On a variational approximation of the heat flow of harmonic maps

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New perspectives in Nonlocal and Nonlinear PDEs, Anacapri

Joint work with: F.H. Lin, Y. Sire, C. Wang

Given (M, g), (N, h) two (closed) Riemannian manifolds, consider

$$E(u) := \frac{1}{2} \int_M |\mathrm{d}u|_g^2 \, \omega_g, \qquad |\mathrm{d}u|_g^2 := g^{ij} \partial_i u \cdot \partial_j u$$

Here $u : M \to \mathbb{R}^k$ with $u(x) \in N$ for any $x \in M$ (N is compact, use Nash's isometric embedding $N \hookrightarrow \mathbb{R}^k$) with
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A critical point of E is an harmonic map and solves

$$P_u(-\Delta u) = 0, \quad P_y : \mathbb{R}^k \to T_y N, \text{ ort. proj.}$$

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Or, equivalently

$$-\Delta u \perp T_{u(x)} N,$$

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$$-\Delta u = g^{ij} A(u)\,(\partial_i u, \partial_j u)\,,$$

A being the 2^{nd} fundamental form of the embedding $N \to \mathbb{R}^k$.

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Examples:

1. $N = \mathbb{R}$ or $\mathbb{R}^k$, harmonic functions are harmonic maps
2. $N = \mathbb{S}^2$ and M a surface, the Gauss map is an harmonic map iff M has constant mean curvature.

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Beside existence, difficult and interesting regularity issues as r.h.s. $\sim |\nabla u|^2$.

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Extensively studied in the last 60 years: Morrey, Eells, Sampson, Schoen, Uhlenbeck, Simon, Giaquinta, Hildebrandt, Jost, Hardt, F.H.Lin, Evans, Hélein, Béthuel, Rivière...

The heat flow of harmonic maps

Problem: *Given* $\bar{u} : M \to N$, *we look for an harmonic map* $u : M \to N$ *in* $[\bar{u}]$, *i.e. the homotopy class of* $\bar{u}$.

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Recall that $\bar{u}$ is homotopic to u_1 if there is a continuous map $H : [0,1] \times M \to N$ such that $H(0,\cdot) = \bar{u}(\cdot)$ and $H(1,\cdot) = u_1(\cdot)$

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Problem: *Given $\bar{u} : M \to N$, we look for an harmonic map $u : M \to N$ in $[\bar{u}]$, i.e. the homotopy class of $\bar{u}$.*

"Deform" $\bar{u}$ via the L^2–gradient flow of E, namely

$$\begin{cases} \partial_t u + P_u(-\Delta u) = 0 & \text{in } M \times (0, +\infty) \\ u(0) = \bar{u} & \text{on } M. \end{cases}$$

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- If $K_N \leq 0$ (Eells & Sampson, Hartman, Hamilton...) unique smooth global solution with $u(t) \xrightarrow{t \to +\infty} u_\infty$ harmonic map with $u_\infty \in [u_0]$.
- If $K_N > 0$ (Struwe, Chen & Struwe, Chen & Lin...)
  - Global weak solution that is partially smooth
  - Singularity formation analysis, finite time blow up...

The gradient flow structure

Let

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Then

$$P_u(-\Delta u) = \mathrm{grad}_{L^2} E(u) = (\partial E(u))^\circ$$

with ∂E the Fréchet subdifferential

$$\xi \in \partial E(u) \quad \text{iff} \quad \liminf_{v\to u} \frac{E(v) - E(u) - \int_M \xi\cdot(v-u)\,\omega_g}{\|v-u\|} \geq 0.$$

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To be precise, $\forall v$ smooth

$$P_\cdot(-\Delta\cdot) : \mathcal{D} \to L^2, \quad \int_M P_u(-\Delta u)\cdot v\;\omega_g = \int_M \nabla u : \nabla P_u v\;\omega_g.$$

Convexity properties and gradient flow

When $K_N \leq 0$ the energy E and

$$d^2(u,v) := \int_M d_N^2(u(x), v(x))\, \omega_g, \quad d_N \text{ geod. distance in } N$$

are convex (along geodesics in $L^2(M;N)$).

