

ON THE DIMENSION OF CAT(0) SPACES WHERE MAPPING CLASS GROUPS ACT

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ABSTRACT. Whenever the mapping class group of a closed orientable surface of genus g acts by semisimple isometries on a complete CAT(0) space of dimension less than g it fixes a point.

1. INTRODUCTION

There are at least two interesting actions of the mapping class group $\text{Mod}(\Sigma_g)$ of a closed orientable surface Σ_g on a complete CAT(0) space. The most classical of these actions comes from the action of $\text{Mod}(\Sigma_g)$ on the first homology of Σ_g : the resulting epimorphism $\text{Mod}(\Sigma_g) \rightarrow \text{Sp}(2g, \mathbb{Z})$ gives an action of $\text{Mod}(\Sigma_g)$ on the Siegel upper half space \mathbf{H}_g , which is the symmetric space for the symplectic group $\text{Sp}(2g, \mathbb{R})$. In this action the Torelli group acts trivially while the Dehn twists in non-separating curves act as neutral parabolics.

The natural action of $\text{Mod}(\Sigma_g)$ on its Teichmüller space \mathcal{T}_g gives rise to the second action. $\text{Mod}(\Sigma_g)$ preserves the Weil-Petersson metric d_{WP} on \mathcal{T}_g , which has non-positive curvature but is not complete [27, 28]. The action of $\text{Mod}(\Sigma_g)$ on the completion of $(\mathcal{T}_g, d_{\text{WP}})$ is faithful if $g > 2$ and is by semisimple isometries; all of the Dehn twists act elliptically, i.e. have fixed points. (See [5, 18] for more details.)

The (real) dimension of \mathbf{H}_g is $g(2g + 1)$ while the dimension of \mathcal{T}_g is $6g - 6$. I do not know if $\text{Mod}(\Sigma_g)$ can act by semisimple isometries on a complete CAT(0) space whose dimension is less than $6g - 6$ without fixing a point. The main purpose of this note is to prove the following weaker bound, which was announced in [5]. I do not claim that this bound is sharp.

Theorem A. *Whenever $\text{Mod}(\Sigma_g)$ acts by semisimple isometries of a complete CAT(0) space of dimension less than g , it must fix a point.*

If X is a polyhedral space with only finitely many isometry types of cells, then all cellular isometries of X are semisimple [3]. Thus Theorem A implies, in

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particular, that $\text{Mod}(\Sigma_g)$ cannot act without a global fixed point on a Bruhat-Tits building of affine type whose rank is at most g . Actions on affine buildings arise from linear representations in a manner that we recall in Section 6. Thus, as Farb points out in [13] p.44, control over such actions leads to constraints on the low-dimensional representation theory of $\text{Mod}(\Sigma_g)$.

Corollary 1. *Let K be a field and let $\rho : \text{Mod}(\Sigma_g) \rightarrow \text{GL}(g, K)$ be a representation. If K has characteristic 0 then the eigenvalues of each $\rho(\gamma)$ ($\gamma \in \Gamma$) are algebraic integers, and if K has positive characteristic they are roots of unity.*

Corollary 2. *If K is an algebraically closed field and $d \leq g$, then there are only finitely many conjugacy classes of irreducible representations $\text{Mod}(\Sigma_g) \rightarrow \text{GL}(d, K)$.*

The value of these corollaries should be weighed against the fact that no representations with infinite image are known to exist in such low degrees. Indeed it seems plausible that there are no such representations below dimension $2g$. Using our results, Louis Funar [15] recently proved that every homomorphism $\text{Mod}(\Sigma_g) \rightarrow \text{SL}(\lfloor \sqrt{g+1} \rfloor, \mathbb{C})$ has finite image.

In the proof of Theorem A, the semisimplicity hypothesis is used to force the Dehn twists to have fixed points. There are two useful ways in which this hypothesis can be weakened. First, it is enough to assume only that there are no neutral parabolics (see Section 5.2). Alternatively, one can simply assume that the Dehn twist in a certain type of separating loop has a fixed point. By exploiting this second observation we prove the following theorem.

Recall that $\gamma_3(G)$, the third term of lower central series of a group G is defined by setting $\gamma_2(G) = [G, G]$ and $\gamma_3(G) = [G, \gamma_2(G)]$. So $G/\gamma_3(G)$ is the freest possible 2-step nilpotent quotient of G , and there is a natural map $\text{Out}(G) \rightarrow \text{Out}(G/\gamma_3(G))$. In the case $G = \pi_1 \Sigma_g$, Morita [25] (cf. [19]) has identified the image of the map $\text{Mod}(\Sigma_g) \cong \text{Out}_+(\pi_1 \Sigma_g) \rightarrow \text{Out}(G/\gamma_3(G))$; it is isomorphic to an explicit semidirect product $A \rtimes \text{Sp}(2g, \mathbb{Z})$ with A abelian.

