Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

Dr. Severin Bunk
Algebra und Zahlentheorie
Fachbereich Mathematik
Universität Hamburg

Algebraic Topology
Summer 2018

Sheet 6

Solutions are due on 18.05.18.

Problem 6.1

Let $f\colon A_* \to B_*$ be a chain map. The *mapping cone of* f is the chain complex $(\mathsf{C}(f), D)$ with

$$\mathsf{C}(f)_n := B_n \oplus A_{n-1} \quad \text{and} \quad D(b,a) := \big(\partial b - f(a),\, -\partial a\big)$$

(a) Show that $(\mathsf{C}(f), D)$ is a chain complex.

(b) Show that there is an exact sequence of complexes

$$0 \longrightarrow B_* \xrightarrow{\ j\ } \mathsf{C}(f)_* \xrightarrow{\ q\ } A[-1]_* \longrightarrow 0\,,$$

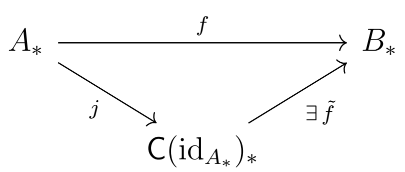
where for $p \in \mathbb{Z}$ we let $(A[p]_*, \partial^{[p]})$ be the shifted complex with

$$A[p]_n := A_{n+p} \quad \text{and} \quad \partial^{[p]}_n = (-1)^p\, \partial_{n+p}\,.$$

Observe that this sequence is split at every level $n \in \mathbb{Z}$ as a short exact sequence of abelian groups, but not as a short exact sequence of chain complexes.

(c) Compute the connecting homomorphism in the induced long exact sequence in homology and explain what the homology of $\mathsf{C}(f)_*$ tells us about f.

(d) Show that f is null-homotopic if and only if f factorises as



i.e. if there exists a chain map $\tilde{f}$ that makes the above diagram commute.

Problem 6.2

For this question you will need to recall the notions of *coverings* and their *lifting properties*. This can be found, for example in the Topologie notes, [Top, Definition 2.7.2], or on p. 56 in Hatcher's book [Hat].

(a) Let X be a path-connected, locally path-connected, and simply connected topological space. Let $p\colon E \to B$ be a covering with E contractible. Prove that every continuous map $f\colon X \to B$ induces only zero maps in reduced homology, i.e. $\tilde{H}_n(f) = 0$ for all $n \in \mathbb{N}_0$.

(b) Show that for $n, m \in \mathbb{N}$ with $m \geq 2$, any map $\mathbb{S}^m \to \mathbb{T}^n = (\mathbb{S}^1)^n$ induces the zero map on all reduced homology groups. Give a counterexample for $m = 1$.

Problem 6.3 Let Y, X, Z be topological spaces and let $\mathcal{T}\mathrm{op}(Y, X)$ denote the set of continuous maps from Y to X. In the proof of Proposition 1.11.14, you have met the *compact-open topology* $\mathcal{T}_{Y,X}$ on the set $\mathcal{T}\mathrm{op}(Y, X)$. A subbasis for $\mathcal{T}_{Y,X}$ is given by the set

$$\mathcal{S}_{Y,X} := \{\mathcal{V}(K, U) \subset \mathcal{T}\mathrm{op}(Y, X) \mid K \subset Y \text{ compact, } U \subset X \text{ open}\}, \quad \text{where}$$
$$\mathcal{V}(K, U) := \{f \in \mathcal{T}\mathrm{op}(Y, X) \mid f(K) \subset U\}.$$

Typical notations for the topological space $(\mathcal{T}\mathrm{op}(Y, X), \mathcal{T}_{Y,X})$ are X^Y, or $\underline{\mathcal{T}\mathrm{op}}(Y, X)$.

(a) Show that if Y is locally compact Hausdorff, then the evaluation map

$$\mathrm{ev}\colon \underline{\mathcal{T}\mathrm{op}}(Y, X) \times Y \to X, \quad (f, y) \mapsto f(y)$$

is continuous.
Hint: consider $(f, y) \in \mathrm{ev}^{-1}(U)$ and use that locally compact Hausdorff spaces are regular to find a compact neighbourhood $\overline{V}$ of y in Y with $\overline{V} \subset f^{-1}(U)$.

(b) Still assuming that Y is locally compact Hausdorff, show that there is a bijection (of sets)

$$\varphi\colon \mathcal{T}\mathrm{op}(Y \times Z, X) \xrightarrow{\cong} \mathcal{T}\mathrm{op}\big(Z, \underline{\mathcal{T}\mathrm{op}}(Y, X)\big), \qquad \big(\varphi(f)(z)\big)(y) := f(y, z).$$

Hint: First, for every $z \in Z$ show that $\varphi(f)(z)$ is a continuous map $Y \to X$ (write $\varphi(f)(z)$ as a composition of continuous maps). Second, to show that $\varphi(f)$ is continuous, restrict your attention to the subbasis sets $\mathcal{V}(K, U)$ for $K \subset Y$ compact and $U \subset X$ open. For fixed $z \in Z$, consider points $(y, z) \in f^{-1}(U)$ and use the product topology to find a suitable open neighbourhood $z \in W \subset Z$ as a finite intersection. Third, for the converse, use the continuity of g and the regularity of Y to find a compact subset $\overline{V}_z$ such that $g(z) \subset \mathcal{V}(\overline{V}_z, U)$.

Remark: if in addition Z is Hausdorff, the map φ is even a homeomorphism

$$\varphi\colon \underline{\mathcal{T}\mathrm{op}}(Y \times Z, X) \xrightarrow{\cong} \underline{\mathcal{T}\mathrm{op}}\big(Z, \underline{\mathcal{T}\mathrm{op}}(Y, X)\big).$$