What can we know from Liouville equations for many-particle systems?

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Liouville equations for many-particle systems

Consider N particles dynamics: (X_0^i, V_0^i) are random variables

$$\begin{cases} \frac{\mathrm{d}X_t^i}{\mathrm{d}t} = V_t^i & \text{(Velocity)} \\ \\ \frac{\mathrm{d}V_t^i}{\mathrm{d}t} = F_t^i(X_t^i, X_t^j, V_t^i, V_t^j) & \text{(Interacting force).} \end{cases}$$

We use f_N to denote the joint distribution on the phase space $\mathbb{R}^{6N}$, and it is constant along the trajectories of the system (Liouville's theorem in statistical mechanics), namely,

$$0 = \frac{\mathrm{d}f_N}{\mathrm{d}t} = \frac{\partial f_N}{\partial t} + \sum_{i=1}^{N} \nabla_{x^i} f_N \cdot V_t^i + \sum_{i=1}^{N} \nabla_{v^i} f_N \cdot F_t^i$$

Liouville equations describe evolution of the joint distribution of N particles. If we further assume particles are identical, then its solution's marginal gives one-particle distribution (m-particle, for fixed $m \in \mathbb{N}$)

$$f_{N,1} = \int f_N \,\mathrm{d}v^2 \,\mathrm{d}x^2 \cdots \mathrm{d}v^N \,\mathrm{d}x^N, \;\; f_{N,m} = \int f_N \,\mathrm{d}v^{m+1} \,\mathrm{d}x^{m+1} \cdots \mathrm{d}v^N \,\mathrm{d}x^N.$$

Liouville equations for many-particle systems

For stochastic system, the Liouville equation is often given by Itô's formula, for test function φ_N on $\mathbb{R}^{6N}$

$$\int \varphi_N f_N(t, \mathrm{d}x^1, \mathrm{d}v^1, \ldots, \mathrm{d}x^N, \mathrm{d}v^N) = \mathbb{E}\left[\varphi_N(X_t^1, V_t^1, \ldots, X_t^N, V_t^N)\right].$$

The Liouville equation is a Fokker-Planck equation (Kolmogorov forward equation), $\partial_t f_N = \mathcal{L}_N f_N$. The solution is a probability measure containing statistical information of N-particle system.

> **Question:** What can we know from the Liouville equation? How does the information pass to one-particle distribution when N is very large?

Heuristically, from Liouville equation $\partial_t f_N = \mathcal{L}_N f_N$ to limiting one-particle distribution $\partial_t \bar{f} = \mathcal{Q}\bar{f}$? as $N \to \infty$, we expect:

- High-dimensional $\longrightarrow$ Low-dimensional
- Linear $\longrightarrow$ Nonlinear
- Nonlocal $\longrightarrow$ Local

BBGKY-hierarchy method

How to connect a high-dimension linear Liouville equation $\partial_t f_N = \mathcal{L}_N f_N$ with a low-dimension limiting nonlinear equation $\partial_t \bar{f} = \mathcal{Q}\bar{f}$?

A systematic approach:

1. Dimension reduction: to get BBGKY (finite) hierarchy for m-marginals ($m = 1, 2, \ldots, N-1$)
$$\partial_t f_{N,1} = \int \mathcal{L}_N f_N \,\mathrm{d}x^2 \,\mathrm{d}v^2 \ldots \mathrm{d}x^N \,\mathrm{d}v^N = \mathcal{L}_1(f_{N,1}) + \int \mathcal{L}_2(f_{N,2}) \,\mathrm{d}x^2 \,\mathrm{d}v^2, \cdots$$
2. Pass to the large N limit: to get infinite hierarchy (existence)
$$\partial_t f_1 = \mathcal{L}_1(f_1) + \int \mathcal{L}_2(f_2) \,\mathrm{d}x^2 \,\mathrm{d}v^2, \quad \partial_t f_2 = \mathcal{L}_2(f_2) + \int \mathcal{L}_3(f_3) \,\mathrm{d}x^3 \,\mathrm{d}v^3, \cdots$$
3. A natural solution of the infinite hierarchy given by $\partial_t \bar{f} = \mathcal{Q}\bar{f}$ that
$$\partial_t (\bar{f})^{\otimes m} = \mathcal{L}_m\Big((\bar{f})^{\otimes m}\Big) + \int \mathcal{L}_{m+1}\Big((\bar{f})^{\otimes m+1}\Big) \,\mathrm{d}x^{m+1} \,\mathrm{d}v^{m+1};$$
4. Show the uniqueness: the unique solution of the hierarchy is given as tensorisation $f_1 = \bar{f}$, $f_2 = \bar{f} \otimes \bar{f}$, $\quad \ldots, \quad f_m = (\bar{f})^{\otimes m}$.
5. Conclude the scaling limit that $f_{N,m} \to \bar{f}^{\otimes m}$ as $N \to \infty$.

BBGKY-hierarchy method

Idea:

Liouville equation $\rightarrow$ BBGKY hierarchy $\rightarrow$ infinite hierarchy $\rightarrow$ nonlinear PDE

Implementation: classical and quantum

- Lanford ('75), Kings ('75), Gallagher-Saint-Raymond-Texier ('13): Boltzmann equation from Boltzmann-Grad scaling
- Erdos-Schlein-Yau (06',07'): Schrödinger equation from a moderate scaling
- Erdos-Schlein-Yau (10'): Gross-Pitaevskii equation from the local interaction scaling
- Mischler-Mouhot (13'): space-homogenous Boltzmann equation from mean-field scaling
- Chen-Holmer (23'): Boltzmann equation from weak-coupling scaling
- Bresch-Jabin-Soler (24'): Vlasov-Fokker-Planck equation from mean-field scaling
- $\cdots$

Space-homogeneous Landau equation

Consider the 3D space-homogeneous Landau equation

$$\partial_t \bar{f} = \mathcal{Q}(\bar{f}), \quad \bar{f}(0) = f^0,$$

which represents the density in the velocity of some kind of gas molecules in a plasma, where the collision operator is given by

$$\begin{aligned}\mathcal{Q}(\bar{f}) &= \mathrm{div}_{v^1} \int_{\mathbb{R}^3} A(v^1 - v^2)\Big(\bar{f}(v^2)\nabla_{v^1}\bar{f}(v^1) - \bar{f}(v^1)\nabla_{v^2}\bar{f}(v^2)\Big)\,\mathrm{d}v^2 \\ &= \mathrm{div}\left[\big(A * \bar{f}\big)\nabla\bar{f} - \big(B * \bar{f}\big)\,\bar{f}\right],\end{aligned}$$

with the matrix A and the vector B given by

$$A(z) = |z|^{\gamma+2}(\mathrm{Id} - \frac{z \otimes z}{|z|^2}), \quad B = \mathrm{div}A = -2z|z|^{\gamma}.$$

- $\gamma \in (0, 1]$: Hard potential;
- $\gamma = 0$: Maxwellian molecules;
- $\gamma \in [-2, 0)$: moderately soft potential;
- $\gamma \in [-3, -2)$: very soft potential;
- $\gamma = -3$: Coulomb potential.

Space-homogeneous Landau equation

Here, we focus on the Kac particle dynamics for the space-homogeneous Landau equation: In the spirit of Kac ('56), Miot-Pulvirenti-Saffirio ('11) analyses the Liouville equation

$$\partial_t f_N = \frac{1}{2N} \sum_{i \neq j} \left(\mathrm{div}_{v^i} - \mathrm{div}_{v^j} \right) \left[A(v^i - v^j) \big(\nabla_{v^i} - \nabla_{v^j} \big) f_N \right],$$

while Carrapatoso ('16) constructs a corresponding stochastic particle system:

$$\mathrm{d} V_t^i = \frac{2}{N} \sum_{j \neq i}^N B(V_t^i - V_t^j) \, \mathrm{d}t + \sqrt{\frac{2}{N}} \sum_{j \neq i}^N A^{\frac{1}{2}} (V_t^i - V_t^j) \, \mathrm{d} Z_t^{i,j},$$

where the matrix A and the vector B given by

$$A(z) = |z|^{\gamma+2} (\mathrm{Id} - \frac{z \otimes z}{|z|^2}), \quad B = \mathrm{div} A = -2z|z|^{\gamma}, \quad \gamma \in [-3, 1],$$

with the 3D-Brownian motions are anti-symmetric such as for $i < j$, $Z_t^{i,j} = W_t^{i,j}$ while $Z_t^{j,i} = -W_t^{i,j}$, and initial data are i.i.d. $\mathrm{Law}(V_0^i) = f^0$.

Conservation:

$$\frac{1}{N} \sum_i V_t^i = \frac{1}{N} \sum_i V_0^i \text{ and } \frac{1}{N} \sum_i |V_t^i|^2 = \frac{1}{N} \sum_i |V_0^i|^2, \quad \text{a.s.}$$

Marginals and hierarchies

Liouville equation for Kac particles:

$$\partial_t f_N = \tfrac{1}{2N} \textstyle\sum_{i\neq j} \mathrm{div}_{v^i - v^j} \left[A(v^i - v^j) \nabla_{v^i - v^j} f_N \right],$$

BBGKY hierarchy: $(\nabla_{v^i - v^j} := \nabla_{v^i} - \nabla_{v^j},\ \mathrm{div}_{v^i - v^j} := \mathrm{div}_{v^i} - \mathrm{div}_{v^j})$

$$\begin{aligned}\partial_t f_{N,m} = &\frac{1}{2N} \sum_{i\neq j\leq m} \mathrm{div}_{v^i - v^j} \left[A(v^i - v^j) \nabla_{v^i - v^j} f_{N,m} \right] \\ &+ \frac{N-m}{N} \sum_{i=1}^{m} \mathrm{div}_{v^i} \int_{\mathbb{R}^3} \left[A(v^i - v^{m+1}) \nabla_{v^i - v^{m+1}} f_{N,m+1} \right] \mathrm{d}v^{m+1},\end{aligned}$$

Landau infinite hierarchy:

$$\partial_t f_m = \sum_{i=1}^{m} \mathrm{div}_{v^i} \left[\int_{\mathbb{R}^3} A(v^i - v^{m+1}) \left(\nabla_{v^i - v^{m+1}} f_{m+1} \right) \mathrm{d}v^{m+1} \right],$$

Landau equation:

$$\partial_t \bar{f} = \mathrm{div}_{v^1} \left[\int_{\mathbb{R}^3} A(v^1 - v^2) \nabla_{v^1 - v^2} \left(\bar{f}(v^1) \bar{f}(v^2) \right) \mathrm{d}v^2 \right].$$

Weak form of the Liouville equation

Let $f_N(0) = (f^0)^{\otimes N}$, for any $T > 0$ and any test function $\varphi_N \in C_b^2(\mathbb{R}^{3N})$, it holds ($\mathrm{d}V_N = \mathrm{d}v^1 \cdots \mathrm{d}v^N$)

$$\int_{\mathbb{R}^{3N}} \varphi_N f_N(T)\,\mathrm{d}V_N - \int_{\mathbb{R}^{3N}} \varphi_N\,(f^0)^{\otimes N}\,\mathrm{d}V_N$$

$$= \frac{1}{N}\sum_{i\neq j}^{N} \int_0^T \int_{\mathbb{R}^{3N}} A(v^i - v^j) : (\nabla^2_{v^i v^i}\varphi_N - \nabla^2_{v^i v^j}\varphi_N) f_N \,\mathrm{d}V_N\,\mathrm{d}t$$

