Wasserstein vs Hilbert: approaches to multi-particle control problems

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In collaboration with *S. Lisini, C. Orrieri, G. Savaré*



Plan

General aim: Study optimal-control problems for a system of interacting particles (or multi-agent system).
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How we formulate the problem: Eulerian description in the space of probability measures $\mathcal{P}_2(\mathbb{R}^d)$; Lagrangian description in $L^2(\Omega;\mathbb{R}^d)$ for some fixed parametrization space $(\Omega,\mathcal{B},\mathbb{P})$.

Tasks:

(Q1). Compare the above formulations;

(Q2). Study of limit theory for both formulations: prove stability and Γ-convergence results for the corresponding finite-particles systems to the mean-field ones;

(Q3). HJB...

Preliminaries: Measure Theory and Optimal Transport

E separable Banach space (usually $\mathrm{E} = \mathbb{R}^d$)
$(\Omega, \mathcal{B})$ standard Borel space, $\mathbb{P}$ non-atomic.

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$\Gamma(\mu, \nu)$: couplings $\gamma \in \mathcal{P}(\mathrm{E} \times \mathrm{E})$ with marginals μ and ν, e.g. $\gamma = (X, Y)_\sharp\mathbb{P}$, $X_\sharp\mathbb{P} = \mu$, $Y_\sharp\mathbb{P} = \nu$.

$$W_2^2(\mu, \nu) := \min \left\{ \int |x - y|^2 \, d\gamma : \gamma \in \Gamma(\mu, \nu) \right\}$$
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\end{aligned}
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If $\mu_n, \mu \in \mathcal{P}_2(\mathrm{E})$ with $W_2(\mu_n, \mu) \to 0$, then there exist $X_n, X \in L^2_{\mathbb{P}}(\Omega; \mathrm{E})$ such that

- $X_{n\sharp} \mathbb{P} = \mu_n$, $X_\sharp \mathbb{P} = \mu$
- $\|X - X_n\|_2 \to 0$

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Main example: discrete measures

$$\mu = \frac{1}{N} \sum_{n=1}^{N} \delta_{x_n}, \quad \Omega := \{\omega_1, \ldots, \omega_N\}, \quad \mathbb{P} := \frac{1}{N} \sum_{n=1}^{N} \delta_{\omega_n}, \quad X(\omega_n) := x_n.$$

Preliminaries: Lift and law invariance

Definition

1. A function $V : L^2_{\mathbb{P}}(\Omega; \mathbb{R}^d) \to \mathbb{R}$ is law invariant if
$$V(X) = V(Y), \quad \text{for all } X, Y \text{ such that } X_\sharp \mathbb{P} = Y_\sharp \mathbb{P}.$$
2. Given $v : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, the lift of v is the function $V : L^2_{\mathbb{P}}(\Omega; \mathbb{R}^d) \to \mathbb{R}$ defined by
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Notice that

- the lift V of v is law invariant
- v is continuous if and only if V is continuous

Preliminaries: continuity equation

Superposition Principle [Ambrosio-Gigli-Savaré]

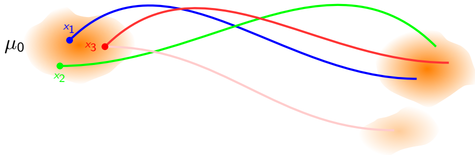
Continuity equation

$$\begin{cases} \partial_t \mu_t + \mathrm{div}(f_t \mu_t) = 0, & t \in (0, T), \\ \mu_{|t=0} = \mu_0 \in \mathcal{P}_2(\mathbb{R}^d). \end{cases}$$

Characteristic system

$$\begin{cases} \dot\gamma(t) = f_t(\gamma(t)), & t \in (0, T), \\ \gamma(0) = x \in \mathrm{supp}(\mu_0). \end{cases}$$

