Limit groups, positive-genus towers and measure equivalence

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Abstract

An ω-residually free tower is positive-genus if all surfaces used in its construction are of positive genus. We prove that every limit group is virtually a subgroup of a positive-genus ω-residually free tower. By combining this with results of Gaboriau, we prove that elementarily free groups are measure equivalent to free groups.

Measure equivalence was introduced by M. Gromov in [8] as a measure-theoretic analogue of quasi-isometry. The motivating examples are commensurable groups and lattices in the same locally compact second countable group. Much progress has been made in distinguishing measure-equivalence classes (see, for example, [1], [3], [4], [5], [6] and [16]), but there have been many fewer constructions of examples of measure equivalent groups. The only groups whose measure-equivalence classes are completely classified are finite groups, amenable groups, and lattices in simply connected Lie groups with finite centre and real rank at least 2 (see [3]). In particular, the measure-equivalence class of free groups is still quite poorly understood.

In [7], D. Gaboriau constructs some new examples of groups measure equivalent to free groups, encapsulated in the following theorem.

Theorem 0.1 *Let Σ be a compact orientable surface of positive genus, with one boundary component. Then the iterated amalgamated product*

$$\pi_1(\Sigma) *_{\langle\partial\Sigma\rangle} \pi_1(\Sigma) *_{\langle\partial\Sigma\rangle} \ldots *_{\langle\partial\Sigma\rangle} \pi_1(\Sigma)$$

is measure equivalent to a free group.

The amalgamated product in the above statement is an example of a *limit group* (indeed, it is an elementarily free group). Gaboriau also asks if all limit groups are measure equivalent to free groups.

Limit groups have been studied under a variety of names: see [10], [11] and [12]. The name limit group was introduced by Z. Sela in his solution of the Tarski problem (see [14] *et seq.*). The *elementary theory* of a group G is the set of first-order sentences that are true in G. The Tarksi problem asks which groups have the same elementary theory as the free group of rank 2. The *existential theory* consists of those sentences that only use one quantifier $\exists$. Limit groups turn out to be precisely the groups with the same existential theory as a free group. Another class, still more closely related to free groups, is the class of *elementarily free groups*: those groups with the same elementary theory as a free group.

Using [14] (cf. [10], [11]), one can give more constructive definitions of limit groups and elementarily free groups. For the purposes of this paper, both classes are defined in terms of ω-residually free towers, which are in turn defined as the fundamental groups of certain complexes, inductively constructed from graphs, surfaces and tori (see section 1). Such a tower is called *positive-genus* if every surface used in its construction is of positive genus. Our main result concerning towers is the following.

Theorem A (Theorem 1.11) *Every limit group is virtually a subgroup of a positive-genus ω-residually free tower.*

Using this and results of [7], we deduce the following theorem, which partially answers Gaboriau's question.

Theorem B (Theorem 2.6) *Every elementarily free group is measure equivalent to a free group.*

This paper is organized as follows. In section 1 we construct various finite-index subgroups of towers and prove theorem A. In section 2 we recapitulate some useful results of [7] and prove that elementarily free groups are measure equivalent to free groups. In section 3 we discuss methods of attacking the case of all limit groups.

1 Limit groups

1.1 ω-residually free towers

For an introduction to the theory of limit groups, see [2].

Definition 1.1 *An* ω-rft space of height 0*, denoted* X_0*, is a one-point union of finitely many compact graphs, tori, and closed hyperbolic surfaces of Euler characteristic less than -1.*

An ω-rft space of height h*, denoted* X_h*, is obtained from an* ω*-rft space* X_{h-1} *of height* $h-1$ *by attaching one of two sorts of blocks.*

1. ***Quadratic block.*** *Let* Σ *be a connected compact hyperbolic surface with boundary, with each component either a punctured torus or having* $\chi \leq -2$*. Then* X_h *is the quotient of* $X_{h-1} \sqcup \Sigma$ *obtained by identifying the boundary components of* Σ *with curves on* X_{h-1}*, in such a way that there exists a retraction* $\rho : X_h \to X_{h-1}$*. The retraction is also required to satisfy the property that* $\rho_*(\pi_1(\Sigma))$ *be non-abelian.*

2. ***Abelian block.*** *Let* T *be an* n*-torus, and fix a coordinate circle* γ*. Fix a loop* c *in* X_{h-1} *that generates a maximal abelian subgroup in* $\pi_1(X_{h-1})$*. Then* X_h *is the quotient of* $X_h \sqcup (S^1 \times [0,1]) \sqcup T$ *obtained by identifying* $S^1 \times \{0\}$ *with* c*, and* $S^1 \times \{1\}$ *with* γ*.*

An ω*-rft space is called* hyperbolic *if no tori are used in its construction.*

Definition 1.2 *An* (ω-residually free) tower of height h*, denoted* L_h*, is the fundamental group of an* ω*-rft space of height* h*.*

The following deep theorem of Sela (see [13]) will, for our purposes, serve as a definition of elementarily free groups.