$$+ \frac{1}{N}\sum_{i\neq j}^{N} \int_0^T \int_{\mathbb{R}^{3N}} B(v^i - v^j) \cdot (\nabla_{v^i}\varphi_N - \nabla_{v^j}\varphi_N) f_N \,\mathrm{d}V_N\,\mathrm{d}t,$$

where

$$RHS \le N\|\nabla^2\varphi_N\|_{L^\infty} \int_0^T \int_{\mathbb{R}^6} |v^1 - v^2|^{2+\gamma} f_{N,2}\,\mathrm{d}v^1\,\mathrm{d}v^2\,\mathrm{d}t.$$

Then,

- For hard potential $\gamma \in (0,1]$, it needs $f_{N,2} \in L^1$ with higher moment estimate $(\ge 2+\gamma)$;
- For very soft potential $\gamma < -2$, we need higher integrability $(> \frac{3}{\gamma+5})$.

Assumptions for initial data

We assume the initial data satisfies the following two groups of conditions

Assumptions for hard potentials

1. $f^0 \in L^1(\mathbb{R}^3)$, and it is supported in the ball centred at the origin and with radius r_0.
2. Finite entropy: $\mathcal{H}(f^0) = \int_{\mathbb{R}^3} f^0 \log f^0 < +\infty$.

Assumption for soft potentials

1. $f^0 \in L^1(\mathbb{R}^3)$ with finite energy:
$$\int_{\mathbb{R}^3} f^0 = 1, \text{ and } \int_{\mathbb{R}^3} f^0 |v|^2 < +\infty;$$
2. Finite entropy: $\mathcal{H}(f^0) = \int_{\mathbb{R}^3} f^0 \log f^0 < +\infty$;
3. Finite Fisher information: $\mathcal{I}(f^0) = \int_{\mathbb{R}^3} f^0 |\nabla \log f^0|^2 < +\infty$.

Liouville equations with hard potentials [Guo 25+]

For $\gamma \in (0, 1]$, under the assumption for hard potentials (compact support+finite entropy), we have:

Proposition 1

Decay of entropy: $\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{N} \int_{\mathbb{R}^{3N}} f_N \log f_N \leq 0$.

Proposition 2

Propagation of the exponential moment uniformly-in-time uniformly-in-N:

$$\sup_{t \in [0,\infty)} \int_{\mathbb{R}^3} \exp\left(\xi_{r_0,\gamma} |v^1|^{\frac{4}{\gamma+2}} \right) f_{N,1}(t, v^1) \, \mathrm{d}v^1 < \infty.$$

Relies on a sharpened version of Povzner's inequality + cut-off in time.

Proposition 3 (Existence)

For any fixed $m \in \mathbb{N}$ and any $T > 0$, as $N \to \infty$,

$$f_{N,m} \rightharpoonup f_m \quad \text{weakly in} \quad L^1([0, T] \times \mathbb{R}^{3m}) \quad \text{up to a subsequence,}$$

where f_m is a weak solution and propagates exponential moment.

Uniqueness of solutions of hierarchy

Proposition 4 (Stability estimate)

For any $t \in [0, T]$, any $0 < \eta < 1$, it holds

$$\mathcal{W}_2^2(f_m(t), \bar{f}^{\otimes m}(t)) \le Cm(1+T)\left(m\mathcal{W}_2^2(f^0, \bar{f}^0)\right)^{1-\eta}.$$

Construct corresponding SDE hierarchy up to level n (large) and couple it with independent particles:

$$u_m(t) := \sum_{i=1}^m \mathbb{E}\left[\left|U_t^i - \bar{U}_t^i\right|^2\right], \quad u_{m+1}(t) := \sum_{i=1}^{m+1} \mathbb{E}\left[\left|U_t^i - \bar{U}_t^i\right|^2\right],$$