- if $f_t(\cdot)$ is locally Lipschitz in x unif. w.r.t. t
  $\Rightarrow \quad \mu_t = \Phi_t \sharp \mu_0$;
- if f satisfies integrability assumptions
  $\Rightarrow \quad \mu_t = e_t \sharp \boldsymbol{\eta}$, with $\boldsymbol{\eta} \in \mathcal{P}(C^0([0,T];\mathbb{R}^d))$ concentrated on characteristics



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with $e_t(\gamma) := \gamma(t)$

Structural assumptions on dynamics and cost

Consider the following metric on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, $p \geq 1$,

$$d((x,\mu),(y,\nu)) := \left(|x-y|^2 + W_2^2(\mu,\nu)\right)^{1/2}.$$

- U a compact subset of a Banach space, called control set;
- $f : \mathbb{R}^d \times U \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ a continuous vector field satisfying
$$|f(x,u,\mu) - f(y,u,\rho)| \leq L\, d((x,\mu),(y,\rho))$$
for some $L > 0$ independent of $u \in U$;
- $\mathcal{C} : \mathbb{R}^d \times U \times \mathcal{P}_2(\mathbb{R}^d) \to [0,+\infty[$, employed in the running cost,
$\mathcal{C}_T : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to [0,+\infty[$, employed in the terminal cost,
be continuous satisfying
$$|\mathcal{C}(x,u,\mu)| \leq D\left(1 + |x|^2 + \mathrm{m}_2^2(\mu)\right)$$
$$|\mathcal{C}_T(x,\mu)| \leq D\left(1 + |x|^2 + \mathrm{m}_2^2(\mu)\right),$$
for some $D > 0$, for all $(x,u,\mu) \in \mathbb{R}^d \times U \times \mathcal{P}_2(\mathbb{R}^d)$,

Different approaches

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$\mu \in \mathrm{AC}([0,T];\mathcal{P}_2(\mathbb{R}^d))$

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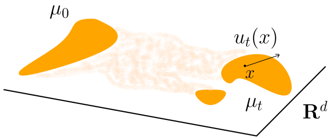

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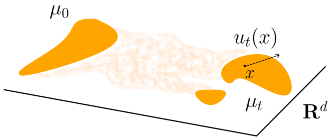

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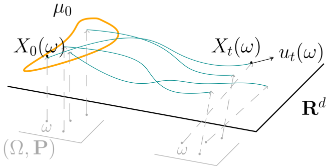

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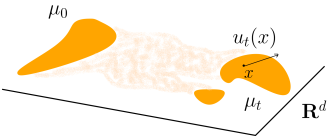

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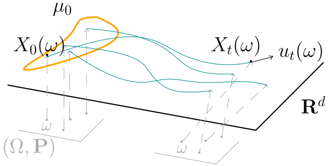

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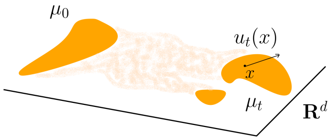

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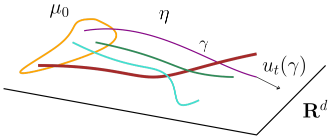

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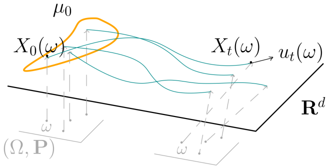

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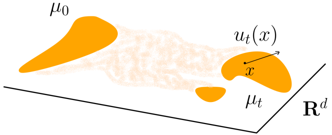

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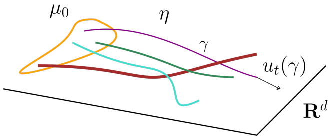

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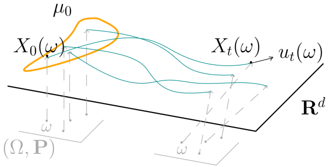

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Tasks

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(Q2). Study of limit theory for both formulations: prove stability and Γ-convergence results for the corresponding finite-particles systems to the mean-field ones;

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Q1: Existence of minimizers for K, E & Equivalence results

We say that the convexity conditions hold if

- U is convex;
- $u \mapsto \mathcal{C}(x, u, \mu)$ is convex;
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Theorem (Existence of minimizers for convex E and K)

Assume the convexity conditions. For each $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ there exist $(\boldsymbol{\eta}, \tilde{u}) \in \mathcal{A}_{\mathbf{K}}(\mu_0)$ and $(\mu, u) \in \mathcal{A}_{\mathbf{E}}(\mu_0)$ such that

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The proof for **E** follows by a suitable compactness property and lower semicontinuity of $J_{\mathbf{E}}$. For **K** use the SP.