Theorem 1.3 *A group is* elementarily free *if and only if it is the fundamental group of a hyperbolic* ω*-rft space.*

Towers are examples of limit groups. Another theorem of Sela [15] and, independently, O. Kharlampovich and A. Myasnikov [11], will serve as a definition of limit groups.

Theorem 1.4 *A group is a* limit group *if and only if it is a finitely generated subgroup of an* ω*-residually free tower.*

We will need a result that is an immediate consequence of the fact that the limit groups are precisely the finitely generated ω-residually free groups.

Lemma 1.5 *Limit groups are* residually free*; that is, if L is a limit group and $g \in L - \{1\}$ then there exists a homomorphism to a free group $f : L \to F$ with $f(g) \neq 1$.*

A key feature of the definition of a tower is the retraction $\rho : X_h \to X_{h-1}$. In the abelian case, the retraction simply projects T onto the coordinate circle γ, and thence to c. In both cases, ρ induces a retraction $\rho_* : L_h \to L_{h-1}$ on the level of fundamental groups.

An ω-rft space X_h has a natural graph-of-spaces[1] decomposition Γ_X, with two vertex spaces, namely X_{h-1} and the block at height h; the edge spaces are circles. We will often use the retraction to pull finite covers back from X_{h-1} to X_h. It is worth noting that such pullbacks inherit a similar graph-of-spaces decomposition from X_h.

Lemma 1.6 *Let X be a CW-complex with a graph-of-spaces decomposition Γ_X, such that there is a retraction $\rho : X \to X'$ to a vertex space. Let $Y' \to X'$ be a connected covering of degree d, and let $Y \to X$ be the connected covering obtained by pulling back along ρ; that is, $\pi_1(Y) = \rho_*^{-1}(\pi_1(Y'))$. Then:*

1. *$Y \to X$ is of degree d and inherits a graph-of-spaces decomposition Γ_Y;*
2. *the pre-image of X' in Y is a (connected) vertex space of Γ_Y homeomorphic to Y';*
3. *$Y \to X$ extends $Y' \to X'$, and Y inherits a retraction to Y' covering ρ.*

A tower L_h inherits, by the Seifert–van Kampen Theorem, a graph-of-groups decomposition Γ_L from the graph-of-spaces decomposition Γ_X of the associated ω-rft space X_h. The decomposition Γ_L is 2-acylindrical [14].

[1] By convention, our graphs of spaces are connected and have connected vertex and edge spaces.

1.2 Positive-genus towers

The purpose of this section is to prove that, up to finite index, the quadratic blocks can be assumed to have positive genus. A compact, connected surface Σ with Euler characteristic $\chi(\Sigma)$ and $b(\Sigma)$ boundary components is of *positive genus* if $\chi(\Sigma) + b(\Sigma) \leq 0$. Note that, in particular, all finite covers of such Σ also have positive genus.

Definition 1.7 *An ω-rft space is* positive-genus *if every quadratic block used in its construction is of positive genus. A tower is* positive-genus *if it is the fundamental group of a positive genus ω-rft space.*

We are going to prove that every elementarily free group is virtually a subgroup of a positive-genus elementarily free group. Our strategy for obtaining positive-genus quadratic blocks is to identify connected p-sheeted coverings (here p is a prime number) that restrict to a p-sheeted covering on each boundary component. We achieve this by passing to a finite-index subgroup of the tower that admits a map to $\mathbb{Z}/p\mathbb{Z}$ which maps each attaching loop of the top quadratic block non-trivially. In particular, we must arrange for the attaching loops to become non-trivial in homology.

Recall that, for X a topological space, $c : S^1 \to X$ a loop, and $Y \to X$ a covering map, the *elevations* of c to Y are the minimal connected covers $\hat{S}^1 \to S^1$ such that $\hat{S}^1 \to X$ lifts to Y. Fixing basepoints, it follows from standard covering space theory that $\pi_1(\hat{S}^1)$ is the pre-image of $\pi_1(Y)$ in $\pi_1(S^1)$.

