An Invitation to Harmonic Analysis

Adam Brown

Abstract

Fourier series are a central topic in the study of differential equations. However, it can be difficult to gain an intuition for these mysterious decompositions. We will explore how Fourier series naturally appear in representation theory, and how they can be used to solve differential equations. Generalizations of our techniques compose an extremely interesting field known as harmonic analysis.

Definition

A *differential equation* is an equation which relates a function with it's derivatives.

For example, if f is a function of the variable x, we could require that f satisfies the following differential equation:

$$\frac{df}{dx} = 2xf(x) \tag{1}$$

A *solution* to a differential equation is a function which satisfies the equation. For example,

$$f(x) = e^{x^2}$$

satisfies (1)

$$\frac{df}{dx} = 2xe^{x^2} = 2xf(x)$$

Definition

If we have a function of several variables, a *partial differential equation* is an equation which relates a function and it's partial derivatives.

For example:

$$\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} = 0 \tag{2}$$

For example, the function $f(x_1, x_2) = x_1 - x_2$ solves (2)

$$\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} = 1 - 1 = 0$$

Mathematicians have been developing methods to solve differential equations for hundreds of years. Solutions of different deferential equations are extremely useful in a vast range of fields such as physics, engineering, mathematical biology, etc. Today we will talk about how we can apply some results from representation theory and harmonic analysis to solve some differential equations.

Heat Equation

The heat equation describes the diffusion of heat through a material. Mathematically, the heat equation is

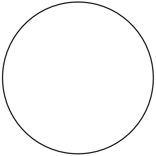
$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = \frac{\partial u}{\partial t}$$

where u is a function of $x_1, \cdots, x_n$ and t. We can think of $x_1, \cdots, x_n$ as our spatial variables, and t as time. If we plug in a particular point in space and a particular time, u should tell us the temperature at that point. If we want to consider heat dissipating through different materials with different specific heats and physical properties, we will have several constants and possibly some other terms. But we will ignore these constants to make our computations more elegant.

Heat Equation on a Circle

Consider a circle.

S^1 :



If we want to study how heat would dissipate through a metal circle like this, we need to solve

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial u}{\partial t} \tag{3}$$

we will denote equation (3) by $u_{\theta\theta} = u_t$.

Heat Equation on a Circle

We need to begin by describing the initial heat distribution at time $t = 0$. This is called the initial condition

$$u(\theta, 0) = f(\theta)$$

Let us consider initial temperature distributions $f(\theta)$ which are square integrable:

$$\int_0^1 f(\theta)^2 d\theta < \infty$$

We can think about that as a requirement that there is a finite amount of thermal energy in our metal circle.

Heat Equation on a Circle

The space of square integrable functions on S^1 is a Hilbert space, ie we can add functions and multiply them by constants. If $f, g \in L^2(S^1)$ and c is a constant, then

$$\begin{aligned} f(\theta) + g(\theta) &\in L^2(S^1) \\ cf(\theta) &\in L^2(S^1) \end{aligned}$$

We can think of $L^2(S^1)$ as an infinite dimensional vector space. So we should be able to write $f(\theta)$ as a linear combination of basis vectors. If we choose our basis vectors carefully, we can try to solve the initial condition problem for each basis vector, and then use the principle of superposition to find the solution for $f(\theta)$.

Representation Theory

Observe that the circle S^1 has a rotational symmetry. If we rotate the circle by an angle θ, the result is still a circle. We may be able to exploit this symmetry to find a basis for the space of functions on the circle.

To simplify the mathematics, lets consider f as a function from the circle S^1 to the complex numbers $\mathbb{C}$. The real part of the image will correspond to the temperature at a particular point on the circle. So

$$f : S^1 \to \mathbb{C}$$

Representation Theory

We can incorporate the rotational symmetry by rotating the domain of our function of by an angle ϕ:

$$f_\phi(\theta) = f(\theta - \phi)$$

Example: If $f(\theta) = \sin(\theta)$, then we can rotate f to get

$$f_{\pi/2}(\theta) = \sin(\theta - \pi/2) = -\cos(\theta)$$

What functions f, have the property that

$$f_\phi(\theta) = g(\phi)f(\theta)$$

for some $g : S^1 \to \mathbb{C}$.
Answer:

Theorem (Peter-Weyl Theorem)

If $f(\theta) = e^{in\theta}$ for an integer $n \in \mathbb{Z}$, then

$$f_\phi(\theta) = e^{in(\theta-\phi)} = e^{-in\phi}e^{in\theta} = g(\phi)f(\theta)$$

Representation Theory

Furthermore, the Peter Weyl theorem tells us that we can think of all the possible initial conditions as a vector space with basis $\{e^{in\theta}\}$. So we can write a generic initial condition as a linear combination of basis elements:

$$f(\theta) = \sum_{n\in\mathbb{Z}} c_n e^{in\theta}$$

This is called the Fourier Series of f.

How do we find the constants c_n?
Projection formula:

$$\begin{aligned}
\mathrm{Proj}_{e^{in\theta}}(f) &= \frac{\langle f, e^{in\theta} \rangle}{||e^{in\theta}||^2} e^{in\theta} \\
&= \left(\frac{1}{2\pi} \int_0^{2\pi} f(\phi) \overline{e^{in\phi}} d\phi \right) e^{in\theta} \\
&= \left(\frac{1}{2\pi} \int_0^{2\pi} f(\phi) e^{-in\phi} d\phi \right) e^{in\theta}
\end{aligned}$$

So we have a formula for the constants:

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) e^{-in\phi} d\phi$$

Heat Equation on a Circle

Now we want to solve $u_{\theta\theta} = u_t$ with initial condition $u(\theta, 0) = e^{in\theta}$. Let us take a guess that the answer will look something like

$$u(\theta, t) = e^{in\theta} g(t)$$

where we need to determine the function g. Solving a differential equation with this method is called separation of variables. If we plug our guess into the differential equation we get

$$\begin{aligned} -n^2 e^{in\theta} g(t) = u_{\theta\theta} &= u_t = e^{in\theta} g'(t) \\ -n^2 e^{in\theta} g(t) &= e^{in\theta} g'(t) \\ -n^2 g(t) &= g'(t) \end{aligned}$$

So in order to find $g(t)$, we just need to solve the ordinary differential equation.

Heat Equation on a Circle

$$
\begin{aligned}
g'(t) &= -n^2 g(t) \\
\frac{dg}{dt} &= -n^2 g \\
\frac{dg}{g} &= -n^2 dt \\
\int \frac{1}{g} dg &= \int -n^2 dt \\
\ln(g) &= -n^2 t + c \\
g(t) &= Ce^{-n^2 t}
\end{aligned}
$$

From our separation of variables assumption, we know that $u(\theta, 0) = e^{in\theta} g(0) = e^{in\theta}$, so $g(0) = 1$ implies that $C = 1$.

Heat Equation on a Circle

Now we can state the solution to the heat equation on a circle with initial condition $u(\theta, 0) = e^{in\theta}$:

$$u(\theta, t) = e^{in\theta} e^{-n^2 t}$$

We can extend this solution to a solution for any initial condition $f(\theta)$ with the Fourier series. If

$$f(\theta) = \sum_{n \in \mathbb{Z}} c_n e^{in\theta}$$

Then the solution to the heat equation with initial condition $f(\theta)$ will be a linear combination of solutions of the heat equation with initial conditions $e^{in\theta}$

Heat Equation on a Circle

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$$u(\theta, t) = \sum_{n\in\mathbb{Z}} c_n e^{in\theta} e^{-n^2 t}$$

Now we need to note that the right hand side of the equation has real and imaginary values. The real values are what we are interested in, because they tell us what the temperature is at a given location and time.