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Features

- $PDE =$ limit of the minimizing movement scheme
- No smoothness of the target is required
- Importance of the joint convexity of E and of d^2

A different point of view (for $K_N \leq 0$, for now)

Following Ilmanen (1994) and De Giorgi (1995), we introduce $\forall \varepsilon > 0$

$$I_\varepsilon[v] := \int_0^\infty \frac{e^{-\varepsilon t}}{\varepsilon} \left(\frac{\varepsilon}{2} \|\partial_t v(t)\|^2 + E(v(t)) \right) \mathrm{d}t,$$

Note that $I_\varepsilon : H^1_{\mathrm{loc}}(\mathbb{R}_+; L^2(M; \mathbb{R}^k)) \to [0, +\infty]$.

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$$\begin{cases} P_{u_\varepsilon}\left(-\varepsilon \partial_t^2 u_\varepsilon + \partial_t u_\varepsilon - \Delta u_\varepsilon\right) = 0, & \text{in } M \times \mathbb{R}_+ \quad \text{(Euler-Lagr. Eq.)} \\ u_\varepsilon(x, 0) = \bar{u} & \text{on } M \times \{0\} \quad \text{(bndry cond.)} \end{cases}$$

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YES, and the proof follows by the general strategy of Rossi, Savaré, S., Stefanelli (2019)

Weighted Energy-Dissipation principle for Gradient Flows in Metric Spaces, JMPA (2019)

W.E.D. approach to evolution

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See U. Stefanelli, *The Weighted Inertia-Energy-Dissipation principle (2025) M3AS*

The convergence proof of (RSSS) in a nutshell

- Direct method of Calc. Var. $\rightsquigarrow$ $\exists$ of u_ε
- Time inner variation $\rightsquigarrow$ $\frac{\mathrm{d}}{\mathrm{d}t}\left(E(u_\varepsilon(t)) - \frac{\varepsilon}{2}\|\partial_t u_\varepsilon(t)\|^2\right) = \|\partial_t u_\varepsilon(t)\|^2$
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- The Value Function $V_\varepsilon(\bar{u}) := \min_{v, v(0)=\bar{u}} I_\varepsilon[v]$

The Value Function satisfies the Dynamic programming principle

$$\boxed{V_\varepsilon(\bar{u}) = \min_{v, v(0)=\bar{u}} \left[\int_0^t \frac{\mathrm{e}^{-\frac{s}{\varepsilon}}}{\varepsilon}\left(\frac{\varepsilon}{2}\|\partial_t v\|^2 + E(v(s))\right)\mathrm{d}s + V_\varepsilon(v(t))\mathrm{e}^{-\frac{t}{\varepsilon}}\right]}$$

and the Hamilton-Jacobi equation

$$\frac{1}{2}\|\mathrm{grad}_{L^2} V_\varepsilon(\bar{u})\|^2 = \frac{1}{\varepsilon}E(\bar{u}) - \frac{1}{\varepsilon}V_\varepsilon(\bar{u}) \quad \forall\, \bar{u} \in H^1(M; N)$$

The convergence proof (cont.)

Combining the Hamilton-Jacobi equation and the Dynamic Programming principle we get

$$\frac{1}{2}\left\|\partial_t u_\varepsilon\right\|^2 + \frac{1}{2}\left\|\mathrm{grad}_{L^2} V_\varepsilon(u_\varepsilon)\right\|^2 + \frac{\mathrm{d}}{\mathrm{d}t} V_\varepsilon(u_\varepsilon(t)) = 0 \qquad \text{a.e. in } (0, +\infty)$$

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$$\int_0^T \|\partial_t u_\varepsilon\|^2 \,\mathrm{d}t \le C(T)E(\bar{u}), \qquad \sup_{t\ge 0} E(u_\varepsilon(t)) \le E(\bar{u})$$

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Thus

$$u_\varepsilon \xrightarrow{\varepsilon\to 0} u \qquad \text{in } C^0([0,T];L^2) \quad \text{for any } T>0,$$
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More in general $L^2(M;N) \rightsquigarrow X$ metric space and $E \rightsquigarrow \varphi : X \to (-\infty,+\infty]$ convex, l.s.c., coercive.