To get inequality hierarchy for large cut-off R ($m \le n-1$) by Itô's formula,

$$u_m(T) \le u_m(0) + C_1 R^\gamma \int_0^T \left[m u_{m+1}(t) - (m-1)u_m(t)\right] \mathrm{d}t + \frac{C_2 T m}{\exp(\xi R^{\frac{4}{\gamma+2}})}.$$

Combinatorics [Lacker 23'], for fixed m and arbitrarily large n (consistency)

$$\begin{aligned}\mathcal{W}_2^2\big(f_m(T), \bar{f}^{\otimes m}(T)\big) &\le u_m(T)\\ &\le \mathcal{W}_2^2(f^0, \bar{f}^0) m \exp(C_1 R^\gamma T) + \frac{C_2 T \exp(C_3 R^\gamma T) m}{\exp[C_4 R^{\frac{4}{\gamma+2}}]} + \frac{C_5 \exp(C_6 R^\gamma T)}{n}.\end{aligned}$$

Existence + Uniqueness $\implies$ Mean-field limits

N-particle Liouville equation:

$$\partial_t f_N = \frac{1}{2N} \sum_{i \neq j} \mathrm{div}_{v^i - v^j} \left[A(v^i - v^j) \nabla_{v^i - v^j} f_N \right], \quad f_N(0) = (f^0)^{\otimes N}.$$

Space-homogenous Landau equation with hard potentials: $\gamma \in (0, 1]$

$$\partial_t \bar{f} = \mathrm{div}_{v^1} \left[\int_{\mathbb{R}^3} A(v^1 - v^2) \nabla_{v^1 - v^2} \left(\bar{f}(v^1) \bar{f}(v^2) \right) \mathrm{d}v^2 \right], \quad \bar{f}(0) = f^0.$$

Landau infinite hierarchy:

$$\partial_t f_m = \sum_{i=1}^{m} \mathrm{div}_{v^i} \left[\int_{\mathbb{R}^3} A(v^i - v^{m+1}) \left(\nabla_{v^i - v^{m+1}} f_{m+1} \right) \mathrm{d}v^{m+1} \right].$$

Under assumption $f^0 \in L^1 \cap L \log L$ with compact support, we have:

Theorem 1 (Well-posedness of hierarchy) [Guo 25+]

For any fixed $m \in \mathbb{N}$, there exist a unique weak solution f_m of the Landau hierarchy, which is given in tensorised form as $f_m = \bar{f}^{\otimes m}$. In particular, for any $t \geq 0$, $\mathcal{W}_2(f_m(t), \bar{f}^{\otimes m}(t)) = 0$.

Theorem 2 (Mean-field limits) [Guo 25+]

The propagation of chaos holds $f_{N,m} \to \bar{f}^{\otimes m}$ in $L^1([0, T], \mathcal{P}(\mathbb{R}^{3m}))$, as $N \to \infty$.

Liouville equations with soft potentials [Carrillo-Guo 25']

Under the assumption for soft potentials (finite energy/ entropy/ Fisher), we have:

Proposition 1

Decay of Fisher information:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{N}\int_{\mathbb{R}^{3N}} f_N |\nabla \log f_N|^2 \leq 0.$$

Two-particle Fisher [Guillen-Silvestre 25'] + Three-particle Fisher

Theorem 3 (Existence)

For any fixed $m \in \mathbb{N}$ and any $T > 0$, for all $t \in [0, T]$, as $N \to \infty$, it holds

$$f_{N,m} \to f_m \quad \text{strongly in} \quad L^1([0,T] \times \mathbb{R}^{3m}) \quad \text{up to a subsequence,}$$

where f_m is a weak solution of the hierarchy with the finite Fisher information.

We obtain by embedding, for example, $f_1 \in L^{3-}(\mathbb{R}^3)$.

Uniqueness of solutions of hierarchy for soft potentials

Uniqueness of solutions of hierarchy for soft potentials

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Grazie mille!