Seminar on Topological Quantum Field Theories

Winter 20/21

In this seminar we will familiarise ourselves with the ideas and several results in the mathematical field of topological quantum field theories (TQFTs). These are an active field of research in mathematics and physics, and ideally participants would get to a point where they can understand and appreciate current problems in this area. We will start by reviewing the mathematical basics (manifolds, monoidal categories) and the physical motivation for TQFTs (the path integral), before studying TQFTs formally. Generally, TQFTs translate gluing relations between manifolds into algebraic data, and one of our main goals will be to understand and classify various flavours of TQFTs. We will thereby learn more about both sides of this translation. Finally, at the end of the semester we will have one or two talks surveying advanced topics that lead on to current research topics. Our main reference will be the lecture notes [Saf].

Contact

Severin Bunk (severin.bunk@uni-hamburg.de)

Time and place

Thursdays, 10am–12 noon, online (BBB or Zoom)

General advice for seminar talks

Seminar talks are different from lectures. The goal of a seminar talk is not to give full proofs of every statement, but to cover a larger amount of material, focussing more on ideas and less on proofs—you simply won't have the time to prove everything—and that's something you should practise here. However, include some nice (not too technical) proofs in your talk to demonstrate to the audience how the technology you are showing them works.

You can give your talks **either as a 55 minute slide show, or as an 80 minute 'board talk'**, which in these times means by somehow sharing your screen or videoing your hand on a sheet of paper. The reason is that slide shows are quicker, so that it can become very hard to follow a longer slide show. You should be able to cover about the same amount of material in either system.

Please approach me so that we can have a **first meeting two weeks before your talk**, where I want you to already have a concrete structure for your talk, and where you should ask about anything that is unclear to you. We will meet **again one week prior to your talk**, and at this point you need to have finalised a draft of your talk, so that we can sort out any remaining issues.

Very important advice for any talk: **practise it before you give it!** That is, run through your talk at least once, say out loud everything you are going to say and write everything you are going to write as if it was in front of an audience. It's the only way to know that you have timed your talk well, and it will make you more confident in the actual talk. **Our goal is to learn mathematics and for**

you to practise and improve your presentation skills. After your talk, I will provide feedback on your presentation.

Program

Talk 1. (05.11.20, Severin) *Introduction, overview, and organisation.*

Talk 2. (12.11.20, Cem Truong) *Manifolds, fibre bundles, and structure groups.*

This is a review of concepts from differential geometry that will be needed to define and study TQFTs. Most important are the notions of smooth manifolds and diffeomorphisms. Also cover fibre bundles, the tangent bundle, differential of a smooth map of manifolds, principal bundles, and reduction of structure group of a bundle. The last point will later feature in the notion of *tangential structures* for TQFTs, enabling us to discuss oriented TQFTs, spin TQFTs, etc. As references, you could use [MT97, Ch. 8, Ch. 15], and [Fre, App. A], but also other introductory books on differential geometry.

Talk 3. (19.11.20, Severin) *The physical origin of TQFTs: the path integral.*

The goal of this talk is to understand the historical motivation of the definition of TQFTs by Atiyah and Segals [Ati88, Seg87]; at the same time, this is where the relation of our seminar to physics lies, and we will see that this relation still poses many open problems.

TQFTs arose from Feynman's formalism of path integrals. These are well-defined in quantum mechanics, and the first part of the talk should briefly review this idea—a good reference is [Ryd96, Sec. 5.1].

The transition to quantum field theory is then conceptually clear: replace zero- and one-dimensional manifolds by d- and $(d+1)$-dimensional manifolds; this will lead to the notion of a *σ-model* QFT. Following [Fre93], introduce the TQFT axioms as direct (higher-dimensional) analogues of how we built the path integral. Atiyah and Segal's idea was now to dispose of the no longer well-defined path integral, instead formalising its properties: the outcome is a TQFT. To explain this to us, cover [Fre93, Secs. 1–2], but also have a look into Aityah's original paper [Ati88]—it is a very nice read! If there is time, briefly do [Fre93, Sec. 3] as an example (this is Dijkgraaf-Witten theory—it will return in Talk 8).

Talk 4. (26.11.20, Jacob Stegemann) *Monoidal categories and the definition of TQFTs (various flavours).*

Based on [Saf, Secs. 1.2–2.1]. This talk starts with the second half of the theoretical underpinnings of TQFTs (after manifolds in Talk 2): introduce/review monoidal categories, functors, and natural transformations, giving various examples (e.g. $\mathcal{V}\mathrm{ect}_k$, $\mathcal{M}\mathrm{od}_R$). The most fundamental example of such a category for us are the *bordism categories* $\mathcal{B}\mathrm{ord}_d$. Building on Talk 2, explain their objects and morphisms, their symmetric monoidal structure, and their composition operation. Briefly cover bordisms with G-structure, and state the formal definition of a topological (quantum) field theory. Show that diffeomorphic manifolds are cobordant, define the mapping class group (π_0 of a diffeomorphism group) and explain how the mapping class group sits in the automorphism group in the bordism categories (we'll discuss this). Finally, cover the zero-dimensional classification theorems [Saf, Thm. 2.1, 2.2].

