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Direct factors of profinite completions and decidability

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Abstract. We consider finitely presented, residually finite groups G and finitely generated normal subgroups A such that the inclusion $A \hookrightarrow G$ induces an isomorphism from the profinite completion of A to a direct factor of the profinite completion of G. We explain why A need not be a direct factor of a subgroup of finite index in G; indeed G need not have a subgroup of finite index that splits as a non-trivial direct product. We prove that there is no algorithm that can determine whether A is a direct factor of a subgroup of finite index in G.

Let G be a finitely generated residually finite group. The inclusion $A \hookrightarrow G$ of any finitely generated subgroup induces a morphism of profinite completions $\iota : \hat{A} \to \hat{G}$. If A is a direct factor of G then ι is injective and we can identify the closure $\bar{A}$ of $\iota(A)$ with $\hat{A}$. In [13] Nikolov and Segal answered a question of Goldstein and Guralnick [11] by showing that the converse of the preceding observation is false: there exist pairs of finitely generated residually finite groups $A \hookrightarrow G$, with A normal in G, such that $\iota : \hat{A} \to \bar{A}$ is an isomorphism, $\bar{A}$ is a direct factor of $\hat{G}$, but A is not a direct factor of G, nor indeed of any subgroup of finite index in G.

Nikolov and Segal proved this by exhibiting an explicit group of the form $G = A \rtimes_\alpha \mathbb{Z}$, where A is finitely generated and α, although not inner, induces an inner automorphism on A/N for every α-invariant subgroup of finite index $N \subset A$.

The first purpose of the present note is to explain how pairs of residually finite groups $A \hookrightarrow G$ settling the Goldstein–Guralnick question also arise from the constructions in [7]. As well as providing a broader range of examples, these constructions allow one to impose extra conditions on A and G (see Subsection 1.2). For example, one can require G to be finitely presented, indeed to be a direct product of torsion-free hyperbolic groups and hence have a finite classifying space. If one drops the requirement that A be normal, one can arrange for both A and G to be finitely presented.

Our main construction also provides a large classes of examples of the Nikolov–Segal type $G = A \rtimes_\alpha \mathbb{Z}$; see Subsection 1.3.

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The second part of this note concerns the decision problem associated to the Goldstein–Guralnick question: a group G is given by a finite presentation $G = \langle X \cup \mathscr{A} \mid R \rangle$; it is guaranteed that G is residually finite and that the subgroup $A \subset G$ generated by $\mathscr{A}$ has the property that the inclusion map induces an isomorphism from $\hat{A}$ to a direct factor of $\hat{G}$; is there an algorithm that, given this data, can determine whether or not A is a direct factor of a subgroup of finite index in G?

Theorem 0.1. *There does not exist an algorithm that, given the above data, can determine whether or not A is a direct factor of any subgroup of finite index in G.*

Theorem 2.2 below provides a more precise formulation of this result.

1 Building examples

In [7] Bridson and Grunewald settled a question of Grothendieck [9] by constructing pairs of finitely presented, residually finite groups $j : P \hookrightarrow \Gamma$ such that $\hat{j} : \hat{P} \to \hat{\Gamma}$ is an isomorphism but P is not isomorphic (or even quasi-isometric to) Γ. Pairs of finitely generated groups with this property had been found earlier by Platonov and Tavkin [14], Bass and Lubotzky [2], and Pyber [15]. A simplified form of the arguments given in [7] provides a flexible technique for constructing finitely generated examples of a different type. The purpose of this section is to explain these arguments and to apply them to the study of direct factors of profinite groups.

1.1 The basic construction.

Theorem 1.1. *If Q is a finitely presented group that is infinite but has no non-trivial finite quotients, and if $H_2(Q, \mathbb{Z}) = 0$, then there is a short exact sequence $1 \to N \to \Gamma \to Q \to 1$ of groups where Γ is finitely presented and residually finite, N is finitely generated but not finitely presentable, and $N \to \Gamma$ induces an isomorphism of profinite completions $\hat{N} \to \hat{\Gamma}$. Moreover there exists an algorithm that, given a finite presentation of Q, will construct a finite presentation of Γ and a generating set for N.*

In order to make this theorem useful one needs a supply of suitable groups Q. This presents no difficulty since one can embed any finitely presented group H in a finitely presented group $\overline{H}$ that has no finite quotients, matching the geometry and complexity of $\overline{H}$ to that of H in various ways [5], and the universal central extension of $\overline{H}$ can then serve as Q (see [7, §8]). Historically speaking, the first suitable group Q is Higman's famous example

$$J = \langle a, b, c, d \mid aba^{-1} = b^2, bcb^{-1} = c^2, cdc^{-1} = d^2, dad^{-1} = d^2 \rangle.$$

Other small examples are constructed in [7].

Theorem 1.1 is a consequence of the following two results.

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Lemma 1.2. *Let* $1 \to N \to \Gamma \to Q \to 1$ *be a short exact sequence of finitely generated groups. If* Q *has no non-trivial finite quotients and* $H_2(Q, \mathbb{Z}) = 0$, *then* $N \to \Gamma$ *induces an isomorphism* $\hat{N} \to \hat{\Gamma}$.

