the element represented by $b_1 \dots b_j$. Note $\gamma_m = 1$ and $\sigma_{\gamma_m} = \sigma_{\gamma_0}$ is the empty word. By definition, in $F(\mathcal{A})$ we have the equality

$$w = \prod_{j=1}^{m} \sigma_{\gamma_{j-1}} b_j \sigma_{\gamma_j}^{-1}.$$

By replacing each factor on the right hand side of this equality with the right hand side of the previous equation (with $\gamma = \gamma_{j-1}$, $b = b_j$, and $\gamma' = \gamma_j$), we obtain the desired quadratic inequality. (To check the bounds on the number of factors note that $d(\gamma_j.x_0, x_0) \leq (2D+1)|w|/2$.) □ □

Remarks (1) In the course of the preceding proof we proved that every relation $w = 1$ is a consequence of the given relations $\mathcal{R}$.

(2) The idea behind the preceding proof becomes more transparent if one translates it into the language of van Kampen diagrams (cf. [28]).

The conjugacy problem

The discussion in this section follows [33, III]. Many of the basic ideas can be found in [68, 69].

The way in which we shall solve the conjugacy problem for groups that act properly and cocompactly by isometries on CAT(0) spaces is motivated by the following geometric observation: given $l > 0$, if two loops of length $\leq l$ are freely homotopic in a complete non-positively curved space, then one can continuously deform one to the other through intermediate loops of length $\leq l$. Free homotopy classes of loops in a space X correspond to conjugacy classes in $\pi_1 X$, and quasifying the above observation we are led to the following definition in which the elements $w_i^{-1} u w_i \in \Gamma$ are the analogues of the intermediate loops in the free homotopy.

Definition 4.3 (q.m.c.) *A group* Γ *with finite generating set* $\mathcal{A}$ *is said to have the quasi-monotone conjugacy property (q.m.c) if there is a constant* $K > 0$ *such that if two words* $u, v \in F(\mathcal{A})$ *are conjugate in* Γ, *then there is a word* $w = a_1 \dots a_n$ *with* $a_i \in \mathcal{A}^{\pm 1}$, *such that* $w^{-1}uw = v$ *and* $d(1, w_i^{-1}uw_i) \leq K \max\{|u|, |v|\}$ *for* $i = 1, \dots, n$, *where* $w_i = a_1 \dots a_i$.

The existence of K does not depend on the choice of generating set $\mathcal{A}$ but its value does.

Proposition 4.4 *If a group* Γ *acts properly and cocompactly by isometries on a* $CAT(0)$ *space, then it has q.m.c.*

Proof[Sketch] Suppose that Γ acts on X. Fix $x_0 \in X$ and let $\mathcal{A}$ and σ_γ be as in the proof of (4.2). Given words u and v such that $\gamma^{-1}u\gamma = v$ in Γ, one verifies that $w := \sigma_\gamma$ satisfies the requirements of (4.3) (with respect to a constant that depends only on the parameters of the quasi-isometry $\gamma \mapsto \gamma.x_0$ described in (1.13)). To perform this verification, one uses the convexity of the metric on X (2.4) to compare the geodesic quadrilateral in X with vertices $\{x_0, \gamma.x_0, u.x_0, \gamma u.x_0 = v\gamma.x_0\}$ to a quadrilateral Q in the Cayley graph $\mathcal{C}_{\mathcal{A}}(\Gamma)$. The vertices of Q are $\{1, \gamma, u, \gamma u = v\gamma\}$, two of its sides are labelled σ_γ, and the other two sides are labelled u and v. □

4.5 An algorithm to determine conjugacy

Let Γ be a group with finite generating set $\mathcal{A}$. Suppose that Γ has a solvable word problem and also has the q.m.c. property. Let the constant K be as in (4.3).

For each positive integer n, we consider the set $B(n)$ of words in $F(\mathcal{A})$ that have length at most n. Because Γ has a solvable word problem, given a pair of words $v_1, v_2 \in B(n)$ one can decide if there exists $a \in \mathcal{A}^{\pm 1}$ such that $a^{-1}v_1a = v_2$ in Γ; if such an a exists we write $v_1 \sim v_2$.

Consider the (algorithmically constructed) finite graph $\mathcal{G}(n)$ with vertex set $B(n)$ that has an edge joining v_1 to v_2 if and only if $v_1 \sim v_2$. The q.m.c. property says precisely that two words u and v are conjugate in Γ if and only if u and v lie in the same path connected component of $\mathcal{G}(n)$, where $n = K \max\{|u|, |v|\}$. Thus we may decide if u and v represent conjugate elements of Γ.