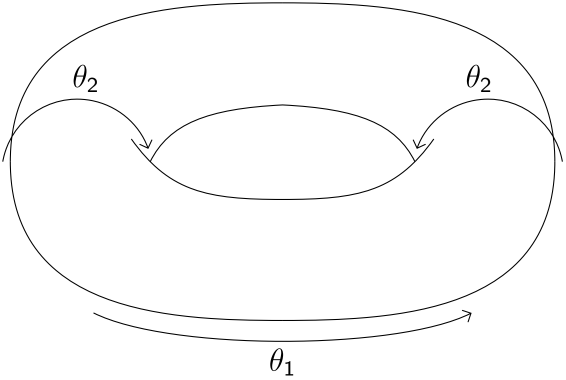
Heat Equation on a Circle

A small calculation gives us

$$\begin{aligned}
\mathcal{R}(u(\theta, t)) &= c_0 + \sum_{n=1}^{\infty} a_n \sin(2\pi n\theta) + b_n \cos(2\pi n\theta) \\
c_0 &= \int_0^{2\pi} f(\phi) d\phi \\
a_n &= 2\int_0^{2\pi} f(\phi) \sin(2\pi n\phi) d\phi \\
b_n &= 2\int_0^{2\pi} f(\phi) \cos(2\pi n\phi) d\phi
\end{aligned}$$

This is called the general solution to the heat equation.

Heat Equation on the Torus



$$u_{\theta_1\theta_1} + u_{\theta_2\theta_2} = u_t$$

Now we can use two different rotational symmetries to find a basis for the space of functions on the torus.

$$f_{(\phi_1,\phi_2)}(\theta_1, \theta_2) = f(\theta_1 - \phi_1, \theta_2 - \phi_2)$$

Which functions have the property that

$$f_{(\phi_1,\phi_2)}(\theta_1,\theta_2) = f(\theta_1 - \phi_1, \theta_2 - \phi_2) = g(\phi_1,\phi_2) f(\theta_1,\theta_2)$$

Answer:

$$f(\theta_1,\theta_2) = e^{in\theta_1} e^{im\theta_2}$$

for $n, m \in \mathbb{Z}$. So we want to solve our PDE with initial conditions $u(\theta_1,\theta_2,0) = e^{in\theta_1} e^{im\theta_2}$

Heat Equation on the Torus

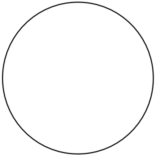
Again, let us assume that our solution will have the form $u(\theta_1, \theta_2, t) = f(\theta_1, \theta_2)g(t)$. Then

$$\begin{aligned}
u_{\theta_1\theta_1} + u_{\theta_2\theta_2} &= u_t \\
(-n^2 e^{in\theta_1} e^{im\theta_2} - m^2 e^{in\theta_1} e^{im\theta_2})g(t) &= e^{in\theta_1} e^{im\theta_2} g'(t) = u_t \\
(-n^2 - m^2)g(t) &= g'(t) \\
g(t) &= e^{-n^2 t} e^{-m^2 t}
\end{aligned}$$

So our general solution has the form

$$\begin{aligned}
u(\theta_1, \theta_2, t) &= \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} c_{n,m} e^{in\theta_1} e^{im\theta_2} e^{-n^2 t} e^{-m^2 t} \\
c_{n,m} &= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f(\theta_1, \theta_2) e^{-in\theta_1} e^{-im\theta_2} d\theta_1 d\theta_2
\end{aligned}$$

Wave Equation on S^1

S^1 : 

$$u_{\theta\theta} = u_{tt}$$

We already found a basis of initial conditions on the circle, $\{e^{in\theta}\}$, so we just need to solve the new differential equation for each of these initial conditions. Assume our solution has the form $u(\theta, t) = e^{in\theta} g(t)$. Then

$$\begin{aligned}
-n^2 e^{in\theta} g(t) = u_{\theta\theta} &= u_{tt} = e^{in\theta} g''(t) \\
-n^2 g(t) &= g''(t) \\
g(t) &= ce^{int} \quad g(0) = 1 \longrightarrow c = 1 \\
u(\theta, t) &= e^{in\theta} e^{int}
\end{aligned}$$

Wave Equation on S^1

So our general solution for the initial condition $u(\theta, 0) = f(\theta)$ is

$$\begin{aligned} u(\theta, t) &= \sum_{n \in \mathbb{Z}} c_n e^{in\theta} e^{int} \\ c_n &= \frac{1}{2\pi} \int_0^{2\pi} f(\phi) e^{-in\phi} d\phi \end{aligned}$$