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Theorem

Assume the convexity conditions and $\mathbb{P}$ without atoms. If $X_0 \in L^2(\Omega; \mathbb{R}^d)$, then $V_{\mathbf{L}}(X_0) = v_{\mathbf{E}}((X_0)_\sharp \mathbb{P})$. In particular, $V_{\mathbf{L}}$ is the lift of $v_{\mathbf{E}}$.

The proof passes through the equivalence with **K**

Q1: Continuity of the value functions

Theorem (Continuity of $v_{\mathbf{E}}$)

Assume the convexity conditions and $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$. If $\{\mu_0^n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}_2(\mathbb{R}^d)$ is a sequence such that $W_2(\mu_0^n, \mu_0) \to 0$ as $n \to +\infty$, then

$$\lim_{n\to+\infty} v_{\mathbf{E}}(\mu_0^n) = v_{\mathbf{E}}(\mu_0).$$

Under the convexity conditions, being $V_{\mathbf{L}}$ the lift of $v_{\mathbf{E}}$, we already observed:

$$v_{\mathbf{E}} \text{ continuous } \Leftrightarrow V_{\mathbf{L}} \text{ continuous}$$

However, we can get the result without assuming the convexity conditions:

Theorem (Continuity of $V_{\mathbf{L}}$)

(No convexity assumption required). Let $\mathbb{P}$ be without atoms and $X_0 \in L^2(\Omega;\mathbb{R}^d)$. If $\{X_0^n\}_{n\in\mathbb{N}} \subseteq L^2(\Omega;\mathbb{R}^d)$ is a sequence such that $\|X_0^n - X_0\|_{L^2} \to 0$ as $n \to +\infty$, then

$$\lim_{n\to+\infty} V_{\mathbf{L}}(X_0^n) = V_{\mathbf{L}}(X_0).$$

Q1: Non existence of minimizers for L

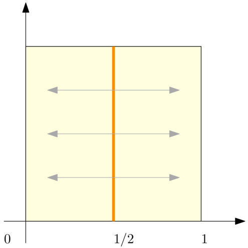
In general, the Lagrangian problem does not have minimizers, even under convexity assumptions. Indeed, the choice of the initial condition X_0 is relevant.

Example:

- $d = 2$, $T = 1$; $U = B_R(0) \subseteq \mathbb{R}^2$;
- $\Omega = [0,1]$, $\mathbb{P} = \mathcal{L}^1_{|[0,1]}$;
- fix a reference measure $\nu = \mathcal{L}^2_{|[0,1]^2} \in \mathcal{P}(\mathbb{R}^2)$ with compact support and set

$$f(x,u,\mu) = u, \quad \mathcal{C}(x,u,\mu) = |u|^2, \quad \mathcal{C}_T(x,\mu) = W_2^2(\mu,\nu);$$

- $X_0 : \Omega \to \mathbb{R}^d$, $X_0(\omega) = (1/2, \omega)$, $\mu_0 = X_{0\sharp}\mathbb{P} = \mathscr{H}^1_{|\{1/2\}\times[0,1]}$.



Tasks

(Q1). Compare the previous formulations;

(Q2). Study of limit theory for both formulations: prove stability and Γ-convergence results for the corresponding finite-particles systems to the mean-field ones;

(Q3). HJB...

Q2: Limit theory - the N-particles case

Lagrangian: we recover the N-particles case $\mathbf{L}^N$ choosing

- $\Omega^N = \{1, \dots, N\}$ with the counting measure $\mathbb{P}^N = \frac{1}{N}\sum_{\omega=1}^{N} \delta_\omega$;
- or, equivalently, $(\Omega, \mathbb{P})$ with $\mathbb{P}$ without atoms, requiring the admissible trajectories and controls to be constant over the elements of a partition $\mathcal{P}^N$ such that $\mathbb{P}(A) = \frac{1}{N}$ for every $A \in \mathcal{P}^N$.