Theorem B. *The image of $\text{Mod}(\Sigma_g) \rightarrow \text{Out}(\pi_1 \Sigma_g / \gamma_3(\pi_1 \Sigma_g))$ fixes a point whenever it acts by isometries on a complete CAT(0) space of dimension less than g .*

This paper is organised as follows. In Section 2 we gather such facts as we need concerning CAT(0) spaces and their isometries. Section 3 contains a variation on the *ample duplication criterion* from [4]. This is a condition on generating sets that leads to the existence of fixed points for group actions via a bootstrap procedure; it is a refinement of the *Helly technique*, cf. [14]. In Section 4 we establish the facts that we need in order to show that the Lickorish generators for $\text{Mod}(\Sigma_g)$ satisfy this criterion. Section 5 contains the proofs of Theorems A and B. Section 6 contains a discussion of Corollaries 1 and 2.

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2. ISOMETRIES OF CAT(0) SPACES

Let X be a geodesic metric space. A geodesic triangle Δ in X consists of three points $a, b, c \in X$ and three geodesics $[a, b]$, $[b, c]$, $[c, a]$. Let $\bar{\Delta} \subset \mathbb{E}^2$ be a triangle in the Euclidean plane with the same edge lengths as Δ and let $\bar{x} \mapsto x$ denote the map $\bar{\Delta} \rightarrow \Delta$ that sends each side of $\bar{\Delta}$ isometrically onto the corresponding side of Δ . One says that X is a CAT(0) space if for all Δ and all $\bar{x}, \bar{y} \in \bar{\Delta}$ the inequality $d_X(x, y) \leq d_{\mathbb{E}^2}(\bar{x}, \bar{y})$ holds.

We refer to Bridson and Haefliger [6] for basic facts about CAT(0) spaces.

In a CAT(0) space there is a unique geodesic $[x, y]$ joining each pair of points $x, y \in X$. A subspace $Y \subset X$ is said to be *convex* if $[y, y'] \subset Y$ whenever $y, y' \in Y$.

We write $\text{Isom}(X)$ for the group of isometries of a metric space X and $\text{Ball}_r(x)$ for the closed ball of radius $r > 0$ about $x \in X$. Given a subset $H \subset \text{Isom}(X)$, we denote its set of common fixed-points

$$\text{Fix}(H) := \{x \in X \mid \forall h \in H, h.x = x\}.$$

Note that if X is CAT(0) then $\text{Fix}(H)$ is closed and convex.

The isometries of a CAT(0) space X divide naturally into two classes: the *semisimple* isometries are those for which there exists $x_0 \in X$ such that $d(\gamma.x_0, x_0) = |\gamma|$ where $|\gamma| := \inf\{d(\gamma.y, y) \mid y \in X\}$; the remaining isometries are said to be *parabolic*. A parabolic γ is *neutral* if $|\gamma| = 0$. Semisimple isometries are divided into *hyperbolics*, for which $|\gamma| > 0$, and *elliptics*, which have fixed points. If γ of X is hyperbolic then there exist γ -invariant isometric copies of \mathbb{R} in X on which γ acts as a translation by $|\gamma|$. Each such subspace is called an axis for γ .

Proposition 2.1. *Let Γ be a group acting by isometries on a complete CAT(0) space X . If $\gamma \in \Gamma$ acts as a hyperbolic isometry then γ has infinite order in the abelianisation of its centralizer $Z_\Gamma(\gamma)$.*

Proof. This is proved on page 234 of [6]. The main points are these: the union of the axes for γ splits isometrically as $Y \times \mathbb{R} \subset X$; this subspace is preserved by $Z_\Gamma(\gamma)$, as is the splitting; the action on the second factor gives a homomorphism from $Z_\Gamma(\gamma)$ to the abelian group $\text{Isom}_+(\mathbb{R})$ and the image of γ is non-trivial. \square

Remark 2.2. A more subtle argument exploiting [20] shows that non-neutral parabolics have infinite image in the abelianisation of their centralizer. This is explained explained in the proof of [5] Theorem 1.

The facts that we need about elliptic isometries rely on the following well-known proposition ([6], II.2.7).

We write $\text{Isom}(X)$ for the group of isometries of a metric space X and $\text{Ball}_r(x)$ for the closed ball of radius $r > 0$ about $x \in X$. Given a subspace $Y \subseteq X$, let $\rho(Y) := \inf\{r \mid Y \subseteq \text{Ball}_r(x), \text{ some } x \in X\}$.

Proposition 2.3. *If X is a complete CAT(0) space and Y is a non-empty bounded subset, then there is a unique point $c_Y \in X$ such that $Y \subseteq \text{Ball}_{\rho(Y)}(c_Y)$.*

Corollary 2.4. *Let X be a complete CAT(0) space. If $H < \text{Isom}(X)$ has a bounded orbit then H has a fixed point.*

Proof. The centre c_O of any H -orbit O will be a fixed point. □

Corollary 2.5. *Let X be a complete CAT(0) space. If the subgroups $H_1, \dots, H_\ell < \text{Isom}(X)$ commute and $\text{Fix}(H_i)$ is non-empty for $i = 1, \dots, \ell$, then $\bigcap_{i=1}^{\ell} \text{Fix}(H_i)$ is non-empty.*

Proof. A simple induction reduces us to the case $\ell = 2$. Since $\text{Fix}(H_2)$ is non-empty, each H_2 -orbit is bounded. As H_1 and H_2 commute, $\text{Fix}(H_1)$ is H_2 -invariant and therefore contains an H_2 -orbit. Orthogonal projection to a closed convex subspace, such as $\text{Fix}(H_1)$, does not increase distances ([6], II.2.4). Thus the centre of this H_2 -orbit is in $\text{Fix}(H_1)$, providing us with a fixed point for $H_1 \cup H_2$. □

3. NERVES, FIXED POINTS, AND BOOTSTRAPS

In this section we isolate from [4] the ideas needed to prove the version of the *ample duplication criterion* that we need.

