Introduction To $K3$ Surfaces (Part 2)

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Calf – 26th May 2005

Abstract

In this second introductory talk, we shall take a look at moduli spaces for certain families of $K3$ surfaces. We introduce the notion of a marking and a lattice poloarisation for a $K3$ surface and construct period domains parametrising these objects.

In the final section, we take a look at the Picard–Fuchs differential equation for a specific family of lattice polarised $K3$ surfaces and use this ODE to explicitly construct a coarse moduli space for our example.

1 Preliminaries

We start by recalling some results that were covered in the first talk. This material can be found in [2].

Definition 1.1 ($K3$ surface). A $K3$ surface is a compact complex surface X with trivial canonical bundle and $\mathrm{H}^1(X, \mathcal{O}_X) = 0$.

All $K3$ surfaces are Kähler and their complex stucture induces a Hodge decomposition

$$\begin{array}{ccccccc} \mathrm{H}^2(X,\mathbb{C}) & \cong & \mathrm{H}^{2,0}(X) & \oplus & \mathrm{H}^{1,1}(X) & \oplus & \mathrm{H}^{0,2}(X) \\ 22 & = & 1 & + & 20 & + & 1 \end{array}$$

with the complex dimensions written on the bottom line. We have isomorphisms

$$\mathrm{H}^{p,q}(X) \cong \mathrm{H}^q(X, \Omega^p_X)$$

where Ω_X^p is the sheaf of germs holomorphic differential p–forms on X.

There exists an even symmetric bilinear form

$$\langle -, - \rangle : \mathrm{H}^2(X, \mathbb{Z}) \times \mathrm{H}^2(X, \mathbb{Z}) \to \mathbb{Z}$$

given by

$$\langle \alpha, \beta \rangle := \int_X \alpha \wedge \beta.$$

Under Poincaré duality, this bilinear form is dual to intersection numbers of cycles.

Definition 1.2 (Lattice). By a *lattice*, we mean a finitely generated, free $\mathbb{Z}$–module together with a symmetric bilinear form. With respect to a choice of basis for the $\mathbb{Z}$–module, the symmetric bilinear form may be represented by a matrix, M say, and we refer to "the lattice M".

Proposition 1.3. *For any $K3$ surface, there exists a choice of basis for* $\mathrm{H}^2(X, \mathbb{Z})$ *with respect to which the symmetric bilinear form is represented by the block matrix*

$$\Lambda_{K3} := U^3 \oplus (-E_8)^2$$

where

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$-E_8 = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{pmatrix}$$

$-E_8$ is the unique negative–definite unimodular even lattice of rank 8, and is the intersection matrix of the E_8 Dynkin diagram.

2 Moduli of K3 surfaces

Definition 2.1. A *marked* $K3$ surface is a $K3$ surface X together with a fixed isomorphism $\sigma\colon \mathrm{H}^2(X,\mathbb{Z}) \to \Lambda_{K3}$.

If we take a non–zero element $\omega \in \mathrm{H}^{2,0}(X) \cong \mathrm{H}^0(X, \Omega^2_X)$, then

$$\mathrm{H}^2(X,\mathbb{C}) \cong \langle\omega\rangle \oplus \mathrm{H}^{1,1}(X) \oplus \langle\bar{\omega}\rangle$$

and the bilinear form, extended to $\mathrm{H}^2(X,\mathbb{C}) = \mathrm{H}(X,\mathbb{Z}) \otimes \mathbb{C}$, takes the following values:

$$\langle\omega,\omega\rangle = \langle\bar{\omega},\bar{\omega}\rangle = 0$$

$$\langle\omega,\bar{\omega}\rangle \in \mathbb{R}_{>0}$$

and, for any $\gamma \in \mathrm{H}^{1,1}(X)$,

$$\langle\omega,\gamma\rangle = 0.$$

Bearing these values in mind, we may define a space parametrising candidate marked $K3$ surfaces.

Definition 2.2 (The Period Space).

$$\Omega = \{\ x \in \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C}) \mid \ \langle x,x\rangle = 0,\ \langle x,\bar{x}\rangle > 0\ \}$$

Theorem 2.3 (Surjectivity of the Period Map, see [1]). *For each $x \in \Omega$, there is some marked $K3$ surface (X,σ) such that $x = \sigma(\mathrm{H}^{2,0}(X))$.*

Also, as a corollary of the so-called weak Torelli theorem for $K3$ surfaces (see [1]), we have an injectivity statement:

Theorem 2.4. *If (X,σ) and (X',σ') are marked $K3$ surfaces with*

$$\sigma(\mathrm{H}^{2,0}(X)) = \sigma'(\mathrm{H}^{2,0}(X')) \in \Omega,$$

then X and X' are isomorphic.