Lemma 1.8 *If X is a connected CW-complex with $\pi_1(X)$ residually free, and $c_1, \ldots, c_m$ is a finite collection of curves in X, then there exists a finite cover $Y \to X$ so that every elevation of each c_i to Y is of infinite order in $H_1(Y)$.*

Proof. Fix a base-point in X, and without loss of generality assume the c_i are based loops representing elements of $L = \pi_1(X)$. Since L is residually free, for each i there exists a homomorphism $f_i : L \to F$ with $f_i(c_i) \neq 1$. By M. Hall's theorem [9], there exists a finite-index subgroup $F_i \subset F$ containing $f_i(c_i)$, such that $f_i(c_i)$ is primitive in $H_1(F_i)$. Let $Y \to X$ be the cover corresponding to the subgroup $\bigcap_i f_i^{-1}(F_i)$. Every elevation d_j of c_i to Y corresponds to a conjugate of a power of $c_i \in L$. Since $f_i(c_i)$ has infinite order in $H_1(F_i)$, it follows that d_j has infinite order in $H_1(Y)$. □

We can now construct a map to $\mathbb{Z}/p\mathbb{Z}$ as required.

Lemma 1.9 *Let Y be a connected CW-complex, and let $d_1, \ldots, d_n$ be a collection of curves in Y that are all of infinite order in $H_1(Y)$. Then, for all sufficiently large primes p, there exists a homomorphism $\varphi : \pi_1(Y) \to \mathbb{Z}/p\mathbb{Z}$ so that $\varphi(d_j)$ is non-trivial for all j.*

Proof. There exists a homomorphism $H_1(Y) \to \mathbb{Z}$ under which each d_j has non-trivial image. This is because $\mathbb{Z}^n$ is ω-residually free. (To prove this, fix a basis for $H_1(Y)$ mod torsion and consider an inner product on the real vector space $V = H_1(Y) \otimes_{\mathbb{Z}} \mathbb{R}$, so that the basis is orthonormal. Then for each j, the unit vectors not normal to $d_j \otimes 1$ are an open subset of full measure on the unit sphere; so the unit vectors not normal to any of the $d_j \otimes 1$ also form an open subset of full measure on the unit sphere; then there exists an integral vector in V not normal to any of the $d_j \otimes 1$. Taking the inner product with this vector defines the required homomorphism $H_1(Y) \to \mathbb{Z}$.)

Now choose a prime p that doesn't divide any of the images of the d_j in $\mathbb{Z}$. In particular, each d_j has non-trivial image under the composition

$$\varphi : \pi_1(Z_h) \to H_1(Z_h) \to \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}.$$

□

We shall apply the preceding lemmas to the height $h-1$ subspace and pull back to the full tower to obtain the positive genus cover that we seek.

Proposition 1.10 *Let X_h be an ω-rft space, constructed by attaching a quadratic block Σ to a space X_{h-1} of height $h-1$. Then there exists a connected cover $Z_h \to X_h$ with an inherited graph-of-spaces decomposition Γ_Z, with one vertex space a connected cover $Z_{h-1} \to X_{h-1}$, and the remaining vertex spaces connected covers $\bar{\Sigma}_i \to \Sigma$, so that each $\bar{\Sigma}_i$ has positive genus. The retraction $\rho : X_h \to X_{h-1}$ pulls back to a retraction $Z_h \to Z_{h-1}$.*

Proof. Let $c_1, \ldots, c_m$ be the images of the boundary curves of Σ in X_{h-1}. Since $\pi_1(X_{h-1})$ is residually free, lemma 1.8 provides a finite covering $Y_{h-1} \to X_{h-1}$, so that if $d_1, \ldots, d_n$ are the elevations of the c_i, each d_j is of infinite order in homology. Let $Y_h \to X_h$ be the covering obtained by pulling back along the retraction ρ, with graph-of-spaces decomposition Γ_Y.

By lemma 1.9, there exists a homomorphism $\varphi : \pi_1(Y_{h-1}) \to \mathbb{Z}/p\mathbb{Z}$ with $\varphi(d_j) \neq 1$ for each j. Let $Z_{h-1} \to Y_{h-1}$ be the covering corresponding to the kernel of φ. Finally, pull the covering $Z_{h-1} \to Y_{h-1}$ back along the retraction $Y_h \to Y_{h-1}$ to give a covering $Z_h \to Y_h$, with graph-of-spaces decomposition Γ_Z.

