Smoothing effect and uniqueness for aggregation-diffusion models

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Smoothing effect and time decay for the porous medium equation

A standard property of porous medium equation

$$\begin{cases} \partial_t u = \Delta u^m & x \in \mathbb{R}^d, \quad t > 0 \\ u(0,x) = u_0(x) \in L^1_+(\mathbb{R}^d) \end{cases} \qquad m > 1$$

is the following L^∞-hypercontractivity [Bénilan 1976, Véron 1979, Vázquez 1982]

$$\|u(t)\|_\infty \le C_{m,d} \, \|u_0\|_1^{\frac{2}{2+d(m-1)}} \left(\frac{1}{t}\right)^{\frac{d}{2+d(m-1)}} \qquad \text{for every } t > 0$$

The estimate is obtained by comparison with the solution with most concentrated datum, i.e., the explicit Barenblatt point source solution

$$\bar{u}(x,t) = t^{-\frac{d}{2+d(m-1)}} \left(K - \tfrac{m-1}{2m(2+d(m-1))} |x|^2 t^{-\frac{2}{2+d(m-1)}} \right)_+^{\frac{1}{m-1}}, \qquad K > 0$$

Task

Obtain sharp L^∞ estimates for aggregation-diffusion models

Outline

- Smoothing effect and time decay for local and nonlocal models
- Hypercontractivity for aggregation-diffusion models
- Gradient flow approach and uniqueness for aggregation-diffusion models

L^p estimate for the porous medium equation $\partial_t u = \Delta u^2,\ d \geq 3$

- Derivative of $\|u(t)\|_p^p$, for $1 < p < +\infty$, along the solution u:

$$\frac{d}{dt}\|u(t)\|_p^p = p\int_{\mathbb{R}^d} u^{p-1}\partial_t u = -p\int_{\mathbb{R}^d} \nabla u^{p-1}\cdot\nabla u^2 = -\frac{8p(p-1)}{(p+1)^2}\int_{\mathbb{R}^d} |\nabla u^{\frac{p+1}{2}}|^2$$

- Sobolev inequality $\|v\|^2_{\frac{2d}{d-2}} \leq S_d\|\nabla v\|_2^2$ yields

$$\frac{d}{dt}\|u(t)\|_p^p \leq -c_p\,\|u(t)\|^{p+1}_{d(p+1)/(d-2)}\,, \qquad c_p := \frac{8p(p-1)}{S_d(p+1)^2}$$

- Interpolation inequality $L^1 - L^p - L^{d(p+1)/(d-2)}$, letting $M := \|u_0\|_1 = \|u(t)\|_1$,

$$\frac{d}{dt}\|u(t)\|_p^p \leq -c_p\ M^{-\frac{2p+d}{d(p-1)}}\left(\|u(t)\|_p^p\right)^{1+\frac{2+d}{d(p-1)}} \qquad \left(y' \leq -c_p M^{-f} y^{\sigma}, \sigma > 1, f > 0\right)$$

- Comparison principle for the differential inequality satisfied by $\|u(t)\|_p^p$:

$$\|u(t)\|_p \leq M^{\frac{2p+d}{p(2+d)}}\left(\frac{d(p-1)}{c_p(2+d)}\,\frac{1}{t}\right)^{\frac{p-1}{p}\,\frac{d}{2+d}}$$

The red constant is unbounded as $p \to +\infty$, not allowing to directly get L^∞ estimate, which requires De Giorgi or Moser iteration method

Iteration method for the L^∞ estimate of $\partial_t u = \Delta u^2,\ d \geq 3$

• Moser iterations: Let $p = 2^j$, interpolate with $L^{2^{j-1}}$ norm instead of L^1 and deduce that $Q_j := \|u(t)\|_{2^j}^{2^j}$ satisfies a one-step recursive estimate of the form

$$Q_j \leq K_d\,(2^d)^j\, t^{-z_j}\, Q_{j-1}^{\gamma_j}, \quad Q_0 = M, \qquad z_j := \frac{2^{j-1}\,d}{2^j + d}, \qquad \gamma_j := \frac{2^{j+1} + d}{2^j + d}$$

Iterating down to $j = 0$ we find

$$\|u(t)\|_{2^j} = Q_j^{1/2^j} \leq C_d\; M^{\frac{2^{j+1}+d}{2^j(2+d)}} \left(\frac{1}{t}\right)^{\frac{d(2^j-1)}{2^j(2+d)}}$$