We need the following standard device for encoding the intersections of families of subsets.

Definition 3.1. Let X be a set. The *nerve* \mathcal{N} of a family of subsets $F_\lambda \subseteq X$ ($\lambda \in \Lambda$) is the abstract simplicial complex with vertex set Λ that has a k -simplex with vertices $\{\lambda_0, \dots, \lambda_k\}$ if and only if $\bigcap_{i=0}^k F_{\lambda_i} \neq \emptyset$.

If the index set Λ is $\mathbf{n} = \{0, \dots, n\}$, where $n \in \mathbb{N}$, then we regard \mathcal{N} as a subcomplex of the standard n -simplex Δ_n .

We write $|\mathcal{N}|$ to denote the geometric realisation of \mathcal{N} .

Helly's classical theorem concerning the intersection of convex subsets in \mathbb{R}^n can be formulated as follows: *Let $\{C_0, \dots, C_N\}$ be a family of closed convex sets in \mathbb{R}^n . If the nerve \mathcal{N} of this family contains the full n -skeleton of Δ_N , then $\mathcal{N} = \Delta_N$.* There are many variations on this theorem in the literature. The following special case of the one proved in [4] will suffice for our purposes.

Theorem 3.2. *If X is a complete CAT(0) space of dimension at most d and \mathcal{N} is the nerve of a family of closed convex subsets of X , then every continuous map $|\mathcal{N}| \rightarrow \mathbb{S}^d$ is homotopic to a constant map.*

Our proof of Theorem A exemplifies the fact that by applying versions of Helly's theorem to fixed point sets of subgroups, one can sometimes prove fixed point theorems for groups of geometric interest (cf. [22], [1]). Forms of this idea appear at various places in the literature. I learned it from Benson Farb [14].

3.1. Joins. Let K_1 and K_2 be abstract simplicial complexes whose vertex sets V_1 and V_2 are disjoint. The *join* of K_1 and K_2 , denoted $K_1 * K_2$ is a simplicial complex with vertex set $V_1 \sqcup V_2$; a subset of $V_1 \sqcup V_2$ is a simplex¹ of $K_1 * K_2$ if and only if it is a simplex of K_1 , a simplex of K_2 , or the union of a simplex of K_1 and a simplex of K_2 . For example, the join of an n -simplex and an m -simplex is an $(n + m + 1)$ -simplex. Note that

$$\dim(K_1 * K_2) = \dim K_1 + \dim K_2 + 1,$$

provided V_1 and V_2 are non-empty.

Corollary 3.3. *Let X be a complete CAT(0) space and let $S_1, \dots, S_\ell \subseteq \text{Isom}(X)$ be subsets such that $[s_i, s_j] = 1$ for all $s_i \in S_i, s_j \in S_j$ ($i \neq j$). If \mathcal{N}_i is the nerve of the family $\mathcal{F}_i = \{\text{Fix}(s_i) \mid s_i \in S_i\}$, then the nerve \mathcal{N} of $\mathcal{F}_1 \cup \dots \cup \mathcal{F}_\ell$ is $\mathcal{N}_1 * \dots * \mathcal{N}_\ell$.*

Proof. It is clear that \mathcal{N} is contained in the join of the \mathcal{N}_i ; we must argue that the converse is true, i.e. that \mathcal{N} has as a simplex the union of each ℓ -tuple of simplices $\sigma_i \subset \mathcal{N}_i$ ($i = 1, \dots, \ell$). The σ_i correspond to an ℓ -tuple of commuting subgroups H_i of $\text{Isom}(X)$, namely the subgroups generated by the elements of S_i indexing the vertices of σ_i . The presence of σ_i in \mathcal{N}_i is equivalent to the statement that $\text{Fix}(H_i)$ is non-empty. Corollary 2.5 then tells us $\text{Fix}(\cup_i H_i) = \cap_i \text{Fix}(H_i)$ is non-empty, as required. \square

Proposition 3.4 (Bootstrap Lemma). *Let k_1, \dots, k_n be positive integers and let X be a complete CAT(0) space of dimension less than $k_1 + \dots + k_n$. Let $S_1, \dots, S_n \subset \text{Isom}(X)$ be subsets with $[s_i, s_j] = 1$ for all $s_i \in S_i$ and $s_j \in S_j$ ($i \neq j$).*

If, for $i = 1, \dots, n$, each k_i -element subset of S_i has a fixed point in X , then for some i every finite subset of S_i has a fixed point.