Talk 5. (03.12.20, Max-Niklas Steffen) *Dualisability in monoidal categories: the category of TQFTs is a groupoid.*

Based on [Saf, Secs. 2.2–2.4]. This talk introduces the concept of *dualisability* in monoidal categories; this is at the heart of many properties of TQFTs (in particular their finiteness properties). Explain the notions of left and right dualisability, make these explicit for various examples, such as $\mathcal{V}\mathrm{ect}_k$, $\mathcal{M}\mathrm{od}_R$, $\mathcal{B}\mathrm{ord}_d$, and bimodules. Explain the relevance of dualisability in TQFTs ([Saf, Prop. 2.12, Cor. 2.14]).

The second part is concerned with the *uniqueness* of duality data. Cover duals of morphisms and dimensions, and summarise the proof that the choice of duality data for a dualisable object is essentially unique (using the functor $\mathrm{DDat}^{full}(\mathcal{C}) \xrightarrow{\sim} (\mathcal{C}^{ld})^{\sim}$). Prove that the category of TQFTs in a fixed dimension with a fixed target is a groupoid [Saf, Prop. 2.28, Cor. 2.29].

Talk 6. (10.12.20, Matthias Dennemann) *1D TQFTs, with and without Morse-Cerf theory.*

Based on [Saf, Secs. 2.5–2.6]. (If required, additional background can be found in [Fre, Lec. 23] and [MT97, Ch. 11–12].) In this talk, we see the first full instances of the Cobordism Hypothesis, a powerful classification theorem for a certain type of TQFTs (see Talk 14). Prove [Saf, Thm. 2.30], introduce the notion of a G-action on a category and of *homotopy fixed points*. Sketch a proof (or prove) [Saf, Lemma 2.36], so that we can understand the 1D unoriented Cobordism Hypothesis [Saf, Thm. 2.37].

In the second part, give an introduction and overview of Morse theory; this is a classical tool in differential topology. Here, it allows us to identify the building blocks of $\mathcal{B}\mathrm{ord}_d$, making it extremely powerful in the context of TQFTs. Use this to sketch another proof of [Saf, Thm. 2.30].

Talk 7. (17.12.20, Luis Vásquez) *Frobenius algebras and the classification of 2D TQFTs.*

Based on [Saf, Secs. 3.1–3.4] (additional reference: [Koc04]). Introduce (commutative) Frobenius algebra objects in monoidal categories, and discuss *separable* Frobenius algebras in $\mathcal{V}\mathrm{ect}_k$. State the Artin-Wedderburn Theorem [Saf, Thm. 3.12] and survey some examples of Frobenius algebras (such as [Saf, E.g. 3.2(1),(2),(4),(5)]). The heart of this talk is the proof of the classification theorem for oriented 2D TQFTs [Saf, Thm. 3.14]: such TQFTs are classified by commutative Frobenius algebra objects. For the remaining time, survey the unoriented and the framed case; these arise as modifications of the above classification theorem.

Talk 8. (07.01.21, Adrien DeLazzer Meunier) *Dijkgraaf-Witten theory.*

Based on [Saf, Sec. 3.5], but see also [DW90]. Dijkgraaf-Witten theory is possibly the prime example of a case where the path integral actually works, the reason being that one works with a finite set of fields, so that ∞-dimensional integrals reduce to finite sums. The fields in this theory are G-bundles with (flat) connection, where G is a finite group. However, following [Saf, Sec. 3.5], we set this theory up in greater generality first: define functions on groupoids, finite groupoids, and prove the important Base Change Formula [Saf, Prop. 3.41]. Explain the TQFT obtained from functions on the groupoid of flat G-bundles in detail, and distil the relation between the 2D case and the representation theory of the finite group G—manifold topology gives us insights into algebra here!

Talk 9. (14.01.21, Till Heine) *Bicategories and full dualisability.*

Based on [Saf, Secs. 4.1–4.3]. We now make the next step towards *locality* of TQFTs: instead of cutting d-manifolds along $(d-1)$-manifolds, we allow a second layer of cutting along $(d-2)$-manifolds. This requires us to extend our toolkit: we have to admit morphisms between morphisms in our categories. Thus, the goal of this talk is to provide necessary theory of bicategories (with an eye towards TQFTs, of course), the next talk will then give several examples.