The conjugacy problem for hyperbolic groups

Hyperbolic groups also satisfy that q.m.c. condition and hence have a solvable conjugacy problem. I shall describe a more efficient way to solve the conjugacy problem in this case. (David Epstein has recently discovered an even quicker algorithm.)

In a *free group* it is easy to solve the conjugacy problem. By definition, a word $w = a_0 \dots a_n$ in the free group on $\mathcal{A}$ is *cyclically reduced* if $a_i^{-1} \neq a_{i+1}$ for $1 = 0, \dots, n-1$, and $a_0^{-1} \neq a_n$. Given two words u and v, in order to check if they define conjugate elements of $F(\mathcal{A})$ one cyclically reduces both words and then looks to see if one of the resulting words is a cyclic permutation of the other.

We wish to quasify this simple solution to the conjugacy problem in order to solve the conjugacy problem for hyperbolic groups. With this is mind, note that a word in $F(\mathcal{A})$ is cyclically reduced if and only if it and all of its cyclic permutations are geodesic words. And recall (1.26) that geodesics in hyperbolic groups are well-approximated by local-geodesics. These facts motivate:

Proposition 4.6 *Let Γ be a group that is δ-hyperbolic with respect to the finite generating set $\mathcal{A}$. There exist constants L and K, depending only on δ, such that if $u, v \in F(\mathcal{A})$ represent conjugate elements of Γ, and if u, v and all of their cyclic permutations are L-local geodesics, then numdepth ¿3 oodeep*

1. $\max\{|u|, |v|\} \leq K$, *or else*
2. *there exists a word $w \in F(\mathcal{A})$ of length at most K such that $w^{-1}u'w = v'$ in Γ, where u' and v' are cyclic permutations of u and v.*

See [33, III] for a proof.

4.7 An algorithm to decide conjugacy in hyperbolic groups

Let Γ be a group that is δ-hyperbolic with respect to the finite generating set $\mathcal{A}$. Given two words u and v over the alphabet $\mathcal{A}^{\pm 1}$, one looks for subwords of length $\leq L$ in u, v and their cyclic permutations that are not geodesic, where L is the constant of (4.6). If such a subword is found, one replaces it by a geodesic word representing the same group element. One continues this process until u and v have been replaced by (conjugate) words u' and v' all of whose cyclic permutations are L-local geodesics. Let u' and v' be the resulting words. The preceding proposition provides a *finite* set of words Σ such that u is conjugate to v in Γ if and only if $w^{-1}u'w = v'$ in Γ for some $w \in \Sigma$. And using Dehn's algorithm one can decide whether any of the putative relations $w^{-1}u'w = v'$ is actually valid in Γ.

The conjugacy problem for subgroups

Suppose that Γ acts properly and cocompactly by isometries on a CAT(0) space. We shall see a little later that Γ may contain finitely presented subgroups with an unsolvable conjugacy problem. On the other hand:

Theorem 4.8 ([29]) *If Γ acts properly and cocompactly by isometries on a CAT(0) space and if Q is a group in which the generalized word problem is solvable, then the kernel of any homomorphism $\Gamma \to Q$ has a solvable conjugacy problem.*

A key point in the proof of this result is that there is an algorithm which given $\gamma \in \Gamma$ will compute a finite set of generators for $C_\gamma(\Gamma)$; this uses ideas of Hamish Short [107].

Isomorphism problems

The ideas used in the proof of the following beautiful theorem of Zlil Sela [104] have not been discussed in these notes.

Theorem 4.9 *The isomorphism problem is solvable among torsion-free hyperbolic groups.*

Distinguishing among non-positively curved manifolds

Closed 2-manifolds were classified in the nineteenth century — they are determined up to homeomorphism by orientability and Euler characteristic. If Thurston's Geometrization Conjecture [112] is true then the homeomorphism problem for compact 3-manifolds is also solvable. In other words, there is an algorithm which takes as input pairs of compact 3-manifolds and answers YES or NO, after a finite amount of time, according to whether or not the manifolds are homeomorphic. (We are implicitly assuming that the manifolds under consideration are described as finite objects. For the sake of argument, let us suppose that they are specified as finite simplicial complexes.)