The key point to observe is that each edge space of Γ_Y is only covered by one edge space of Γ_Z. Indeed, $Z_h \to Y_h$ is a covering of degree p, but any edge cycle d_j has order p under the map $\mathbb{Z}/p\mathbb{Z}$, so an elevation of it to Z_h covers d_j with degree p. Thus d_j only has one elevation to Z_h.

It follows that the underlying graphs of Γ_Y and Γ_Z are the same. Consider a surface vertex $\bar{\Sigma}_i$ of Γ_Z, covering a surface vertex Σ_i of Γ_Y. By construction $b(\bar{\Sigma}_i) = b(\Sigma_i)$, so we have

$$\chi(\bar{\Sigma}_i) + b(\bar{\Sigma}_i) = p\,\chi(\Sigma_i) + b(\Sigma_i).$$

Since $\chi(\Sigma_i) \leq -1$ and $\chi(\Sigma_i) + b(\Sigma_i) \leq 2$, $\bar{\Sigma}_i$ has positive genus for $p \geq 3$. □

The above result is enough to prove that elementarily free groups are measure equivalent to free groups. It is perhaps more cleanly expressed, however, in terms of the following theorem, which we believe to be of independent interest.

Theorem 1.11 *Every limit group L has a finite-index subgroup M that is a subgroup of a positive-genus tower P. If L is elementarily free then P can be taken to be elementarily free.*

By theorem 1.4 it suffices to prove the theorem for towers. More precisely, we prove the following.

Proposition 1.12 *Let L_h be a tower of height h. Then there exists a finite-index subgroup $M_h \subset L_h$ contained in a positive-genus tower P_h. If L_h is elementarily free then P_h can be taken to be elementarily free. If $A \subset M_h$ is a maximal abelian subgroup then A is also maximal abelian in P_h.*

Proof. The proof is by induction on height. By definition, every level 0 tower is positive-genus. Consider L_h the fundamental group of an ω-rft space X_h of height h, obtained as usual by attaching a block to a height $h-1$ space X_{h-1} with fundamental group L_{h-1}.

First we consider the case of a quadratic block Σ. By induction, L_{h-1} has a finite-index subgroup M_{h-1} that is a subgroup of a positive-genus tower P_{h-1}. By proposition 1.10, L_h has a finite-index subgroup $K_h = \pi_1(Z_h)$ with graph-of-groups decomposition Γ_K, with one vertex labelled by K_{h-1} a finite-index subgroup of L_{h-1} and the remaining vertices labelled by the fundamental groups of surfaces of positive genus, amalgamated with K_{h-1} along boundary components. Set $M_h = \rho_*^{-1}(K_{h-1} \cap M_{h-1})$. Then M_h inherits a graph-of-groups decomposition Γ_M, with one vertex labelled by $K_{h-1} \cap M_{h-1}$ and the remainder by fundamental groups of surfaces of positive genus, amalgamated with $K_{h-1} \cap M_{h-1}$ along boundary components. The retraction $\rho_* : L_h \to L_{h-1}$ restricts to a retraction $M_h \to K_{h-1} \cap M_{h-1}$. Enlarge Γ_M to Γ_P by replacing $K_{h-1} \cap M_{h-1}$ with P_{h-1}. Extending ρ_* by the identity on P_{h-1}, it is clear that $P_h = \pi_1(\Gamma_P)$ is a positive-genus tower.

The case of an abelian block T is similar. By induction, there exists a finite-index subgroup $M_{h-1} \subset L_{h-1}$ that embeds in a positive genus tower P_{h-1}. The pullback $M_h = \rho_*^{-1}(M_{h-1})$ inherits a graph-of-groups decomposition Γ_M, with one vertex labelled by M_{h-1} and the remainder by finitely generated free abelian groups. Each abelian vertex has a coordinate factor amalgamated with a cyclic maximal abelian subgroup of M_{h-1}. Enlarge Γ_M to Γ_P by replacing M_{h-1} by P_{h-1}. Since cyclic maximal abelian subgroups of M_{h-1} are maximal abelian in P_{h-1}, the resulting fundamental group $P_h = \pi_1(\Gamma_P)$ is again a tower.