Our new contribution (1)

$u_\varepsilon : M \times \mathbb{R}_+ \to \mathbb{R}^k$ is a minimizing harmonic map with values in N. Indeed,

$$u_\varepsilon \in \operatorname{argmin} I_\varepsilon[v] = \operatorname{argmin} \int_{M\times\mathbb{R}_+} |\mathrm{d}_{x,t} v|^2_{g_\varepsilon}\, \omega_{g_\varepsilon}$$

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namely u_ε is a miminizing harmonic map from the cylinder $(M \times \mathbb{R}_+, g_\varepsilon)$ with values in N

$$g_\varepsilon = e^{\varphi_\varepsilon(t)}\left(\varepsilon g + \mathrm{d}t^2\right), \qquad \varphi_\varepsilon(t) = \frac{2}{m-1}\left(-\frac{t}{\varepsilon} - \frac{m}{2}\log\varepsilon\right)$$

As such u_ε is smooth for any fixed $\varepsilon > 0$ (see Jost).

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$u_\varepsilon : M \times \mathbb{R}_+ \to \mathbb{R}^k$ is a minimizing harmonic map with values in N. Indeed,

$$u_\varepsilon \in \operatorname{argmin} I_\varepsilon[v] = \operatorname{argmin} \int_{M\times\mathbb{R}_+} |\mathrm{d}_{x,t} v|^2_{g_\varepsilon}\, \omega_{g_\varepsilon}$$

namely u_ε is a miminizing harmonic map from the cylinder $(M \times \mathbb{R}_+, g_\varepsilon)$ with values in N

$$g_\varepsilon = e^{\varphi_\varepsilon(t)}\left(\varepsilon g + \mathrm{d}t^2\right), \qquad \varphi_\varepsilon(t) = \frac{2}{m-1}\left(-\frac{t}{\varepsilon} - \frac{m}{2}\log\varepsilon\right)$$

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Problem: Is is possible to obtain the smoothness of the limit FROM the approximation scheme?

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For any $\varepsilon > 0$ set $e_\varepsilon(u_\varepsilon)(x,t) := \frac{\varepsilon}{2}\left|\partial_t u_\varepsilon\right|^2(x,t) + \frac{1}{2}\left|\mathrm{d}u_\varepsilon\right|_g^2(x,t)$.

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$$-\varepsilon\partial_t^2 e_\varepsilon(u_\varepsilon) + \partial_t e_\varepsilon(u_\varepsilon) - \Delta e_\varepsilon(u_\varepsilon) \leq C\, e_\varepsilon(u_\varepsilon) \qquad \text{in } M \times \mathbb{R}_+.$$

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Thus, with some work (energy estimate + Moser iteration scheme), the Harnack-type inequality

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The energy estimate is inspired by a similar one in
Audrito "On the existence and Hölder regularity of solutions to some nonlinear Cauchy-Neumann problems" J. Evol. Equ (2023)

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Therefore, when $\varepsilon \to 0$, u is unique and verifies

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since
$$\left| g^{ij} A(u)[\partial_i u, \partial_j u] \right| \leq C \left| \nabla u \right|^2 \in L^\infty(M \times \mathbb{R}_+),$$
u is smooth in $M \times \mathbb{R}_+$.

Our new contribution (4): Eells & Sampson at level ε

We start from the simple observation:

$$V_\varepsilon(v) \leq E(v) \quad \forall v \qquad \text{and} \quad \frac{1}{2}\left\|\mathrm{grad}_{L^2} V_\varepsilon(v)\right\|^2 = \frac{E(v) - V_\varepsilon(v)}{\varepsilon}$$

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Thus $\exists\, u_\infty^{(\varepsilon)}$ such that

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The E.V.I. for V_ε gives, for any $v \in D(V_\varepsilon)$,

$$V_\varepsilon(u_\varepsilon(t)) \leq V_\varepsilon(v) + \frac{d^2(\bar{u}, v)}{2t} \ \ \rightsquigarrow \ \ V_\varepsilon(u_\infty^{(\varepsilon)}) \leq V_\varepsilon(v).$$

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$\leadsto$ u_∞ is homotopic to $\bar{u}$.

$u_\varepsilon(\cdot, t + t_n) \in [\bar{u}]$ via $E.V.I.$ and for $n >> 1$ $d_N(u_\varepsilon(x, t + t_n), u_\infty(x)) \leq i_N$
$\leadsto$ $u_\varepsilon(\cdot, t + t_n) \in [u_\infty]$ via the unique geodesic.

Conclusion & Perspectives

- New proof (with a novel geometric interpretation) of the celebrated Eells & Sampson Theorem
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