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Eulerian: we denote by $\mathbf{E}^N$ the Eulerian problem with the additional constraint that admissible trajectories satisfy: $\mu_t \in \mathcal{P}^N(\mathbb{R}^d)$ for every $t \in [0, T]$, where

$$\mathcal{P}^N(\mathbb{R}^d) := \left\{ \mu = \frac{1}{N}\sum_{i=1}^N \delta_{x_i} \text{ for some } x_i \in \mathbb{R}^d \right\}.$$

RMK: asking $\mu_0 \in \mathcal{P}^N(\mathbb{R}^d)$ doesn't imply that $\mu_t \in \mathcal{P}^N(\mathbb{R}^d)$ for every $t \in [0, T]$... the control is not Lipschitz!

Q2: Equivalence result for N-particles

Proposition (Equivalence result for N-particles)

For any $X_0 \in (\mathbb{R}^d)^N$, denoting by $\mu_0 := \frac{1}{N}\sum_{i=1}^N \delta_{X_0^i}$, we have

$$V_{\mathbf{L}^N}(X_0) = v_{\mathbf{E}^N}(\mu_0).$$

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Theorem (empirical Superposition Principle)

Let $t \mapsto \mu_t$ absolutely cont., $\mu_t \in \mathcal{P}^N(\mathbb{R}^d)$ for every $t \in [0, T]$. Then

- $\exists!$ (integrable) $w : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ s.t. μ solves

$$\partial_t \mu_t + \mathrm{div}(w_t \mu_t) = 0, \quad \text{in } [0, T] \times \mathbb{R}^d.$$

- $\exists\, \eta \in \mathcal{P}^N(\Gamma_T)$, $\eta = \dfrac{1}{N}\displaystyle\sum_{i=1}^N \delta_{\gamma_i}$, s.t. $(e_t)_\sharp \eta = \mu_t$ for every $t \in [0, T]$ and for any $i = 1, \ldots, N$, γ_i solves

$$\dot{\gamma}_i(t) = w_t(\gamma_i(t)) \quad \text{for a.e. } t \in [0, T].$$

Q2: Finite particles approximation for L and E

Theorem

Let $\mathbb{P}$ be without atoms and assume the convexity conditions.

- If $X_0 \in L^2(\Omega; \mathbb{R}^d)$ and $X_0^N : \Omega^N \to \mathbb{R}^d$, $N \in \mathbb{N}$, satisfy $X_0^N \to X_0$ as $N \to +\infty$, then
$$\lim_{N \to +\infty} V_{\mathbf{L}^N}(X_0^N) = V_{\mathbf{L}}(X_0).$$
- If $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ and $\mu_0^N \in \mathcal{P}^N(\mathbb{R}^d)$, $N \in \mathbb{N}$, satisfy $W_p(\mu_0^N, \mu_0) \to 0$ as $N \to +\infty$, then
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$$\lim_{N\to+\infty} v_{\mathbf{E}^N}(\mu_0^N) = v_{\mathbf{E}}(\mu_0).$$
- In particular, if $(X_0^N)_\sharp \mathbb{P}^N = \mu_0^N$ it holds that
$$\lim_{N\to+\infty} V_{\mathbf{L}^N}(X_0^N) = v_{\mathbf{E}}(\mu_0).$$

Tasks

(Q1). Compare the previous formulations;

(Q2). Study of limit theory for both formulations: prove stability and Γ-convergence results for the corresponding finite-particles systems to the mean-field ones;

(Q3). HJB...