It remains to show that any maximal abelian subgroup A of M_h is maximal abelian in P_h. For this we need a little Bass–Serre Theory. Suppose $g \in P_h$ commutes with every element of A. If g is elliptic in Γ_P then $g \in A$ by induction on height, so assume that g acts hyperbolically on the Bass–Serre tree T_P of Γ_P, preserving an axis l. In this case, A also preserves l.

If A were conjugate into a vertex of Γ_P then A would fix l pointwise. But this would contradict the fact that Γ_P is 2-acylindrical, since in an acylindrical tree the stabilizer of a line is trivial.

Therefore there is some $a \in A$ which acts hyperbolically on T_P, also with axis l. Since the edge groups of Γ_P are precisely the images of the edge groups of Γ_M, the Bass–Serre tree T_M of Γ_M is the minimal M_h-invariant subtree of T_P and contains l. Fix an edge e in l. Then ge is an edge of l, so lies in T_M. There is only one M_h-orbit of e in T_M, so there exists $m \in M_h$ such that $me = ge$. The stabilizer of e lies in M_h, so it follows that $g \in M_h$. Since A was maximal abelian in M_h, $g \in A$. □

2 Measure equivalence

We are now in a position to use the results of [7] to prove that elementarily free group are measure equivalent to free groups. For motivation and background, we refer the reader to the papers of Damien Gaboriau, particularly [5].

2.1 Definition and properties

Definition 2.1 *Two countable groups G_1, G_2 are* measure equivalent *if there exist commuting, measure-preserving, (essentially) free actions on some measure space (Ω, m), such that the action of G_i admits a finite measure fundamental domain. Write*

$$G_1 \overset{\mathrm{ME}}{\sim} G_2.$$

The standard examples of measure-equivalent groups are commensurable groups and lattices in the same locally compact second countable group. We will not use the definition of measure equivalence directly, but deduce our result from the following properties.

Theorem 2.2 ($\mathbf{P_{ME}7}$ in [7]) *If G_1 and G_2 are measure equivalent to a free group then so is $G_1 * G_2$.*

Theorem 2.3 ($\mathbf{P_{ME}8}$ in [7]) *If G is measure equivalent to a free group and $H \subset G$ is a subgroup then H is measure equivalent to a free group.*

Theorem 0.1 is a special case of:

Theorem 2.4 (Corollary 3.18 of [7]) *Consider a countable group G measure equivalent to a free group, and $C \subset G$ an infinite cyclic subgroup. If Σ is a compact orientable surface of positive genus with a single boundary component then $G *_{C=\langle \partial\Sigma \rangle} \pi_1(\Sigma)$ is also measure equivalent to a free group.*

We generalize theorem 2.4 to the case of multiple boundary components.

Corollary 2.5 *Consider a path-connected space X with $G = \pi_1(X)$ measure equivalent to a free group. Let Σ be a compact, orientable surface of positive genus with non-empty boundary. Let X' be the quotient of $X \sqcup \Sigma$ obtained by identifying the boundary curves of Σ with loops in X that generate infinite cyclic subgroups of $\pi_1(X)$. Then $\pi_1(X')$ is measure equivalent to a free group.*

Proof. By cutting Σ along a certain simple closed curve γ, we can decompose it as $\Sigma_1 \cup_\gamma \Sigma_2$, where Σ_1 is a punctured sphere and Σ_2 is of positive genus and has one boundary component. X' acquires a similar decomposition as $X_1 \cup_\gamma \Sigma_2$, where X_1 is obtained from X by amalgamating loops on X with all of the boundary curves of Σ_1 except γ. Note that Σ_1 deformation retracts onto the graph formed by the boundary circles $c_1, \ldots, c_n$ other than γ, together with a disjoint collection of arcs α_j ($j = 2, \ldots, n$) connecting c_1 to c_j. This deformation retraction extends to a deformation retraction of X_1 onto the union of X and the arcs α_j. It follows from theorem 2.2 that $\pi_1(X_1) \cong \pi_1(X) * F_{n-1}$ is measure equivalent to a free group. Thus $\pi_1(X') = \pi_1(X_1) *_{\langle \partial\Sigma_2 \rangle} \pi_1(\Sigma_2)$ is measure equivalent to a free group, by theorem 2.4. □

We are now ready to prove that elementarily free groups are measure equivalent to free groups.

2.2 Elementarily free groups

Theorem 2.6 *Every elementarily free group is measure equivalent to a free group.*

Proof. By theorem 1.11, it suffices to prove the result for positive-genus elementarily free groups.