Letting $j \to \infty$

$$\|u(t)\|_\infty \leq C_d\; M^{\frac{2}{2+d}} \left(\frac{1}{t}\right)^{\frac{d}{2+d}}$$

Nonlocal porous media equations

$$\partial_t u = \mathrm{div}\,(u\,\nabla \mathfrak{p}_u) \quad x \in \mathbb{R}^d,\ \ t > 0, \qquad \mathfrak{p}_u \text{ is the pressure (Darcy's law)}$$

Nonlocal diffusion models: $\mathfrak{p}_u = W * u$, for suitable radially decreasing convolution kernel W (typical choices are Newton, Riesz or Bessel kernel).

$$\frac{d}{dt}\|u(t)\|_p^p = p\int_{\mathbb{R}^d} u^{p-1}\partial_t u = -(p-1)\int_{\mathbb{R}^d} \nabla u^p \cdot \nabla \mathfrak{p}_u$$

For instance if $W(x) = k_{d,s}|x|^{2s-d}$, $0 < s < 1$ (Riesz), by fractional Sobolev inequality:

$$\frac{d}{dt}\|u(t)\|_p^p \le -c_p\,M^{-f}\,\left(\|u(t)\|_p^p\right)^{\sigma}, \qquad \sigma > 1,\ f > 0 \text{ depend on } d, s, p$$

yielding L^p estimate. De Giorgi or Moser iteration methods provide L^∞ estimate:

$$\|u(t)\|_\infty \le C_{d,s}\,M^{\frac{2(1-s)}{d+2(1-s)}}\left(\frac{1}{t}\right)^{\frac{d}{d+2(1-s)}}, \qquad t > 0,$$

which is consistent with porous medium equation $\partial_t u = \Delta u^2$:

$$\|u(t)\|_\infty \le C_d\,M^{\frac{2}{2+d}}\left(\frac{1}{t}\right)^{\frac{d}{2+d}}, \qquad t > 0$$

[Caffarelli-Vázquez 2011, Caffarelli-Soria-Vázquez 2013, Serfaty-Vázquez 2013, Biler-Imbert-Karch 2015, Lisini-M.-Segatti 2018, Dao-Diaz 2020]

Aggregation-diffusion models

$$(KS) \qquad \begin{cases} \partial_t u = \Delta u^m - \chi \operatorname{div}(u \nabla K * u) & x \in \mathbb{R}^d, \quad t > 0, \\ u(0) = u_0 \in L^1_+(\mathbb{R}^d) \end{cases}$$

with $d \geq 2$, $m \geq 1$, $\chi \geq 0$, $K(x) = \begin{cases} c_d |x|^{2-d} & \text{if } d \geq 3 \\ -\frac{1}{2\pi} \log |x| & \text{if } d = 2 \end{cases}$ is Newtonian kernel

Free energy:
$$\mathcal{G}(u) = \begin{cases} \dfrac{1}{m-1} \displaystyle\int_{\mathbb{R}^d} u^m - \frac{\chi}{2} \int_{\mathbb{R}^d} u \, K * u & \text{if } m > 1 \\ \displaystyle\int_{\mathbb{R}^d} u \log u - \frac{\chi}{2} \int_{\mathbb{R}^d} u \, K * u & \text{if } m = 1 \end{cases}$$

The model is a Wassertein gradient flow

$$\partial_t u = \operatorname{div}(u \nabla \mathfrak{p}_u), \qquad \mathfrak{p}_u = \frac{\delta \mathcal{G}}{\delta u} = \begin{cases} \dfrac{m}{m-1} u^{m-1} - \chi K * u & \text{if } m > 1 \\ \log u - \chi K * u & \text{if } m = 1 \end{cases}$$

Three regimes for aggregation-diffusion models

Let $m > 1$.

By dilation $u_\lambda(x) := \lambda^d u(\lambda x)$, $\qquad \mathcal{G}(u_\lambda) = \lambda^{d(m-1)} \dfrac{1}{m-1} \displaystyle\int_{\mathbb{R}^d} u^m - \lambda^{d-2} \dfrac{\chi}{2} \int_{\mathbb{R}^d} u\, K * u$

Threshold exponent $m_c := 2 - 2/d$.

