Universität Hamburg
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Algebraic Topology
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Sheet 12

Solutions are due on 06.07.18.

Problem 12.1

(a) Let R be a commutative ring with unit 1_R and let I be a directed poset. Show that if

$$0 \longrightarrow (A_i) \xrightarrow{(\phi_i)} (B_i) \xrightarrow{(\psi_i)} (C_i) \longrightarrow 0$$

is a short exact sequence of directed systems of R-modules indexed by I, then the sequence of R-modules

$$0 \longrightarrow \varinjlim A_i \xrightarrow{\phi} \varinjlim B_i \xrightarrow{\psi} \varinjlim C_i \longrightarrow 0$$

is short exact.

Note that this implies that the direct limit of a directed system preserves kernels and cokernels.

(b) Prove that if $(A_i)_{i \in I}$ is a directed system of chain complexes, then

$$\varinjlim H_m(A_i) \cong H_m(\varinjlim A_i)\,.$$

Problem 12.2

(a) Let X be path-connected and non-compact and let G be an arbitrary abelian group. Compute $H^0_c(X;G)$.

(b) Show that

$$H^n_c(X \times \mathbb{R}^m; G) \cong H^{n-m}_c(X;G) \qquad \forall\, m \leq n\,.$$

Which earlier result in the lectures does this resemble closely?

Problem 12.3

Let M be a compact connected orientable 3-manifold without boundary. Suppose that the fundamental group $\pi_1(M)$ is known. Determine the homology groups $H_*(M)$ as explicitly as possible from this information.

Problem 12.4

We define the m-dimensional halfspace

$$\mathbb{R}^m_- := \left\{(x_1, \ldots, x_m) \in \mathbb{R}^m \,|\, x_1 \leq 0\right\}.$$

As a subset of $\mathbb{R}^m$, its topological boundary $\partial\mathbb{R}^m_-$ consists of those points with $x_1 = 0$.

An m-dimensional *topological manifold with boundary* is a Hausdorff space M with a countable basis of its topology, together with homeomorphisms $\varphi_i \colon U_i \to V_i$ with the following properties. We have $U_i \subset M$ and $V_i \subset \mathbb{R}^m_-$ are open, and $(U_i)_{i \in I}$ forms an open cover of M. A point $x \in M$ is called a *boundary point of* M if there is a homeomorphism $\varphi \colon U \to V$ with $U \subset M$ open, $x \in U$, $V \subset \mathbb{R}^m_-$ open, and $\varphi(x) \in \partial\mathbb{R}^m_-$. The set of boundary points of M is denoted by ∂M.

Compute the following boundaries of topological manifolds with boundary:

(a) $\partial([0,1])$ and $\partial([0,1] \times [0,1])$.

(b) $\partial(\mathbb{D}^2 \times \mathbb{S}^1)$.

(c) $\partial(\mathbb{D}^2 \times \mathbb{D}^1)$ and $\partial(\mathbb{D}^2 \times \mathbb{D}^2)$.

(d) $\partial\,\mathbb{S}^n$ and $\partial\,\mathbb{R}P^n$.

(e) $\partial(\mathbb{S}^1 \times \mathbb{S}^1 \setminus \mathring{\mathbb{D}}^2)$, i.e. determine the boundary of the complement of an open disc in a 2-torus.

(f) $\partial(M \times N)$, where M and N are manifolds with boundary.