Q3: Hamiltonians for L and E

Given $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, the classical Hamiltonian $H_\mu : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is defined by

$$H_\mu(x, q) := \sup_{\bar{u} \in U} \left\{ -f(x, \bar{u}, \mu) \cdot q - \mathcal{C}(x, \bar{u}, \mu) \right\}.$$

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The Hamiltonian for **L**, $\mathcal{H}_{\mathbf{L}} : L^2_{\mathbb{P}}(\Omega;\mathbb{R}^d) \times L^2_{\mathbb{P}}(\Omega;\mathbb{R}^d) \to \mathbb{R}$, is given by

$$\mathcal{H}_{\mathbf{L}}(X,\zeta) = \int_\Omega H_{X_\sharp\mathbb{P}}(X(\omega),\zeta(\omega))\,\mathrm{d}\mathbb{P}(\omega) = \int_{\mathbb{R}^d\times\mathbb{R}^d} H_{X_\sharp\mathbb{P}}(x,q)\,\mathrm{d}\left[(X,\zeta)_\sharp\mathbb{P}\right](x,q).$$

Given $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and $\xi \in L^2_\mu(\mathbb{R}^d;\mathbb{R}^d)$, we define the Hamiltonian for **E**

$$\mathcal{H}_{\mathbf{E}}(\mu,\xi) := \int_{\mathbb{R}^d} H_\mu(x,\xi(x))\,\mathrm{d}\mu(x) = \int_{\mathbb{R}^d\times\mathbb{R}^d} H_\mu(x,q)\,\mathrm{d}\left[(\mathrm{id},\xi)_\sharp\mu\right](x,q).$$

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Proposition

- $\mathcal{H}_{\mathbf{L}}$ is law invariant in $L^2_{\mathbb{P}}(\Omega;\mathbb{R}^d) \times L^2_{\mathbb{P}}(\Omega;\mathbb{R}^d)$.
- For any $X \in L^2_{\mathbb{P}}(\Omega;\mathbb{R}^d)$, $\xi \in L^2_{X_\sharp \mathbb{P}}(\mathbb{R}^d;\mathbb{R}^d)$, it holds

$$(\star) \quad \mathcal{H}_{\mathbf{L}}(X,\xi \circ X) = \mathcal{H}_{\mathbf{E}}(X_\sharp \mathbb{P},\xi).$$

Q3: Important remarks

Proposition [Lions, Gangbo-Tudorascu, Jimenez-Marigonda-Quincampoix]

Let $V : L^2_{\mathbb{P}}(\Omega; \mathbb{R}^d) \to \mathbb{R}$ be law invariant and differentiable at $\bar{X}$. Then there exists $p \in \mathrm{Tan}_{\bar{X}_\sharp \mathbb{P}}(\mathbb{R}^d)$ such that

$$DV(X) = p \circ X \quad \text{for any } X \text{ s.t. } X_\sharp \mathbb{P} = \bar{X}_\sharp \mathbb{P}.$$

In this case, we denote $D\, v(\bar{\mu}) = p$.

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Notice that, differently from

Gangbo - Tudorascu

Jimenez - Marigonda - Quincampoix

Jimenez

in our case, we have that

- both $V_{\mathbf{L}}$ and $\mathcal{H}_{\mathbf{L}}$ are not defined in an abstract way from $v_{\mathbf{E}}$ and $\mathcal{H}_{\mathbf{E}}$ (by lift or by $(\star)$), but they come from an optimal control problem in $L^2_{\mathbb{P}}(\Omega;\mathbb{R}^d)$;

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- in particular, $\mathcal{H}_{\mathbf{L}}$ is already defined over the whole $L^2_{\mathbb{P}}(\Omega;\mathbb{R}^d) \times L^2_{\mathbb{P}}(\Omega;\mathbb{R}^d)$ and thus it doesn't need to be extended.