- $m = m_c$: fair competition regime (critical mass appears)
- $m > m_c$: diffusion dominated regime
- $m < m_c$: attraction dominated regime

Global-in-time solutions exist for $m > m_c$. Also for $m = m_c$ if the mass is subcritical [Calvez, Carrillo 2006], [Sugiyama 2007], [Blanchet, Carrillo, Laurencot 2009]

Classical Keller-Segel model is obtained for $d = 2$, $m = m_c = 1$: critical mass $= 8\pi/\chi$ [Jäger, Luckhaus 1992], [Dolbeault, Perthame 2004], [Blanchet-Dolbeault-Perthame 2006]

From now on, we assume $m \geq m_c$. We assume that the mass is subcritical if $m = m_c$

Hypercontractivity for aggregation diffusion models ($d \geq 3$, $m = 2$)

For a solution u to the equation $\partial_t u = \Delta u^2 - \chi\, \mathrm{div}\,(u\, \nabla K * u)$

$$\frac{d}{dt}\|u(t)\|_p^p = -\frac{8p(p-1)}{(p+1)^2}\int_{\mathbb{R}^d} |\nabla u^{\frac{p+1}{2}}|^2 + \chi(p-1)\int_{\mathbb{R}^d} u^{p+1}$$

By suitably applying Young and Sobolev inequalities, and interpolation with L^1 norm

$$\frac{d}{dt}\|u(t)\|_p^p \leq -\bar{c}_p\, M^{-\frac{2p+d}{d(p-1)}} \left(\|u(t)\|_p^p\right)^{1+\frac{2+d}{d(p-1)}} + \chi\, \tilde{c}_p\, M^{p+1}, \quad \left(y' \leq -\bar{c}_p\, M^{-f}\, y^{\sigma} + \chi\, \tilde{c}_p\, M^{p+1}\right)$$

Comparison principle entails

$$\|u(t)\|_p \leq M^{\frac{2p+d}{p(2+d)}} \left(\frac{d(p-1)}{\bar{c}_p(2+d)}\,\frac{1}{t}\right)^{\frac{p-1}{p}\frac{d}{2+d}} + (\chi\, \hat{c}_p)^{1/p}\; M$$

The constants $\dfrac{d(p-1)}{2+d}$ (same of porous media) and $\hat{c}_p^{1/p}$ are unbounded as $p \to +\infty$

Gradient flow approach: JKO scheme ($d \geq 3$, $m = 2$)

Let $\mathcal{W}$ the quadratic Wasserstein distance and $\mathcal{G}(u) = \int_{\mathbb{R}^d} u^2 - \frac{\chi}{2} \int_{\mathbb{R}^d} u\, K * u$.

Given $u_\tau^0 \sim u_0$ a mass density with finite second moment and $\tau > 0$, for all $k \in \mathbb{N}$

u_τ^k minimises $\quad u \mapsto \dfrac{1}{2\tau} \mathcal{W}^2(u, u_\tau^{k-1}) + \mathcal{G}(u)$ among mass densities (with mass $M = \|u_0\|_1$)

Typical regularization: $u_\tau^0 = u_0 * N_\tau$, where N_τ is a Gaussian kernel

Discrete solution: $u_\tau(t) := u_\tau^{[t/\tau]}$ (piecewise constant interpolation). $[\cdot]$ is integer part

Lemma (L^p estimate - discrete version)

$$\|u_\tau(t)\|_p \leq M^{\frac{2p+d}{p(2+d)}} \left(\frac{d(p-1)}{\bar{c}_p(2+d)} \, \frac{1}{t} \right)^{\frac{(p-1)}{p} \frac{d}{2+d}} + (\chi\, \hat{c}_p)^{1/p} M \; + \mathcal{R}(\tau), \qquad \mathcal{R}(\tau) \to 0$$

It can be proved with the flow interchange method by [McCann, Matthes, Savaré 2009].

The L^∞ estimate

$$(KS) \qquad \begin{cases} \partial_t u = \Delta u^m - \chi \,\mathrm{div}\,(u \nabla K * u) & x \in \mathbb{R}^d, \quad t > 0, \\ u(0) = u_0 \in L^1_+(\mathbb{R}^d) \end{cases}$$

Theorem

Let $d \geq 2$ and $m > m_c$. Let $u_0 \in L^m(\mathbb{R}^d)$ have mass M and finite second moment.