Indeed, the period space Ω is a fine moduli space for marked $K3$ surfaces and it is observed that $\dim_{\mathbb{C}}(\Omega) = 20$.

2.1 Lattice Polarised K3 surfaces

Recall that $\mathrm{Pic}(X) = \mathrm{H}^{1,1}(X) \cap \mathrm{H}^2(X, \mathbb{Z})$, and hence $0 \leq \mathrm{rank}(\mathrm{Pic}(X)) \leq 20$. Also, the embedding $\mathrm{Pic}(X) \subset \mathrm{H}^2(X, \mathbb{Z})$ is a primitive embedding of lattices.

In the first part of this introduction, we saw that the symmetric bilinear form has signature $(3, 19)$ on $\mathrm{H}^2(X, \mathbb{C})$ and, by Hodge's index theorem, signature $(1, 19)$ when restricted to $\mathrm{H}^{1,1}(X) \cap \mathrm{H}(X, \mathbb{R})$. If ρ denotes the rank of the Picard lattice, then the signature of $\mathrm{Pic}(X)$ is $(1, \rho - 1)$ whenever X admits some embedding in a projective space and signature $(0, \rho)$ whenever no such embedding exists.

We focus now on families of algebraic $K3$ surfaces with similar Picard lattices. A more detailed discussion of the following material can be found in [3].

Definition 2.5 (Lattice polarised $K3$ surface). Let M be a lattice of signature $(1, r - 1)$ that can be primitivley embedded in the $K3$ lattice Λ_{K3}.

An M–polarised $K3$ surface is a projective $K3$ surface, X, together with a primitive embedding

$$i\colon M \hookrightarrow \mathrm{Pic}(X)$$

such that $i(M)$ contains an pseudo–ample element of $\mathrm{Pic}(X)$.

Example 2.6. Let $M = \langle 4 \rangle$. In other words, $M \cong \mathbb{Z}$ with $\langle 1, 1 \rangle = 4$. Then $\langle 4 \rangle$–polarised $K3$ surfaces coincide with quartic hypersurfaces in $\mathbb{P}^3$ (after forgetting about their specific embedding in $\mathbb{P}^3$).

There is a coarse moduli space of M–polarised $K3$ surfaces which is constructed as follows. Fix and embedding $M \hookrightarrow \Lambda_{K3}$ and define

$$\Omega_M = \{\ x \in \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C}) \mid \langle x, x \rangle = 0,\ \langle x, \bar{x} \rangle > 0, \langle x, m \rangle = 0\ \forall m \in M\}.$$

Bearing in mind that $\langle \omega, \gamma \rangle = 0$ for all $\gamma \in \mathrm{Pic}(X)$ and $\omega \in \mathrm{H}^{2,0}(X)$, this is a space of candidate M–polarised, marked $K3$ surfaces. As before, we have surjectivity and injectivity results and this time, coarse moduli space for M–polarised $K3$ surfaces (forgetting the marking) is constructed as the quotient

$$\mathcal{M}_{K3,M} := \Omega_M / \{\varphi \in \mathrm{Aut}(\Lambda_{K3}) \mid \varphi(M) = M\}.$$

Notice that $\dim_{\mathbb{C}}(\mathcal{M}_{K3,M}) = 20 - \mathrm{rank}(M)$.

3 Extended Example

We shall look at an example of a family of M-polarised $K3$ surfaces where $\mathrm{rank}(M) = 19$ so that the corresponding moduli space has dimension 1. Consider the family of quartic hypersurfaces

$$X_\lambda : \left(\lambda(x_0^4 + x_1^4 + x_2^4 + x_3^4) = (x_0^2 + x_1^2 + x_2^2 + x_3^2)^2\right) \subset \mathbb{P}^3.$$

For general λ, the surface X_λ is nonsingular and hence is a $K3$ surface. In fact, this family is singular at $\lambda = 1, 2, 3, 4$ where X_λ has DuVal singularities, and at $\lambda = 0$ where the surface is seen to be a double quadric.