Explain the fundamental 2-groupoid of a topological space, and abstract from this the notion of a bicategory, their functors, natural transformations, and modifications. Define the homotopy category of a bicategory and survey (various flavours of) monoidal bicategories. Introduce dualisability for objects in monoidal bicategories (which uses only $\mathrm{Ho}(\mathcal{C})$), and adjoints for morphisms. This leads up to the very important definition of *fully dualisable objects* [Saf, Def. 4.21]. To have a criterion for full dualisability, explain the Serre morphism and state (and prove, if time) [Saf, Prop.4.24]. Finally, define actions of topological groups on bicategories, explain [Saf, Prop. 4.27–4.29], and define (sketch) the bicategory of homotopy fixed points.

Talk 10. (21.01.21, Sebastian Heinrich) *Two models for 2-vector spaces and extended 2D TQFTs*

Based on [Saf, Secs. 4.4–4.7]. Introduce the Morita bicategory of algebras, bimodules, and intertwiners, and study full dualisability in this bicategory; in particular, prove (or sketch) [Saf, Prop.4.34].

Briefly recall/introduce the notion of an abelian category (we'll discuss this). Introduce the bicategory of 2-vector spaces and the tensor product of 2-vector spaces, and study dualisability in $2\mathcal{V}\mathrm{ect}_k$, leading to Tillmann's theorem [Saf, Prop.4.44].

Define (sketch) the extended bordism category $\mathcal{B}\mathrm{ord}_{[n-2,n]}$, explain the action of $\mathrm{Diff}(K)$ on K [Saf, E.g. 4.58], and define extended TQFTs. Prove that every object in $\mathcal{B}\mathrm{ord}_{[n-2,n]}$ is fully dualisable [Saf, Prop. 4.64], and state the Cobordism Hypothesis for 2D TQFTs [Saf, Thm. 4.67]. Prove [Saf, Prop. 4.73]. If there is time, briefly comment on the framed and G-structured versions of the Cobordism Hypothesis.

Talk 11. (28.01.21, Hannes Knötzele) *(Multi-)Fusion categories.*

Based on [Saf, Secs. 5.1–5.3]. We turn to the study of extended 3D TQFTs, in particular valued in $2\mathcal{V}\mathrm{ect}_k$. This requires us to familiarise ourselves with (multi-)fusion categories. Introduce these and survey 2-vector spaces with property R, leading to [Saf, Thm. 5.14]: 2-vector spaces with property R are the same as multi-fusion categories. Introduce rigid, balanced and ribbon categories, and survey their dimension and traces, giving examples (Fibonacci category and possibly say something about quantum groups associated with a Lie algebra).

Talk 12. (04.02.21, Alexander Koch) *Extended 3D TQFTs and knot invariants*

Based on [Saf, Secs. 5.4–5.6]. The relation between 3D TQFTs and knot invariants (more precisely Chern-Simons theory and the Jones polynomial) were an essential part of the contributions that earned Edward Witten his Fields Medal [Wit94]. Here we take a first look into the notions involved and this relation.

Briefly comment on generators and relations of $\mathcal{B}\mathrm{ord}_3$. Define the $\mathrm{HH}^0(\mathcal{C})$ for a k-linear category and the Chern character $\mathrm{K}(\mathcal{C}) \to \mathrm{HH}^0(\mathcal{C})$. Using this, define modular (multi-)tensor categories, their S- and T-matrices, and explain that $\mathrm{K}(\mathcal{C}) \otimes_{\mathbb{Z}} k$ carries an action of the modular group. State the classification theorem for oriented 3D TQFTs [Saf, Thm. 5.47] and explain the following remarks.

Define knots and links, and reduce the problem of studying the topology of knots to an algebraic problem using Reidemeister moves. Define the Kauffmann bracket and the Jones polynomial and explain Markov moves, as well as the construction of the oriented link invariant $I_{\mathcal{C},x}$ form an object x in a ribbon category $\mathcal{C}$. Combine both parts of the talk to explain how 3D TQFTs valued in anomaly-free modular tensor categories give rise to knot invariants.

Talk 13. (11.02.21, Max Demirdilek) *Combinatorial constructions of TQFTs: Turaev-Viro theories*

The classification of TQFTs is an important problem. However, even given a result like the Cobordism hypothesis, it still remains extremely hard to construct TQFTs. One powerful idea to achieve such constructions is to decompose the data of a manifold into combinatorial data, using triangulations and topological complexes (like CW complexes).