For each $n \geq 4$, there is an algorithm that associates to any finite presentation $\mathcal{P}$ a closed n-manifold with fundamental group $|\mathcal{P}|$. Using this fact, Markov [84] proved that there does not exist an algorithm to decide homeomorphism among (smooth, PL or topological) manifolds in general.

On the other hand, if one restricts one's attention to manifolds of negative curvature then the homeomorphism problem is solvable. To see this one needs the following deep topological rigidity theorem of Farrell and Jones [61].

Theorem 4.10 *Let $n \geq 5$ and let M and N be closed non-positively curved n-manifolds. If $\pi_1 M \cong \pi_1 N$, then M and N are homeomorphic.*

As Sela pointed out in [104], combining (4.9) and (4.10) one gets:

Theorem 4.11 *Let $n \geq 5$ be an integer. There exists an algorithm which takes as input two compact n-manifolds that support metrics of negative curvature, and which (after a finite amount of time) will stop and answer YES or NO according to whether or not the manifolds are homeomorphic.*

Cautious readers should interpret (4.12) as a statement regarding the homeomorphism problem for recursive classes of negatively curved manifolds.

Question 4.12 *Is there an algorithm to decide homeomorphism among closed non-positively curved manifolds in each dimension $n \geq 4$?*

I strongly suspect that the answer to this question is no. In the light of (4.11), in order to provide a negative answer one should show that the isomorphism problem for the fundamental class of such manifolds is unsolvable. For the moment I cannot even prove this with manifolds replaced by complexes. The only result in this direction is:

Theorem 4.13 ([13]) *There is a closed polyhedral manifold M of non-positive curvature and a recursive sequence of covering spaces $\widehat{M}_i \to M$, $i \in \mathbb{N}$, with each $\pi_1 \widehat{M}_i$ finitely presented, such that there does not exist an algorithm to determine whether or not M_i is homotopy equivalent to M_0.*

The subgroup question

Let me now turn to the basic question that I articulated at the beginning of this lecture:

Question 4.14 (The subgroup question[10]) *Let X be a compact non-positively curved space and let $\Gamma \leq \pi_1 X$ be a subgroup. Under what circumstances can one deduce that Γ is the fundamental group of a compact non-positively curved space?*

One might regard this question as a special case of the broader question: might there exist an algebraic characterization of the groups Γ that act properly and cocompactly by isometries on CAT(0) spaces? However (4.14) has other facets. For example, since any compact non-positively space X has the homotopy type of $K(\pi_1 X, 1)$, Question (4.14) can be rephrased as: under what circumstances does a finitely presented covering space of a non-positively curved space have the homotopy type of a compact non-positively curved space? In turn, this question is related to the existence of compact cores, in the sense of [103].

Subgroups of hyperbolic groups

Recall the modified Rips construction (3.9). *In the following discussion we shall maintain the notation established in (3.9).*

We first apply (3.9) in tandem with a theorem of Robert Bieri [19]:

Theorem 4.15 *If a normal subgroup of a group of cohomological dimension* 2 *is finitely presented, then that subgroup is either free or of finite index.*

It is not hard to show that the group N in the Rips construction (3.9) is not free if G is infinite and therefore:

Corollary 4.16 *In the notation of (3.9), the subgroup N is finitely presentable if and only if G is finite.*

Remarks (1) Baumslag, Miller and Short [14] observed that since there is no algorithm to decide if a group given by an arbitrary finite presentation is finite, (4.16) can be used to show that there is no algorithm to decide if a finitely generated subgroup of a hyperbolic group is finitely presented. See [14] for further results in this vein.

(2) We have just seen that one can produce finitely generated subgroups of hyperbolic groups that are not finitely presented and hence not hyperbolic. It is much more difficult to construct finitely presented subgroups of hyperbolic groups which are not themselves hyperbolic. At present the only construction of such subgroups is that given by Noel Brady [25]. In [62] Steve Gersten shows that if G is a hyperbolic group of cohomological dimension two, then every finitely presented subgroup of G is also hyperbolic.

[10] In order to allow torsion, one ought to restate this question in terms of proper cocompact actions by groups of isometries. I have chosen the simpler form for expository reasons.