Let $u_\tau(t)$ a discrete solution from the JKO scheme. Then there is a vanishing sequence (τ_n) such that u_{τ_n} converges in $L^2_{loc}((0,+\infty) \times \mathbb{R}^d)$ to a a global-in-time weak solution u of the aggregation diffusion problem (KS): $\int_0^T \int_{\mathbb{R}^d} |\nabla u^{m-1}|^2 u \, dx \, dt < +\infty$ and

$$\int_0^{+\infty} \int_{\mathbb{R}^d} (\partial_t \varphi \, u - \nabla \varphi \cdot \nabla u^m + (\nabla \varphi \cdot \nabla K * u)\, u) \,\mathrm{d}x \,\mathrm{d}t = 0, \quad \forall \, \varphi \in C_c^\infty((0,+\infty) \times \mathbb{R}^d).$$

Furthermore, there exists a constant $C_{\chi,d,m}$ depending only on χ, d and m such that

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \leq C_{\chi,d,m} \, M^{\frac{2}{2+d(m-1)}} \left(\frac{1}{t}\right)^{\frac{d}{2+d(m-1)}} + C_{\chi,d,m} \, M^{\frac{2}{2+d(m-2)}} \qquad \text{for every } t > 0.$$

Remarks

The theorem yields existence of a global-in-time solution that satisfies

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \leq C_{\chi,d,m}\, M^{\frac{2}{2+d(m-1)}} \left(\frac{1}{t}\right)^{\frac{d}{2+d(m-1)}} + C_{\chi,d,m}\, M^{\frac{2}{2+d(m-2)}} \qquad t > 0$$

- the estimate extends previous results [Blanchet-Dolbeault-Perthame 2006, Bian-Liu-Zou 2014, Egaña Fernández-Mischler 2016, Liu-Wang 2016]
- a similar result can be obtained for $m = m_c$ as well (with subcritical mass)
- if $u_0 \in L^\infty(\mathbb{R}^d)$, we also get $\|u(t)\|_{L^\infty(\mathbb{R}^d)} \leq \bar{C}(M, \|u_0\|_\infty, \chi, d, m)$ for every $t > 0$, consistently with L^∞ bounds from [Kowalczyk 2005, Di Marino-Santambrogio 2022]
- If $u_0 \in L^q(\mathbb{R}^d)$ for some finite q , the power of $1/t$ improves to $\dfrac{d}{2q + d(m-1)}$
- Proof by Moser iterations: if $m = 2$ then $Q_j := \|u(t)\|_{2^j}^{2^j}$ satisfies recursive relation

$$Q_j \leq \left(K_d\,(2^d)^j\, t^{-z_j}\, Q_{j-1}^{\gamma_j}\right) \vee \left(\chi\, Q_{j-1}^2\right), \quad Q_0 = M, \quad z_j := \frac{2^{j-1}\, d}{2^j + d}, \quad \gamma_j = \frac{2(2^j + d/2)}{2^j + d}$$

The gradient flow-EVI structure of (KS) and uniqueness

We exploit the quasi-Lipschitz estimate as in [Yudovich 1963]. For $\rho \in L^1 \cap L^\infty(\mathbb{R}^d)$

$$|\nabla K * \rho(x) - \nabla K * \rho(y)| \leq C_\rho\, \omega(|x-y|), \qquad C_\rho := K_d(\|\rho\|_{L^1(\mathbb{R}^d)} + \|\rho\|_{L^\infty(\mathbb{R}^d)})$$

where $\quad \omega(x) \sim x\,|\log x|$ for small x, $\quad \omega(x) \sim x$ for large x

Theorem (Carrillo-Lisini-M. 2014)

Let u a solution to (KS) and $\bar u \in L^\infty(\mathbb{R}^d)$ a given density (with same mass M as u). Then

$$\frac{1}{2}\frac{d}{dt} W_2^2(u(t), \bar u) \leq \mathcal{G}(\bar u) - \mathcal{G}(u(t)) + C(t)\, \omega(W_2^2(u(t), \bar u)), \quad t > 0$$

where $C(t) := K_d(M + \|\bar u\|_\infty + \|u(t)\|_\infty)$.

Then for two solutions $u(t), \zeta(t)$, let $y(t) = W_2^2(u(t), \zeta(t))$. For small t by L^∞ estimate

$$\frac{1}{2}\frac{d}{ds} y(s)\Big|_{s=t} \leq \frac{1}{2}\frac{d}{ds} W_2^2(u(s), \zeta(t))\Big|_{s=t} + \frac{1}{2}\frac{d}{ds} W_2^2(u(t), \zeta(s))\Big|_{s=t}$$

$$\lesssim (M + \|u(t)\|_\infty + \|\zeta(t)\|_\infty)\, \omega(W_2^2(u(t), \zeta(t))) \lesssim t^{-\frac{d}{2+d(m-1)}}\, \omega(y(t))$$

$y' \leq g(t)\, \omega(y(t))$, ω is log-Lipschitz, g is integrable at 0, so $y(0) = 0$ implies $y \equiv 0$

Thanks for the attention