Proof. Suppose that the conclusion of the proposition were false. Then for $i = 1, \dots, n$ there would be a smallest integer $k'_i \geq k_i$ such that some $(k'_i + 1)$ -element subset $T_i = \{s_{i,1}, \dots, s_{i,k'_i+1}\}$ in S_i did not have a common fixed point.

Since any k'_i elements of T_i have a common fixed point, the nerve of the family $\mathcal{F}_i = \{\text{Fix}(s_{i,1}), \dots, \text{Fix}(s_{i,k'_i+1})\}$ would be the boundary of a k'_i -simplex $\partial\Delta_{k'_i}$. Hence, by Corollary 3.3, the nerve of $\mathcal{F}_1 \cup \dots \cup \mathcal{F}_n$ would be the join $\partial\Delta_{k'_1} * \dots * \Delta_{k'_n}$. But this contradicts Theorem 3.2, because the realisation of this join is homeomorphic to a sphere of dimension $(\sum_{i=1}^n k'_i) - 1 \geq \dim X$. \square

¹i.e. is the vertex set of a simplex in the geometric realisation

Since isometries of finite order have fixed points, taking $k_i = 1$ we get:

Corollary 3.5. *If each of the groups $\Gamma_1, \dots, \Gamma_n$ has a finite generating set consisting of elements of finite order, then at least one of the Γ_i has a fixed point whenever $D = \Gamma_1 \times \dots \times \Gamma_n$ acts by isometries of a complete CAT(0) space of dimension less than n .*

When applying the above proposition one has to overcome the fact that the conclusion only applies to *some* S_i . A convenient way of gaining more control is to restrict attention to conjugate sets.

Corollary 3.6 (Conjugate Bootstrap). *Let k and n be positive integers and let X be a complete CAT(0) space of dimension less than nk . Let S_1, \dots, S_n be conjugates of a subset $S \subset \text{Isom}(X)$ with $[s_i, s_j] = 1$ for all $s_i \in S_i$ and $s_j \in S_j$ ($i \neq j$).*

If each k -element subset of S has a fixed point in X , then so does each finite subset of S .

3.2. Some surface topology. The reader will recall that, given a closed orientable surface Σ and two compact homeomorphic subsurfaces with boundary $T, T' \subset \Sigma$, there exists an automorphism of Σ taking T to T' if and only if $\Sigma \setminus T$ and $\Sigma \setminus T'$ are homeomorphic. In particular, two homeomorphic subsurfaces are in the same orbit under the action of $\text{Homeo}(\Sigma)$ if the complement of each is connected.

The relevance of this observation to our purposes here is the following lemma, which will be used in tandem with the Conjugate Bootstrap.

Lemma 3.7. *Let H be the subgroup of $\text{Mod}(\Sigma)$ generated by the Dehn twists in a set of loops all of which are contained in a compact subsurface $T \subset \Sigma$ with connected complement. If Σ contains m mutually disjoint subsurfaces T_i homeomorphic to T , each with connected complement, then $\text{Mod}(\Sigma)$ contains m mutually-commuting conjugates H_i of H .*

Proof. Let ϕ_i be the mapping class of a homeomorphism $\Sigma \rightarrow \Sigma$ carrying T to T_i . The subgroups $H_i := \phi_i H \phi_i^{-1}$ are supported in disjoint subsurfaces and therefore commute. \square

4. THE LICKORISH GENERATORS

We fix a closed orientable surface Σ_g of genus $g \geq 2$. Whenever we speak of a *subsurface* it is to be understood that the subsurface is compact and connected.

Building on classical work of Max Dehn [12], Raymond Lickorish [23] proved that the mapping class group of Σ_g is generated by the Dehn twists in $3g - 1$ curves α_i, β_i ($1 \leq i \leq g$), γ_i ($1 \leq i < g$) portrayed in figure 1. These are pairwise disjoint except that $|\alpha_i \cap \beta_i| = |\beta_i \cap \gamma_i| = |\beta_i \cap \gamma_{i-1}| = 1$. Let Λ denote this set of curves.

We say that a subset $S \subset \Lambda$ is *connected* if the union $U(S)$ of the loops in S is connected.

Let $A_{ij} = \{\alpha_k \mid i \leq k \leq j\}$, let $B_{ij} = \{\beta_k \mid i \leq k \leq j\}$, and let $C_{ij} = \{\gamma_k \mid i \leq k \leq j\}$. We distinguish the following connected subsets of Λ .

$$(4.1) \quad \begin{aligned} [\beta_i, \beta_j] &:= A_{i+1, j-1} \cup B_{ij} \cup C_{i, j-1} \\ [\beta_i, \alpha_j] &:= [\beta_i, \beta_j] \cup \{\alpha_j\} & [\alpha_i, \beta_j] &:= \{\alpha_i\} \cup [\beta_i, \beta_j] \\ [\beta_i, \gamma_j] &:= [\beta_i, \beta_j] \cup \{\gamma_j\} & [\gamma_i, \beta_j] &:= [\beta_i, \beta_j] \setminus \{\beta_i\} \\ [\alpha_i, \alpha_j] &:= \{\alpha_i\} \cup [\beta_i, \alpha_j] & [\gamma_i, \gamma_j] &:= [\beta_i, \gamma_j] \setminus \{\beta_i\} \\ [\gamma_i, \alpha_j] &:= [\gamma_i, \beta_j] \cup \{\alpha_j\} & [\alpha_i, \gamma_j] &:= [\alpha_i, \beta_j] \cup \{\gamma_j\}. \end{aligned}$$

Roughly speaking, these are the sets of loops in Λ that one encounters as one proceeds clockwise in figure 1 from a loop with lesser index i to one of greater index j .