Positive answers to the subgroup question in low dimensions

In low dimensions there are clear positive answers to the subgroup closure question (4.14). If X is 1-dimensional then in order to get the desired action it is sufficient to assume that $\Gamma \subset \pi_1 X$ is finitely generated — subgroups of free groups are free! In (4.16) we saw that in dimension 2 it is not sufficient to assume that Γ is finitely generated — if Γ is to be the fundamental group of a compact non-positively curved space then it must be finitely presented. However the absence of finite presentability turns out to be the only obstruction in dimension 2. This follows from a very general result that I learnt from Peter Shalen:

Theorem 4.17 (Subgroup closure theorem) *Let $\mathbb{K}$ be a class of combinatorial CW complexes that is closed under passage to connected subcomplexes and covering spaces. If $K \in \mathbb{K}$ and H is a finitely presented subgroup of $\pi_1 K$, then there is a compact $X \in \mathbb{K}$ of dimension at most 2 such that $\pi_1 X \cong H$.*

By definition, a combinatorial map between CW complexes is one that takes open cells homeomorphically onto cells. A complex is called combinatorial if all of its attaching maps are combinatorial.

When endowed with the unique path metric that makes the covering map a local isometry, a covering of any non-positively curve space is obviously non-positively curved. Thus one can apply the above theorem to any class of non-positively curved complexes that is closed under passage to subcomplexes. It is clear from the link condition (3.4) that 2-dimensional complexes satisfy this condition, thus:

Corollary 4.18 *If X is a compact non-positively curved 2-complex, then every finitely presented subgroup of $\pi_1 X$ is also the fundamental group of such a complex.*

One proves the Subgroup closure theorem by means of a tower argument. Towers were developed in the context of 3-manifolds by C.D. Papakyriakopoulos [94] and adapted to the setting of combinatorial complexes by Jim Howie [74].

Definition 4.19 (of a tower) *A tower map (of height h) is a map $g : K_h \to L_0$ between connected CW complexes that admits a decomposition*

$$g = i_0 \circ p_1 \circ \cdots \circ p_h \circ i_h,$$

where each $i_r : K_r \to L_r$ is the inclusion of a connected subcomplex, and each $p_r : L_r \to K_{r-1}$ is a connected covering. A tower lifting of a continuous map $f : Y \to L$ of CW-complexes is a factorization $f : Y \xrightarrow{f'} K \xrightarrow{g} L$ where g is a tower map.

Howie [74, 3.1] gives an integer valued measure of complexity for combinatorial maps that is reduced by passage to proper coverings. Thus he proves:

Lemma 4.20 *If Y and L are connected CW complexes and Y is compact, then every combinatorial map $f : Y \to L$ has a (maximal) tower lift $f : Y \xrightarrow{f'} K \xrightarrow{g} L$ with $f'_* : \pi_1 Y \to \pi_1 K$ surjective.*

One uses (4.20) in tandem with the following enhanced version of van Kampen's Lemma [77], a detailed proof of which can be found in [33]; see also [62].

Lemma 4.21 *If X is a combinatorial complex, G is a finitely presented group and $\psi : G \to \pi_1(X, x_o)$ is a homomorphism, then there exists a compact 2-complex Y with $\pi_1(Y, y_o) \cong G$ and a combinatorial map $f : Y \to X$ such that $f(y_o) = x_o$ and the following diagram commutes:*

$$\begin{array}{ccc} G & \xrightarrow{\quad\psi\quad} & \pi_1(X, x_0) \\ \downarrow \cong & & \Vert \\ \pi_1(Y, y_0) & \xrightarrow{\quad f_*\quad} & \pi_1(X, x_0). \end{array}$$

When taken together, these lemmas prove the Subgroup closure theorem.

3-manifolds

The examples described in (4.23) show that (4.18) cannot be extended to complexes of any dimension greater than 2. But in the setting of manifolds there is a subgroup closure result in dimension 3.

Theorem 4.22 *If M is a compact non-positively curved 3-manifold, then every finitely generated subgroup of $\pi_1 M$ is the fundamental group of a compact non-positively curved 3-manifold.*

This theorem is proved in [29]; it is a consequence of Scott's Compact Core Theorem [103] and (3.15). The manifolds considered may have boundary.

Cohomological obstructions

Example 4.23 *Let F be a free group of rank two, let P_n be the direct product of n copies of F and let $h_n : P_n \to \mathbb{Z}$ be a map whose restriction to each factor is a surjection. Let $SB_n = \ker h_n$.*

Following earlier work of Stallings [109] on the case $n = 3$, in [20] Bieri showed that if $n \geq 3$ then SB_n is finitely presented and that there is an Eilenberg-Maclane space $K(SB_n, 1)$ with a finite $(n-1)$-skeleton, but $H^n(SB_n, \mathbb{Z})$ is not finitely generated and hence there is no $K(SB_n, 1)$ with a finite n-skeleton. It follows from 2.1(7) that SB_n is not the fundamental group of any compact non-positively curved space. (It does however satisfy the other properties listed in (2.1) — see [30].)

