Massey Products in Symplectic Geometry

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Massey Products

Ingredients

- v_{12}, v_{23}, v_{34} closed forms;

- $[v_{12}v_{23}] = [v_{23}v_{34}] = 0$;

- $\overline{v} = (-1)^{|v|}v$;

- $dv_{13} = \overline{v_{12}}v_{23}$; $dv_{24} = \overline{v_{23}}v_{34}$;

- $\overline{v_{12}}v_{24} + \overline{v_{13}}v_{34}$ is a closed form;

Definition

$\langle [v_{12}], [v_{23}], [v_{34}] \rangle := [\overline{v_{12}}v_{24} + \overline{v_{13}}v_{34}]$

$\langle [v_{12}], [v_{23}], [v_{34}] \rangle \in H^{\bullet}/\mathcal{I}([v_{12}, v_{34}])$

The dd^c-lemma (property)

A complex manifold M satisfies the dd^c-lemma if the following are equivalent

- a form α is d-exact and d^c-closed;
- a form α is d^c-exact and d-closed;
- $\alpha = dd^c\beta$

Kähler manifolds have this property.

d^c is d twisted by the complex structure.

- Deligne *et al* – 1975

Complex manifold		Massey products vanish
$+$	$\Rightarrow$	
dd^c-lemma		(uniformily)

- relies on the fact that

$$\Omega^{p,q} \wedge \Omega^{p',q'} \subset \Omega^{p+p',q+q'}.$$

Massey Products in Symplectic Geometry

- Thurston (1976): Symplectic non-Kähler manifold (symplectic fibrations);
- McDuff (1984): 1-connected symplectic non-Kähler manifold (symplectic blow-up);

Both examples have nonvanishing Massey products!

Lefschetz property in (M^{2n}, ω)

$$\omega^i : H^{n-i}(M) \xrightarrow{\cong} H^{n+i}, \forall i$$

- (Brylinski – 1988) A new differential operator

  $$\delta = \Lambda d - d\Lambda;$$

  $\Lambda = -\sum \partial_{x_i} \wedge \partial_{y_i}$.

- δ is d twisted by the symplectic structure

- (Yan & Mathieu – 1996)) Lefschetz property gives a decomposition of cohomology into primitives

- Primitives are symplectic analogous of complex p, q decomposition.

(Merkulov – 1998) Lefschetz property is equivalent to

Symplectic $d\delta$-lemma: the following are equivalent

- a form α is d-exact and δ-closed;
- a form α is δ-exact and d-closed;
- $\alpha = d\delta\beta$.

(Gualtieri – 2003) Gen Cplx Geometry:
d^c and δ are particular cases of a general rule.

Remark: Neither Thurston's nor McDuff's examples satisfy the Lefschetz property.

- Merkulov's result does not imply Massey products vanish!

- Produt of primitives is not primitive

- (Babenko-Taimanov – 2000) Conjecture:

Lefschetz property $\Rightarrow$ *vanishing of Massey products*

The cohomology of the blow-up:

$$M^{2d} \hookrightarrow X^{2n}$$

- $H^\bullet(\tilde{X}) \cong H^\bullet(X) + aH^{\bullet-2}(M) + \cdots + a^{k-1}H^{\bullet-2k+2}(M)$;

- $a^k = -PD(M) - ac_{k-1} - \cdots - a^{k-2}c_2 - a^{k-1}c_1$.

- Symplectic form: $\tilde{\omega} = \omega + \varepsilon a$.

Blowing up Massey Products

- If $\langle \alpha, \beta, \gamma \rangle \neq 0$ is a MP in $X \Rightarrow$ Nonzero MP in $\tilde{X}$:

$$\langle \alpha, \beta, \gamma \rangle \neq 0 \text{ in } \tilde{X}.$$

- If $\langle \alpha, \beta, \gamma \rangle \neq 0$ is a MP in M and co-dimension $> 6 \Rightarrow$ Nonzero MP in $\tilde{X}$:

$$\langle a\alpha, a\beta, a\gamma \rangle \neq 0 \text{ in } \tilde{X}.$$

Blowing up the Lefschetz property

- The map $\tilde{\omega}^i$ depends on how M sits inside X;

- ε provides a 1-parameter family of such maps;

- The kernel of $\tilde{\omega}$ is defined by a closed condition.

Blowing up the Kernel – M^2

- for H^i, $i > 2$,

  $$\dim(\ker(\tilde{\omega}^{n-i})) = \dim(\ker(\omega^{n-i})),$$

  Lefschetz holds at level i in $\tilde{X}$ iff, it does so in X;

- for H^2
  if $\exists v \in \ker(\omega^{n-2})$ st $i^*(v) \neq 0$ then

  $$\dim(\ker(\tilde{\omega}^{n-2})) = \dim(\ker(\omega^{n-2})) - 1,$$

  otherwise these kernels have the same dimension;

- for H^1
  if $\exists v_1, v_2 \in \ker(\omega^{n-1})$ st $i^*(v_1 \wedge v_2) \neq 0$, then

  $$\dim(\ker(\tilde{\omega}^{n-2})) \leq \dim(\ker(\omega^{n-2})) - 2,$$

  otherwise

  $$\dim(\ker(\tilde{\omega}^{n-2})) \leq \dim(\ker(\omega^{n-2})).$$

Blowing up the Kernel – M^{2d}

Assume M is Lefschetz

- for H^i, $i > 2d$,

  $$\dim(\ker(\tilde{\omega}^{n-i})) = \dim(\ker(\omega^{n-i})),$$

  Lefschetz holds at level i in $\tilde{X}$ iff, it does so in X;

- for H^{2d}
  if $\exists v \in \ker(\omega^{n-2d})$ st $i^*(v) \neq 0$ then

  $$\dim(\ker(\tilde{\omega}^{n-2})) = \dim(\ker(\omega^{n-2})) - 1,$$

  otherwise these kernels have the same dimension;

- for H^i, $i < 2d$

  $$\dim(\ker(\tilde{\omega}^{n-i})) \leq \dim(\ker(\omega^{n-i})),$$

Overall

X, M *Lefschetz* $\Rightarrow$ $\tilde{X}$ *Lefschetz.*

Examples

- Let $\mathbb{H}$ be the 3-d Heisenberg group.
- Let $M^3 = \mathbb{H}/\sim$. (0,0,12)
- M^3 has a nonvanishing Massey product.
- $S^1 \hookrightarrow M^3$
- $M^3 \times M^3$ has a nonvanishing Massey product and a symplectic form

$$\omega = e_{15} + e_{36} + e_{24};$$

- the blow-up, N^6, of $M^3 \times M^3$ along $S^1 \times S^1$ has the Lefschetz property and nonvanishing Massey products.

A simply connected example

- The blow-up of $\mathbb{C}P^7$ along N^6 has the Lefschetz property and nonvanishing Massey products.

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