4.1. Good neighbourhoods of subsets of Λ . In order to describe “good neighbourhoods” we need the following notation. First, for $i < j$ we define δ_{ij} to be a loop that cobounds with α_i a subsurface $T(i, j)$ of genus $j - i$ with two boundary components that contains all of the loops $[\gamma_i, \beta_j]$; in the language of figure 1, it consists of one half of handle i and all of the handles with index $i + 1$ to j inclusive. The definition of δ_{ij} for $j < i$ is similar except that the subsurface $T(i, j)$ that it cobounds with α_i now consists of one half of handle i and all of the handles with index from j to $i - 1$ inclusive.

δ_{13} is shown in figure 1 and δ_{31} is obtained from δ_{13} by rotating the surface through an angle π about the axis of symmetry that lies in the plane of the paper and intersects α_2 .

Finally, we need a loop c_{ij} that separates the handles i to j from the remainder of the surface and a loop d_{ij} that, together with α_i and α_j , separates handles $(i + 1)$ to $(j - 1)$ from the remainder of the surface; c_{13} and d_{13} are shown in figure 1.

Each of the following six lemmas is an elementary observation, but their cumulative import is non-trivial.

Lemma 4.1. *The union of the loops in $[\alpha_i, \alpha_j] \subset \Lambda$ is contained in a subsurface of genus $j - i + 1$ with boundary component c_{ij} .*

Lemma 4.2. *The union of the loops in $[\gamma_i, \gamma_j] \subset \Lambda$ is contained in a subsurface of genus $j - i$ with boundary $\alpha_i \cup d_{i, j+1} \cup \alpha_{j+1}$.*

Lemma 4.3. *The union of the loops in $[\alpha_i, \gamma_j] \subset \Lambda$ is contained in the subsurface $T(j, i)$, which has genus $(j - i + 1)$ and boundary $\alpha_j \cup \delta_{ji}$.*

Lemma 4.4. *The union of the loops in $[\gamma_i, \alpha_j] \subset \Lambda$ is contained in the subsurface $T(i, j)$, which has genus $(j - i + 1)$ and boundary $\alpha_i \cup \delta_{ij}$.*

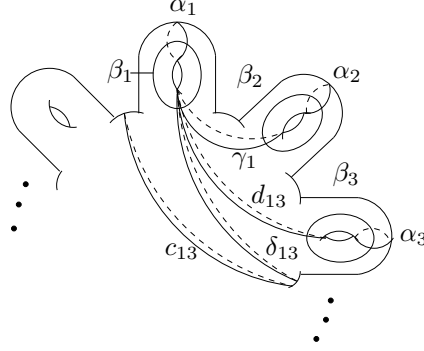


FIGURE 1. The loops used in the main argument

An m -chain in a surface Σ_g is a finite sequence of simple closed loops C_1, \dots, C_m such that $|C_i \cap C_{i+1}| = 1$ and $C_i \cap C_j = \emptyset$ if $|i - j| > 1$.

Lemma 4.5. *If m is even, then a closed regular neighbourhood of an m -chain is a surface of genus $m/2$ with one boundary component. If m is odd, then such a neighbourhood is a surface of genus $(m-1)/2$ with two boundary components.*

In the odd case one distinguishes between *non-separating* and *separating* chains according to whether or not the complement in Σ_g of the union of the loops in the chain is connected or not.

Lemma 4.6. *The only separating m -chains in Λ are those that are obtained from $[\alpha_i, \alpha_j]$ by deleting the loops α_k with $i < k < j$, where $j = i + \frac{1}{2}(m-3)$.*

Recall that $S \subset \Lambda$ is said to be *connected* if the union $U(S)$ of the loops in S is connected.

Proposition 4.7. *Let $S \subset \Lambda$ be a connected subset.*

- (1) *If $|S| = 2\ell$ is even, then $U(S)$ is either contained in a subsurface of genus ℓ with 1 boundary component, or else in a non-separating subsurface of genus at most $\ell - 1$ with 3 boundary components.*
- (2) *If $|S| = 2\ell + 1$ is odd, then $U(S)$ is either contained in a non-separating subsurface of genus ℓ with at most 2 boundary components, or else is contained in a non-separating subsurface of genus at most $\ell - 1$ that has at most 3 boundary components.*

Proof. If S is an m -chain then we can appeal to Lemma 4.5 except in the case where S is separating and $|S| = 2\ell + 1$ is odd. In this case $S \subset [\alpha_i, \alpha_j]$, by Lemma 1:badChains, where $j = i + \ell - 1$. In this case we appeal to Lemma 4.1.