Proposition 3.1. *This above family of $K3$ surfaces has an M–polarisation for some rank 19 lattice M (although we don't attempt to find out exactly what M is in this example).*

Proof. This family of $K3$ surfaces is highly symmetric and is invariant under alternating permutations of the coordinates and additionally under the action of the matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \alpha & 0 & 0 \\ -\alpha & \alpha & 0 & 0 \\ 0 & 0 & \alpha & \alpha \\ 0 & 0 & -\alpha & \alpha \end{pmatrix}$$

where $\alpha = 1/\sqrt{2}$. These transformations generate a group G of order 96. We can show that $\mathrm{H}^{2,0}(X)$ and $\mathrm{H}^{0,2}(X)$ are invariant under G and that

$$(\mathrm{H}^2(X, \mathbb{Z})^G)^\perp \subset \mathrm{Pic}(X_\lambda)$$

and the hyperplane section $H = [\mathbb{P}^2 \cap X_\lambda] \in \mathrm{Pic}(X_\lambda)$ generate a sublattice

$$M = \left\langle H, (\mathrm{H}^2(X, \mathbb{Z})^G)^\perp \right\rangle \subset \mathrm{Pic}(X_\lambda)$$

of rank 19 (The details are omitted). □

3.1 Periods and the Picard–Fuchs equation

Sticking with our example family of $K3$ surfaces, we take an alternative approach to constructing a coarse moduli space for M–polarised $K3$ surfaces. The reader should be warned that some of the details of this construction are skimmed over at speed, particularly near the end.

Definition 3.2 (Period Point). The embedding $X_\lambda \subset \mathbb{P}^3$ induces a consistent marking

$$\sigma_\lambda\colon \mathrm{H}^2(X_\lambda, \mathbb{Z}) \to \Lambda_{K3}$$

and so induces a choice of basis $\{\gamma_{\lambda,1}, \ldots, \gamma_{\lambda,22}\}$ for $\mathrm{H}^2(X_\lambda, \mathbb{Z})$ that varies smoothly with λ.

Letting $\langle\omega\rangle = \mathrm{H}^{2,0}(X_\lambda)$, define the *period point* of X_λ to be

$$p(\lambda) = (\langle\omega, \gamma_{\lambda,1}\rangle, \ldots, \langle\omega, \gamma_{\lambda,22}\rangle) \in \mathbb{P}^{21}.$$

This is a point in *projective* space because ω is only defined up to a scalar multiple. Since $\langle\omega, \gamma\rangle = 0$ whenever $\gamma \in \mathrm{Pic}(X_\lambda) \supset M$, we can show that the period point $p(\lambda)$ lies in some linear subspace of dimension $21 - \mathrm{rank}(M)$. So, in our example,

$$p(\lambda) \in \mathbb{P}^2 \subset \mathbb{P}^{21}.$$

In fact, $\lambda \in \mathbb{P}^1$ traces out a nonsingular conic $p(\lambda) \in \mathbb{P}^2$ whose equation is determined by the restriction of the symmetric bilinear form to the orthogonal complement of M in Λ_{K3}. Since all nonsingular conics in $\mathbb{P}^2$ are projectively equivalent, up to an automorphism of $\mathbb{P}^2$, the coordinates of $p(\lambda) = (p_0, p_1, p_2)$ satisfy $p_0 p_1 = p_2^2$.

Definition 3.3 (Picard–Fuchs differential equation). Writing $n = 21 - \mathrm{rank}(M)$, the $n + 2$ points

$$p(\lambda), \frac{d}{d\lambda}p(\lambda), \ldots, \frac{d^{n+1}}{d\lambda^{n+1}}p(\lambda) \in \mathbb{P}^n = \mathbb{C}^{n+1}/\sim$$

must satisfy a linear dependence relationship (whose coefficients depend on λ).

Hence, the periods of the family X_λ satisfy a differential equation of degree $n + 1 = 22 - \mathrm{rank}(M)$, called the *Picard–Fuchs* differential equation.