Bad subgroups via fibre products

In this section I shall explain how Baumslag, Miller, Short and I exploited the modified Rips construction (3.9) in [13].

The Rips construction encodes the pathologies of arbitrary finite group presentations into pairs of groups $N \subset \Gamma$, where Γ is the fundamental group of a compact negatively curved complex and N is a finitely generated subgroup. Here is an example of how this pathology gets transmitted:

Proposition 4.24 *In the notation of (3.9): if $G = \Gamma/N$ has an unsolvable word problem, then $N \subset \Gamma$ (which is finitely generated) has an unsolvable conjugacy problem.*

Proof We continue with the notation of (3.9). Fix $a \in \underline{a}$. Since Γ is torsion-free and hyperbolic, the centralizer of a is cyclic (1.36). Moreover, one can arrange for a to be the generator of its centralizer (see [33, II.5] or [13]).

Given a word w in the generators $\mathcal{X}$ one can use the relations U to rewrite $w^{-1}aw$ as a word $\hat{w}$ in the generators $\underline{a}$. If $w \in N$ then obviously $\hat{w}$ is conjugate to a in N. Conversely, if $n^{-1}an = \hat{w}$, then $n^{-1}w$ centralizes a and hence $w = na^r$ for some r. It follows that $w \in N$ if and only if $\hat{w}$ is conjugate to a in N. But w is in N if and only if $w = 1$ in $G = \Gamma/N$, and we are assuming that there is no algorithm to decide if $w = 1$ in G. □

Constructing finitely *presented* subgroups which display the sort of pathologies described in (4.24) is much harder. The key to doing so is the following technical theorem. A group G is said to be of type F_3 if there is a $K(G, 1)$ with a finite 3-skeleton.

Theorem 4.25 (The 1-2-3 theorem) *Suppose that $1 \to N \to \Gamma \overset{p}{\to} G \to 1$ is exact, and consider the fibre product:*

$$P := \{(\gamma_1, \gamma_2) \mid p(\gamma_1) = p(\gamma_2)\} \subseteq \Gamma \times \Gamma.$$

If N is finitely generated, Γ is finitely presented and G is of type F_3, then P is finitely presented.

The name of this theorem comes from the fact that the groups N, Γ and G are assumed to be of type F_1, F_2 and F_3 respectively. (Results similar to this, cast in the language of pictures, can be found in [5].)

The product of non-positively curved complexes is non-positively curved, so we may use the 1-2-3 theorem in tandem with the modified Rips construction in order to construct non-positively curved 4-dimensional complexes whose finitely presented subgroups display various pathologies.

Theorem 4.26 *There exist hyperbolic groups Γ and finitely presented subgroups $P \subset \Gamma \times \Gamma$ such that there is no algorithm to decide membership of P, and the conjugacy problem for P is unsolvable. (And Γ is the fundamental group of a compact negatively curved 2-complex.)*

A subtle argument involving (4.26), a judicious choice of HNN extension and a careful analysis of centralizers, yields:

Theorem 4.27 *There exists a compact non-positively curved 4-complex X and a recursive class of finitely presented subgroups $H_n \subset \pi_1 X$ ($n \in \mathbb{N}$) such that there is no algorithm to determine if H_n is (abstractly) isomorphic to H_0.*

By a process of relative hyperbolization one can isometrically embed X in a polyhedral manifold, hence (4.13).

Let me finish be stating a result intended to illustrate how subtle the Subgroup question (4.14) can be:

Theorem 4.28 ([29]) *There exists a closed non-positively curved manifold of dimension nine whose fundamental group has a subgroup $H \subset \Gamma$ such that: numdepth ¿3 oodeep*

1. *H has a finite $K(H, 1)$,*
2. *H satisfies a quadratic isoperimetric inequality,*
3. *H has a solvable conjugacy problem,*
4. *the centralizer $C_H(h)$ of every $h \in H$ is finitely presented, but*
5. *there exist $h \in H$ such that $C_H(h)$ does not satisfy a quadratic isoperimetric inequality, and therefore H does not act properly and cocompactly by isometries on any $CAT(0)$ space.*

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