Proof. The surjectivity of $\hat{N} \to \hat{\Gamma}$ is an immediate consequence of the observation that since Q has no finite quotients, if $f : \Gamma \to F$ is a homomorphism onto a finite group then $f(N) = F$. For injectivity, it is enough to consider finite index subgroups $I \subset N$ that are normal in Γ and to find a finite index subgroup $H \subset \Gamma$ that intersects N in I. The action of Γ by conjugation on N induces a map to the automorphism group of N/I, with kernel M say. Since M has finite index in Γ, it maps onto Q and we have a central extension

$$1 \to (N/I) \cap (M/I) \to M/I \to Q \to 1.$$

Since Q is super-perfect (that is, $H_1 Q = H_2 Q = 0$), this extension splits (see [12, pp. 43–47]. Setting H equal to the kernel of the resulting homomorphism $M \to M/I \to (N/I) \cap (M/I)$ completes the proof. □

The second ingredient in the proof of Theorem 1.1 is Wise's variation on the Rips construction (see [16], [17]).

Theorem 1.3. *There exists an algorithm that, given a finite presentation of a group* $Q = \langle X | R \rangle$, *constructs a finite presentation* $\langle X \cup \{v_1, v_2, v_3\} \mid S \rangle$ *for a torsion-free, residually finite group* Γ, *of cohomological dimension* 2, *that is hyperbolic in the sense of Gromov. The subgroup* $N \subset \Gamma$ *generated by* $\{v_1, v_2, v_3\}$ *is normal but not free, and* $\Gamma/N \cong Q$.

The Rips–Wise algorithm, which is based on small-cancellation theory, is extremely explicit—see [7, §7].

The only assertion of Theorem 1.1 that does not follow immediately from Lemma 1.2 and Theorem 1.3 is the fact that N is not finitely presentable. This is a special case of Bieri's theorem that a finitely presentable normal subgroup of a group of cohomological dimension 2 is either free or of finite index [4].

1.2 Answer to the Goldstein–Guralnick question. The following examples $A \hookrightarrow G$ provide a negative answer to the question of Goldstein and Guralnick described in the introduction. This question arose from a desire to weaken the hypotheses in their generalisation [11] of Ayoub's splitting theorem [1].

Example 1.4. Let N and Γ be as in Theorem 1.1 and let B be any finitely presented residually finite group. Let $G = \Gamma \times B$ and let $A = N \times \{1\}$. Then $A \to G$ induces an isomorphism $\hat{A} \to \bar{A}$ and $\hat{G} = \hat{\Gamma} \times \hat{B} = \bar{A} \times \hat{B}$. But A is not (isomorphic to) a direct factor of G or any subgroup of finite index in G, since such direct factors are finitely presentable.

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Example 1.5. To obtain an example where G and its subgroups of finite index have no non-trivial direct product decompositions whatsoever, one takes $B = \Gamma$ in the above construction and instead of defining G to be $\Gamma \times \Gamma$ one takes it to be the fibre product $P \subset \Gamma \times \Gamma$ of the map $\pi : \Gamma \to Q$ in Theorem 1.1. More explicitly, $P = \{(\gamma_1, \gamma_2) \mid \pi(\gamma_1) = \pi(\gamma_2)\}$. It is proved in [3] that if Q has an Eilenberg–Maclane space $K(Q, 1)$ with a finite 3-skeleton (as Higman's group J does, for example), then P will be finitely presented. And it is proved in [7] that $P \to \Gamma \times \Gamma$ induces an isomorphism of profinite completions. By examining centralizers one can see that P does not virtually split as a direct product (see [7, §6]). It follows that the inclusion of $A = N \times \{1\}$ into P maps $\hat{A}$ isomorphically onto the first factor of $\hat{P} = \hat{\Gamma} \times \hat{\Gamma}$, but neither P nor any of its subgroups of finite index decomposes as a non-trivial direct product.

Example 1.6. With P and Γ as in the preceding example, $P \times \{1\} \hookrightarrow (\Gamma \times \Gamma) \times \Gamma$ provides us with a pair of *finitely presented* groups $A \hookrightarrow G$ with $\hat{A} \cong \bar{A}$ a direct factor of $\hat{G}$ but A not a direct factor of any subgroup of finite index in G. But in this example A is not normal in G.

1.3 Examples of Nikolov–Segal type $G = A \rtimes \mathbb{Z}$. We continue with the notation established in the proof of Lemma 1.2, insisting now that Q be infinite. Since Q has no finite quotients, the image of Γ in the automorphism group of N/I must coincide with that of N, in other words Γ must act on N/I by inner automorphisms. Suppose that Γ is torsion-free and hyperbolic, and note that N cannot be cyclic as it has infinite index in its normalizer [8]. In this case, no element $\gamma \in \Gamma \backslash N$ can act on N (by conjugation in Γ) as an inner automorphism, for if it acted as conjugation by $n \in N$, say, then $n^{-1}\gamma$ would centralize N, whereas non-cyclic subgroups of torsion-free hyperbolic groups have trivial centralizers [8].