In our example, we can use an algorithm due to Morrison, [4], to determine that this ordinary differential equation is

$$c_3 p''' + c_2 p'' + c_1 p' + c_0 p = 0 \tag{1}$$

where

$$\begin{aligned} c_3 &= \lambda^3(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4) \\ c_2 &= 3\lambda^3(2\lambda-5)(\lambda^2-5\lambda+5) \\ c_1 &= 3\lambda(2\lambda^4-10\lambda^3+13\lambda^2-6\lambda+6) \\ c_0 &= 9(\lambda-2). \end{aligned}$$

Because the periods satisfy a nondegenarate quadratic relationship, the solutions of the Picard–Fuchs differential equation satisfy this relation. There is a "symmetric square root" of our Picard–Fuchs equation. That is, a differential equation

$$a_2 f'' + a_1 f' + a_0 f = 0 \tag{2}$$

where (in our example)

$$\begin{aligned} a_2 &= 2\lambda^2(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4) \\ a_1 &= 2\lambda^2(2\lambda^3-15\lambda^2+35\lambda-25) \\ a_0 &= (2\lambda-3)(\lambda-3) \end{aligned}$$

that has a basis of solutions f_0, f_1 such that

$$\begin{aligned} p_0 &= f_0^2 \\ p_1 &= f_1^2 \\ p_2 &= f_0 f_1 \end{aligned}$$

(satisfying $p_0 p_1 = p_2^2$) is a basis of solutions to the original differential equation (1).

This differential equation allows us to explicitly construct a coarse moduli space for our family of M–polarised $K3$ surfaces. Let $\mathrm{Sol}(z)$ denote the 2–dimensional complex vector space of solutions to (2) defined in some disk centred at the point $z \in D = \mathbb{P}^1 \setminus \{0,1,2,3,4\}$. The solutions of the ODE (2) are multiple valued fuctions on D and analytic continuation of a basis of $\mathrm{Sol}(z_0)$ around a closed loop in D leads to another basis of $\mathrm{Sol}(z_0)$. This determines a monodromy representation

$$\rho\colon \pi_1(D, z_0) \to \mathrm{Gl}_2(\mathbb{C})$$

where the image of a loop is the corresponding change of basis matrix. Since the periods (ie. the solutions of 2) are only defined up to scalar multiples, we are only interested in the projective monodromy group

$$\Gamma := \rho(\pi_1(D))/\{\mu I_2\} \subset \mathrm{PGl}_2(\mathbb{C}).$$

In our example (and more generally), up to conjugation, the projective monodromy group actually lies in $\mathrm{PSl}_2(\mathbb{R})$ and acts on the upper half–plane, $\mathbb{H}$. The moduli space is equal to the quotient

$$\mathcal{M}_{K3,M} = \mathbb{H}/\Gamma.$$

We may visualise this quotient by finding a fundamental domain for the action. This is shown in figure 1. The transformations generating the projective monodromy group are hyperbolic rotations by 180 degrees about the four marked points and about the centre point. These five fixed points correspond to the five degenerate points in the original family of $K3$ surfaces.

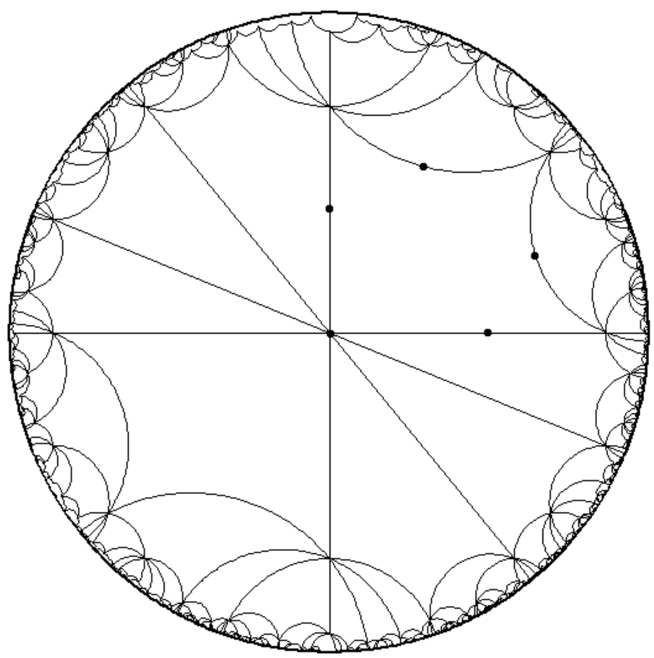

Figure 1: Fundamental domain for the projective monodromy group

References

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