If S is not itself a chain, then there is an m -chain C_1, \dots, C_m in Λ , with m strictly less than $|S|$, such that S is contained in the set $I = [C_1, C_m] \subset \Lambda$, where $[C_1, C_m]$ is one of the ‘‘interval like’’ subsets described in (4.1). We

analyze each of the possibilities using the subsurfaces given in Lemmas 4.1 to 4.4 to enclose $U(S)$ in a subsurface of controlled type.

First suppose that $m \leq 2\ell - 1$ is odd.

If $I = [\alpha_i, \alpha_j]$ then $j \leq i + \ell - 2$ and I is contained in a surface of genus at most $(\ell - 1)$ with 1 boundary component (Lemma 4.1).

If $I = [\beta_i, \beta_j]$ then we can enclose it in $[\alpha_i, \alpha_j]$, and $j \leq i + \ell - 1$. Thus I is contained in a surface of genus at most ℓ with 1 boundary component (Lemma 4.1).

If $I = [\gamma_i, \gamma_j]$ then $j \leq i + \ell - 1$ and I is contained in a surface of genus at most $(\ell - 1)$ with 3 boundary components (Lemma 4.2).

If $I = [\alpha_i, \gamma_j]$ then $j \leq i + \ell - 2$ and I is contained in a surface of genus at most $(\ell - 1)$ with 2 boundary components (Lemma 4.3).

If $I = [\gamma_i, \alpha_j]$ then $j \leq i + \ell - 1$ and I is contained in a surface of genus at most $(\ell - 1)$ with 2 boundary components.

Now suppose that $m = 2k$ is even.

If $I = [\alpha_i, \beta_j]$ or $I = [\beta_i, \alpha_j]$ then we can embed I in $[\alpha_i, \alpha_j]$ with $j \leq i + k - 1$, and this is contained in a surface of genus at most k with 1 boundary component (Lemma 4.1).

If $I = [\beta_i, \gamma_j]$ then we enclose it in $[\alpha_i, \gamma_j]$ with $j \leq i + k - 1$, which is contained in a surface of genus at most k with 2 boundary components (Lemma 4.3).

Finally, if $I = [\gamma_i, \beta_j]$ then we enclose it in $[\gamma_i, \alpha_j]$ with $j \leq i + k$, which is contained in a surface of genus at most k with 2 boundary components (Lemma 4.4).

To complete the proof of (1) note that if $m = 2k$ is even then $k < \ell$, hence the required estimate. In case (2) we have $k \leq \ell$, so the above estimates are again sufficient. \square

We write $\Sigma_{h,n}$ to denote the compact connected orientable surface of genus h with $n > 0$ boundary components.

Lemma 4.8.

- (1) Σ_g contains $\lfloor g/\ell \rfloor$ disjoint subsurfaces homeomorphic to $\Sigma_{\ell,1}$.
- (2) Σ_g contains $\lfloor g/\ell \rfloor$ disjoint non-separating subsurfaces homeomorphic to $\Sigma_{\ell-1,3}$.
- (3) Σ_g contains $\lfloor (g-1)/\ell \rfloor$ disjoint non-separating subsurfaces homeomorphic to $\Sigma_{\ell,2}$.

Proof. To prove (1), one expresses Σ_g as the connected sum of \mathbb{S}^2 with q disjoint copies of Σ_ℓ and one copy of $\Sigma_{\ell'}$, where $q = \lfloor g/\ell \rfloor$ and $\ell' = g - q\ell$.

For (3), note first that given q compact orientable surfaces U_1, \dots, U_q of genus h_1, \dots, h_q , each with 2 boundary components $\partial U_i = \partial U_i^+ \cup \partial U_i^-$, one obtains a closed surface of genus $1 + \sum_i h_i$ by arranging the U_i in cyclic order and identifying ∂U_i^+ with ∂U_{i+1}^- (indices mod q).

More generally, given an integer $g \geq 1 + \sum_i h_i$, one can obtain a closed surface of genus g by including into the above cyclic assembly an additional surface of genus $g - 1 - \sum_i h_i$ with 2 boundary curves.

Reversing perspective, we deduce from the preceding construction that given a closed surface Σ of genus g and an integer h , one can find $\lfloor \frac{g-1}{h} \rfloor$ disjoint, non-separating subsurfaces of genus h , each with 2 boundary components. This proves (3).

To prove (2), we assemble Σ_g from $q = \lfloor g/\ell \rfloor$ copies F_i of $\Sigma_{\ell-1,3}$ and one copy of $\Sigma_{\ell,q}$, where $\ell' = g - q\ell$: label the 3 boundary components of F_i as ∂_i^+ , ∂_i^0 , ∂_i^- and identify ∂_i^+ with ∂_{i+1}^- (indices mod q) to obtain an orientable surface of genus $q(\ell - 1) + 1$ with q boundary components, then cap-off the boundary by attaching $\Sigma_{\ell,q}$.