Thus, if Q is torsion-free (Higman's group, for example) and Γ is constructed as in Theorem 1.3, then for every $\gamma \in \Gamma \backslash N$ we have $\langle N, \gamma \rangle = N \rtimes \langle \gamma \rangle \cong N \rtimes_\alpha \mathbb{Z}$ where no power of α is inner but α acts as an inner automorphism on N/I for every characteristic subgroup of finite index $I \subset N$ (and hence every α-invariant subgroup of finite index K, as one sees by considering the intersection I of all subgroups of index $|N/K|$).

2 The recognition problem for direct factors

In this section we shall prove (a more precise version of) Theorem 0.1. The seed of undecidability that we shall exploit in order to prove Theorem 0.1 comes from the following theorem, which is proved in [6].

Theorem 2.1. *There exists a finitely generated free group $F = F(X)$ and a recursive sequence of finite subsets $\mathcal{R}_n \subset F$ so that there is no algorithm to determine which of the groups $Q_n = F/\langle\langle \mathcal{R}_n \rangle\rangle$ is trivial, but each of the groups has the following properties*:

(1) $H_1(Q_n; \mathbb{Z}) = H_2(Q_n; \mathbb{Z}) = 0$;

(2) *Q_n has no non-trivial finite quotients.*

(*If $Q_n \neq 1$ then Q_n is infinite*.)

This theorem is proved in three stages. First one constructs a sequence of finite group-presentations $\Pi_n \equiv \langle Y | \Sigma_n \rangle$ so that the groups presented are torsion-free and there is no algorithm that can determine which are trivial. Secondly, one modifies these presentations in an algorithmic manner so as to ensure that none of the groups presented has any proper subgroups of finite index. Finally, an additional algorithm is implemented that replaces Π_n', the modified Π_n, with a finite presentation $\tilde{\Pi}_n \equiv \langle X | \mathscr{R}_n \rangle$ for the universal central extension of the group presented by Π_n'. See [6] for details.

2.1 The proof of Theorem 0.1. Consider the sequence of pairs of groups $N_n \hookrightarrow \Gamma_n$ obtained by applying the Rips–Wise algorithm (Theorem 1.3) to the presentations $\langle X | \mathscr{R}_n \rangle$ from Theorem 2.1. The output of the algorithm is a recursive sequence of finite presentations $\mathscr{P}_n \equiv \langle X \sqcup \{v_1, v_2, v_3\} \,|\, S_n \rangle$ for Γ_n, with $N_n \subset \Gamma_n$ given as the subgroup generated by $\{v_1, v_2, v_3\}$. Augmenting $\mathscr{P}_n$ with an additional generator t and the relations $[t, x] = 1$ for all $x \in X \cup \{v_1, v_2, v_3\}$ gives a recursive sequence of presentations $\mathscr{P}_n^+$, for $G_n := \Gamma_n \times \mathbb{Z}$ with $A_n := N_n \times \{1\}$ the subgroup generated by $\{v_1, v_2, v_3\}$.

If $Q_n = 1$ then $N_n = \Gamma_n$. If $Q_n \neq 1$ then Theorem 1.1 assures us that the inclusion $N_n \hookrightarrow \Gamma_n$ still induces an isomorphism $\hat{N}_n \to \hat{\Gamma}_n$, but N_n is not finitely presentable. Thus $A_n \hookrightarrow G_n$ always maps $\hat{A}_n$ isomorphically to the first factor of $\hat{G}_n = \hat{\Gamma}_n \times \hat{\mathbb{Z}}$, but A_n is a direct factor of G_n (equivalently, some subgroup of finite index in G_n) if and only if $Q_n \neq 1$. And there is no algorithm that can determine for which n the group Q_n is trivial. $\square$

The version of Theorem 0.1 stated in the introduction was crafted so as to be immediately comprehensible and free of technical jargon. We close with a more technical statement that has greater precision. This is what is actually proved by the preceding argument.

Theorem 2.2. *There exists a finite set $Y = X \sqcup \{v_1, v_2, v_3, t\}$ and a recursive sequence (S_n) of finite sets of words in the letters $Y^{\pm 1}$ so that the groups $G_n := F(Y)/\langle\!\langle S_n \rangle\!\rangle$, the subgroup $A_n \subset G_n$ generated by the image of $\{v_1, v_2, v_3\}$, and the subgroup $B_n \subset G_n$ generated by the image of $\{t\}$, have the following properties*:

(1) *each $G_n := F(Y)/\langle\!\langle S_n \rangle\!\rangle$ is residually-finite and torsion-free*;

(2) *each B_n is infinite and $\hat{G} = \bar{A}_n \times \bar{B}_n$*;

(3) *the inclusion $A_n \hookrightarrow G_n$ induces an isomorphism $\hat{A}_n \to \bar{A}_n$*;

(4) *the set $\{n | A_n$ is a direct factor of $G_n\} \subset \mathbb{N}$ is not recursive*;

(5) *if A_n is not a direct factor of G_n then neither is it a direct factor of any subgroup of finite index in G_n.*

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