Here we will look at the Turaev-Viro construction of 3D TQFTs (sometimes also called Turaev-Viro-Barrett-Westbury theory). Survey [BKJ, Secs. 1–5]; alternative references are [Pas, Sch]: introduce spherical fusion categories, their calculus (use some diagrams), and explain the notion of a Drinfeld centre of a fusion category. Survey the Reshetikhin-Turaev invariant [BKJ, Thm. 2.4]. Then give an idea of triangulations and polytope decompositions of manifolds, and explain the construction of the Turaev-Viro TQFT [BKJ, Def. 4.2, Thm. 4.3, Thm. 4.4]. If there is time, say some brief words about the proof of the latter theorem and point out that the TQFT thus obtained even forms part of an extended TQFT.

Talk 14. (18.02.21, Álvaro Jiménez) (∞, n)-*categories and the Cobordism Hypothesis.*

This final talk will give a *survey* of the general theory involved in stating the most general version of the Cobordism Hypothesis—the focus in this talk is on understanding ideas rather than details. We have seen that Cobordism Hypothesis exists for TQFTs whose lowest-dimensional manifolds lie in $d = 0$. If we want to have bordisms of dimension n, we thus need to have n layers of morphisms in our categories.

Introduce the simplex category $\mathbb{\Delta}$ and define simplicial sets as functors $X : \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$. Explain how categories give rise to simplicial sets using the nerve functor, define the notion of a Kan complex, and how these can be seen as $(\infty, 0)$-categories, i.e. categories with 1-morphisms, 2-morphisms, 3-morphisms ... up to infinity, but where every morphism is invertible (use the nerve of a groupoid and the singular complex of a topological space as examples).

Follow [Saf, Sec. 6] to define complete Segal spaces and interpret them as $(\infty, 1)$-categories. Define (∞, n)-categories by iterating this construction, and describe equivalences of (∞, n)-categories as Dwyer-Kan equivalences. Define symmetric monoidal (∞, n)-categories as in [Saf, Def. 6.15] and point out that there exist bordism categories $\mathcal{B}\mathrm{ord}_{d;n}$ of $(d-n)$-manifolds and bordisms up to dimension d. Briefly discuss dualisability in symmetric monoidal (∞, n)-categories [Saf, Def. 6.22] and, finally, state the full Cobordism Hypothesis [Saf, Thm. 6.24].

References

[Ati88] M. Atiyah. Topological quantum field theories. *Inst. Hautes Études Sci. Publ. Math.*, 68:175–186, 1988.

[BKJ] B. Balsam and A. Kirillov Jr. Turaev-Viro invariants as an extended TQFT. `arXiv:1004.1533`.

[DW90] R. Dijkgraaf and E. Witten. Topological gauge theories and group cohomology. *Comm. Math. Phys.*, 129(2):393–429, 1990.

[Fre] D. S. Freed. Bordism: Old and New. URL: `https://web.ma.utexas.edu/users/dafr/bordism.pdf`.

[Fre93] D. S. Freed. Lectures on topological quantum field theory. In *Integrable systems, quantum groups, and quantum field theories (Salamanca, 1992)*, volume 409 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 95–156. Kluwer Acad. Publ., Dordrecht, 1993. URL: `https://web.ma.utexas.edu/users/dafr/OldTQFTLectures.pdf`.

[Koc04] Joachim Kock. *Frobenius algebras and 2D topological quantum field theories*, volume 59 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2004.

[MT97] I. Madsen and J. Tornehave. *From calculus to cohomology*. Cambridge University Press, Cambridge, 1997. de Rham cohomology and characteristic classes.

[Pas] A. G. Passegger. Notes on Turaev-Viro-Barrett-Westbury invariants and TQFT. URL: `https://personal-homepages.mis.mpg.de/passegge/doc/turaev-viro-notes.pdf`.

[Ryd96] L. H. Ryder. *Quantum field theory*. Cambridge University Press, Cambridge, second edition, 1996.

[Saf] P. Safronov. Topological Quantum Field Theories. URL: `https://drive.google.com/file/d/0B3Hq3GkR_m3iT0pQcVpOZDFnQms/view`.

[Sch] C. Schweigert. Hopf algebras, quantum groups and topological field theory. *Lecture Notes*. URL: `https://www.math.uni-hamburg.de/home/schweigert/ws12/hskript.pdf`.

[Seg87] G. Segal. The definition of conformal field theory. In *Schloss Ringberg, March 1987: Links between Geometry and Mathematical Physics*, volume 58 of *The MPIM preprint series*, pages 13–17, 1987.

[Wit94] E. Witten. Quantum field theory and the Jones polynomial. In *Braid group, knot theory and statistical mechanics, II*, volume 17 of *Adv. Ser. Math. Phys.*, pages 361–451. World Sci. Publ., River Edge, NJ, 1994.