In each of the above constructions, the relevant subsurfaces that we described are not quite disjoint — they intersect along their boundaries — but this is remedied by an obvious retraction of each subsurface away from its boundary. \square

4.2. A numerical lemma.

Lemma 4.9. *Let $g \geq 1$ and $k \in [2, 2g]$ be integers. If k is even then $(k - 1)\lfloor 2g/k \rfloor \geq g$. If k is odd then $(k - 1)\lfloor 2(g - 1)/(k - 1) \rfloor \geq g$.*

Proof. Fix k even. For every positive integer s , the function $\phi_k(x) = (k - 1)\lfloor 2x/k \rfloor$ is constant on $[sk/2, (s + 1)k/2)$, so if the inequality $\phi_k(g) \geq g$ were to fail for some $g \geq k/2$ then it would fail at $g = \frac{1}{2}(s + 1)k - 1$, with $s \geq 1$. But at this value $\phi_k(g) = (k - 1)s$, and $(k - 1)s < \frac{1}{2}(s + 1)k - 1$ implies $s(k - 2) < k - 2$, which is nonsense.

The proof for k odd is similar. \square

5. PROOF OF THE MAIN THEOREMS

We shall deduce Theorems A and B from the following more technical result.

Definition 5.1. A *handle-separating loop* on a closed orientable surface Σ_g of genus $g \geq 2$ is a simple closed curve c such that one of the components of $\Sigma_g \setminus c$ has genus 1.

Theorem 5.2. *If the mapping class group $\text{Mod}(\Sigma_g)$ of a closed orientable surface of genus $g \geq 2$ acts by isometries on a complete CAT(0) space X of dimension less than g and the Dehn twist in a handle-separating loop fixes a point of X , then $\text{Mod}(\Sigma_g)$ fixes a point of X .*

Lemma 5.3. *Let $g \geq 2$ and consider an action of $\text{Mod}(\Sigma_g)$ by isometries on a complete CAT(0) space X of dimension less than g . Let C_0 be a handle-separating loop on Σ_g .*

If the Dehn twist about C_0 has a fixed point in X , then so does the subgroup generated by the Dehn twists in any pair of simple closed curves C, C' on Σ_g with $|C \cap C'| = 1$.

Proof. Each pair of simple closed curves C, C' on Σ_g with $|C \cap C'| = 1$ is contained in a subsurface W of genus 1 with one boundary component. Thus it will be enough to show that there is a point fixed by the subgroup $M_W \subset \text{Mod}(\Sigma_g)$ consisting of the mapping classes of homeomorphisms that stabilize W and act trivially on its complement.

As in the proof of Lemma 4.8, we express Σ_g as the union of g subsurfaces W_i of genus 1, each with 1 boundary component, and a sphere with g discs deleted. Let C_1, \dots, C_g be the boundary curves of these subsurfaces. Each of the C_i lies in the $\text{Mod}(\Sigma_g)$ -orbit of C_0 and hence the Dehn twist T_{C_i} is an elliptic isometry of X . Since the loops C_i are disjoint, these Dehn twists commute and hence the set F of points fixed by all of them is non-empty (corollary 2.5). $F \subset X$ is closed and convex, and hence is itself a complete CAT(0) space of dimension less than g .

The centralizer of $\langle T_{C_1}, \dots, T_{C_g} \rangle$ in $\text{Mod}(\Sigma_g)$ preserves F . This centralizer contains $D = M_1 \times \dots \times M_g$, where for brevity we have written M_i in place of M_{W_i} . Note that each M_i is conjugate to M_W .

We are interested in the action of D on $F \subset X$. Since the T_{C_i} act trivially on F , this action factors through the product of the groups $M_i / \langle T_{C_i} \rangle$, each of which is isomorphic to the mapping class group of a genus 1 surface with one puncture. This last group is isomorphic to $\text{SL}(2, \mathbb{Z})$, which is generated by a pair of torsion elements. Thus Corollary 3.5 tells us that one of the $M_i / \langle T_{C_i} \rangle$ has a fixed point in F . Hence M_i has a fixed point in $F \subset X$. And since M_W is conjugate to M_i in $\text{Mod}(\Sigma_g)$, it too has a fixed point in X . \square

5.1. The proof of Theorem 5.2. A complete CAT(0) space of dimension 1 is an \mathbb{R} -tree, and if $g \geq 2$ then $\text{Mod}(\Sigma_g)$ has a fixed point whenever it acts by isometries on an \mathbb{R} -tree [11]. Thus we may assume that $g \geq 3$.

Let X be a complete CAT(0) space of dimension less than g on which $\text{Mod}(\Sigma_g)$ acts by isometries so that the Dehn twist in a handle-separating loop fixes a point of X . We write $T(S)$ to denote the subgroup of $\text{Mod}(\Sigma_g)$ generated by the Dehn twists in a set of loops $S \subset \Lambda$. We must show that $\text{Mod}(\Sigma_g)$ fixes a point of X . This is equivalent to showing that $T(S)$ has a fixed point for every subset $S \subseteq \Lambda$.

We proceed by induction, considering a subset $S \subset \Lambda$ such that $T(S')$ has a fixed point for all subsets $S' \subset \Lambda$ of cardinality less than $|S|$. Lemma 5.3 covers the case $|S| \leq 2$.

Suppose first that S is not connected, say $S = S_1 \cup S_2$ with $U(S_1) \cap U(S_2) = \emptyset$. The subgroups $T(S_1)$ and $T(S_2)$ commute, and each has a fixed point since $|S_i| < |S|$, so Corollary 2.5 tells us that $T(S)$ has a fixed point.