At height 0, X_0 is a one-point union of graphs and hyperbolic surfaces. Hyperbolic surface groups are lattices in $\mathrm{PSL}_2(\mathbb{R})$, so are measure equivalent to a free group. Thus, by corollary 2.2, $\pi_1(X_0)$ is measure equivalent to a free group.

At height h, assume that X_h is obtained as usual by gluing a surface Σ to X_{h-1}. By induction, $\pi_1(X_{h-1})$ is measure equivalent to a free group. There are two cases to consider.

If Σ is orientable, then the result is given by corollary 2.5.

If Σ is non-orientable, then it has an orientable double cover $\Sigma' \to \Sigma$ of positive genus, with twice the number of boundary components. The amalgam of Σ' with two disjoint copies of X_{h-1} gives a double cover $X'_h \to X_h$. Identify a point in each copy of X_{h-1} to create a space Y. By proposition 2.2, $\pi_1(X_{h-1} \vee X_{h-1}) = \pi_1(X_{h-1}) * \pi_1(X_{h-1})$ is measure equivalent to a free group.

We have built Y by gluing the orientable surface of positive genus Σ' to $X_{h-1} \vee X_{h-1}$, and each boundary component of Σ' defines an element of infinite order in one of the free factors of $\pi_1(X_{h-1} \vee X_{h-1}$. It follows by corollary 2.5 that $\pi_1(Y)$ is measure equivalent to a free group. Since $\pi_1(Y) \cong \pi_1(X'_h) * \mathbb{Z}$, the result follows from theorem 2.3. □

3 The case of arbitrary limit groups

In the light of theorem 2.3, to show that all limit groups are measure equivalent to free groups it would suffice to prove that ω-residually free towers are measure equivalent to free groups. Even the case of $F_C = F *_{C=Z} \mathbb{Z}^n$, where C is a maximal cyclic subgroup of F and Z is a direct factor of $\mathbb{Z}^n$, seems non-trivial. The methods of the proof of theorem 2.4 in [7] suggest a possible approach.

Let (X, μ) be a probability measure space, and consider an essentially free measure-preserving action α of the group G on X. The *orbit relation* of the action is the equivalence relation given by the orbits of G, and is denoted $\mathcal{R}_\alpha$. There is a notion of free products for equivalence relations, motivated by the normal form for free products of groups.

Definition 3.1 *Consider measured equivalence relations $\mathcal{R}, \mathcal{A}$ and $\mathcal{B}$ on X. Write $\mathcal{R} = \mathcal{A} * \mathcal{B}$ if:*

1. *$\mathcal{R}$ is generated by $\mathcal{A}$ and $\mathcal{B}$;*
2. *for almost all $2p$-tuples $(x_i)_{i \in \mathbb{Z}/2p\mathbb{Z}}$ such that*

$$x_{2j-1} \sim_{\mathcal{A}} x_{2j} \sim_{\mathcal{B}} x_{2j+1}$$

for each j, one has $x_i = x_{i+1}$ for some i.

Gaboriau defines a subgroup $H \subset G$ to be a *measure free factor* if there exists a free probability-measure-preserving action α of G and a subrelation $\mathcal{S}$ of $\mathcal{R}_\alpha$ so that

$$\mathcal{R}_\alpha = \mathcal{R}_{\alpha|H} * \mathcal{S}.$$

It follows from the normal form theorem for free products that free factors are measure free factors. In [7], Gaboriau constructs a non-trivial example: the boundary circle of a positive-genus orientable surface with one boundary component generates a measure free factor. Theorem 2.4 is a special case of:

Theorem 3.2 (Theorems 3.13 and 3.17 of [7]) *If G and G' are measure equivalent to free groups, $C \subset G$ and $C' \subset G'$ are infinite cyclic subgroups, and C is a measure free factor in G, then $G *_{C=C'} G'$ is measure equivalent to a free group.*

At present, the only non-trivial C for which we know that $F_C \overset{\mathrm{ME}}{\sim} F_2$ is that given by Gaboriau's example, in which C is generated by a boundary component of an orientable surface. It is natural to ask if each maximal cyclic subgroup of F is a measure free factor. If this were so, then it would follow from theorem 3.2 that every F_C is measure equivalent to a free group. It is also natural to generalize the question to towers, and ask if each maximal abelian cyclic subgroup of a tower is a measure free factor. Again, if so, it would follow that every limit group is measure equivalent to a free group.

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