Suppose now that S is connected. If $|S| = 2\ell$ is even then Proposition 4.7 tells us that $U(S)$ is contained either in a subsurface of genus ℓ with 1 boundary component or else in a non-separating subsurface of genus $\ell - 1$ with 3 boundary components. Lemma 4.8 tells us that in either case one can fit $\lfloor g/\ell \rfloor$ disjoint copies of this subsurface into Σ_g . Lemma 3.7 then provides us with $\lfloor g/\ell \rfloor$ mutually-commuting conjugates of $T(S)$. As all proper subsets of S are assumed to have a fixed point, the Conjugate Bootstrap (Corollary 3.6) tells us that $T(S)$ will have a fixed point provided that the dimension of X is less than $(2\ell - 1)\lfloor g/\ell \rfloor$. And since we are assuming that $\dim X < g$, Lemma 4.9 tells us that this is the case.

The argument for $|S|$ odd is entirely similar. \square

5.2. The proof of Theorem A. The following lemma explains why Theorem A is a consequence of Theorem 5.2 and the fact that $\text{Mod}(\Sigma_2)$ has property FR, by [11].

Lemma 5.4. *If $g \geq 3$, then whenever $\text{Mod}(\Sigma_g)$ acts by isometries on a complete $\text{CAT}(0)$ space, the Dehn twist T in each simple closed curve either fixes a point or acts as a neutral parabolic (i.e. T is not ballistic in the terminology of [10]).*

Proof. The abelianisation of the centralizer of T in $\text{Mod}(\Sigma_g)$ is finite (cf. [21] and [5] Proposition 2), so Proposition 2.1 and Remark 2.2 imply that the translation number of T is zero. \square

5.3. The proof of Theorem B. The Dehn twists in separating curves lie in the kernel of the natural map $\text{Mod}(\Sigma_g) \rightarrow \text{Out}(\Sigma_g/\gamma_3(\Sigma_g))$. (Johnson [19] proved that they generate the kernel but we do not need that.) Thus Theorem B is an immediate consequence of Theorem 5.2. \square

6. CONSTRAINTS ON THE REPRESENTATION THEORY OF $\text{Mod}(\Sigma_g)$

In Section I.6.2 of [26] Serre proves that if a countable group G has property FA and if k is a field, then for every representation $\rho : G \rightarrow \text{GL}(2, k)$ and every $g \in G$, the eigenvalues of $\rho(g)$ are integral over \mathbb{Z} , i.e. each is the solution of a monic polynomial with integer coefficients. For the convenience of the reader we reproduce Serre's argument, making the slight modifications needed to prove the following generalisation (which is well known to experts). We refer to the seminal paper of Bruhat and Tits for the theory of affine buildings [9]; see also Chapter 9 of [24] and Chapter VI of [8].

Proposition 6.1. *If G is a finitely generated group that has a fixed point whenever it acts by semisimple isometries on a complete $\text{CAT}(0)$ space of dimension less than n , and k is a field, then for every representation $\rho : G \rightarrow \text{GL}(n, k)$ and every $g \in G$, the eigenvalues of $\rho(g)$ are integral over \mathbb{Z} .*

Proof. The subfield $k_\rho \subset k$ generated by the entries of $\rho(G)$ is finitely generated over the prime field \mathbb{Q} or \mathbb{F}_p . Let v be a discrete valuation on k_ρ — i.e. an epimorphism $k_\rho^* \rightarrow \mathbb{Z}$ such that $v(xy) = v(x)v(y)$ and $v(x+y) \geq \min\{v(x), v(y)\}$ for all $x, y \in k_\rho$, with the convention that $v(0) = +\infty$. The corresponding valuation ring is $\mathcal{O}_v = \{x \in k_\rho \mid v(x) \geq 0\}$.

G is perfect, since it can't act freely on a line by isometries, and hence it lies in the kernel of $v \circ \det : \mathrm{GL}(n, k_\rho) \rightarrow \mathbb{Z}$. This kernel acts by isometries on the geometric realisation of a Bruhat-Tits building of affine type; in this action each element that fixes a point fixes a simplex pointwise. This building is a complete CAT(0) space of dimension $n - 1$, so $\rho(G)$ fixes a simplex. Hence $\rho(G)$ is conjugate in $\mathrm{GL}(n, k_\rho)$ to a subgroup of $\mathrm{GL}(n, \mathcal{O}_v)$, since each vertex stabilizer is. It follows that for each $g \in G$ the coefficients of the characteristic polynomial of $\rho(g)$ lie in the intersection $\bigcap_v \mathcal{O}_v$. This intersection is precisely the set of elements of k_ρ that are integral over \mathbb{Z} (see [16], p.140, corollary 7.1.8). Hence, by the transitivity of integrality, the eigenvalues of $\rho(g)$ are integral over \mathbb{Z} . \square

Bass, [2] Proposition 5.3, proves that a finitely generated group satisfying the conclusion of Proposition 6.1 has only finitely many conjugacy classes of irreducible representations $G \rightarrow \mathrm{GL}(n, K)$ for an arbitrary algebraically closed field K .

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