Q3: Sub-differential in $(\mathcal{P}_2(\mathbb{R}^d), W_2)$ and sub-test function in $L^2_{\mathbb{P}}(\Omega;\mathbb{R}^d)$

Definition (subdifferential in $(\mathcal{P}_2(\mathbb{R}^d), W_2)$)

Let $v : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$. Given $(\bar{t}, \bar{\mu}) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$, we say that the pair $(p_{\bar{t}}, p_{\bar{\mu}}) \in \mathbb{R} \times L^2_{\bar{\mu}}(\mathbb{R}^d)$ is a viscosity superdifferential of v at $(\bar{t}, \bar{\mu})$, writing $(p_{\bar{t}}, p_{\bar{\mu}}) \in D^+ v(\bar{t}, \bar{\mu})$, if

- $p_{\bar{\mu}}$ is an optimal (anti)displacement from $\bar{\mu}$;
- for all $(t,\mu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$ and $\sigma \in \Gamma(\bar{\mu}, \mu)$,

$$v(t,\mu) - v(\bar{t}, \bar{\mu}) \leq p_{\bar{t}}\,(t - \bar{t}) + \int_{\mathbb{R}^d} \langle p_{\bar{\mu}}(x), y - x \rangle \mathrm{d}\sigma(x,y) + o\left(\Delta_{t,\sigma}\right).$$

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$$v(t,\mu)-v(\bar t,\bar\mu)\le p_{\bar t}\,(t-\bar t)+\int_{\mathbb{R}^d}\langle p_{\bar\mu}(x), y-x\rangle\mathrm{d}\sigma(x,y)+o\left(\Delta_{t,\sigma}\right).$$

Definition (subtest function in $L^2_{\mathbb{P}}(\Omega;\mathbb{R}^d)$)

Let $V : [0,T]\times L^2_{\mathbb{P}}(\Omega;\mathbb{R}^d)\to\mathbb{R}$. Given $(\bar t,\bar X)\in[0,T]\times L^2_{\mathbb{P}}(\Omega;\mathbb{R}^d)$, we say that $\Phi : [0,T]\times L^2_{\mathbb{P}}(\Omega;\mathbb{R}^d)\to\mathbb{R}$ is a super-test function of V at $(\bar t,\bar X)$ if

- Φ is law invariant, continuous and differentiable at $(\bar t,\bar X)$;
- $\Phi(\bar t,\bar X)=V(\bar t,\bar X)$ and $(\bar t,\bar X)$ is a local maximum point for $V-\Phi$

Q3: Viscosity solutions in $(\mathcal{P}_2(\mathbb{R}^d), W_2)$ and in $L^2_{\mathbb{P}}(\Omega; \mathbb{R}^d)$

Definition (viscosity solution for HJB in $(\mathcal{P}_2(\mathbb{R}^d), W_2)$)

Let $v : [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and consider the equation

$$-\partial_t v(t, \mu) + \mathcal{H}(\mu, \partial_\mu v(t, \mu)) = 0, \quad \text{in } (0, T) \times \mathcal{P}_2(\mathbb{R}^d). \tag{1}$$

v is a viscosity subsolution of (1) if v is u.s.c. and

$$-p_t + \mathcal{H}(\mu, p_\mu) \leq 0,$$

for all $(t, \mu) \in (0, T) \times \mathcal{P}_2(\mathbb{R}^d)$, $(p_t, p_\mu) \in D^+ v(t, \mu)$.
Analogously, we define supersolution and solution...

Definition (viscosity solution for HJB in $L^2_{\mathbb{P}}(\Omega; \mathbb{R}^d)$)

Let $V : [0, T] \times L^2_{\mathbb{P}}(\Omega; \mathbb{R}^d) \to \mathbb{R}$ and consider the equation

$$-\partial_t V(t, X) + \mathcal{H}(X, \partial_X V(t, X)) = 0, \quad \text{in } (0, T) \times L^2_{\mathbb{P}}(\Omega; \mathbb{R}^d). \tag{2}$$

V is a viscosity subsolution of (2) if V is u.s.c. and (2) holds with $\leq$ for any $(t, X) \in (0, T) \times L^2_{\mathbb{P}}(\Omega; \mathbb{R}^d)$ and any super-test function Φ of V at (t, X).
Analogously, we define supersolution and solution...

Q3: Results for HJB

Theorem

Under the convexity conditions, the value function $v_{\mathbf{E}}$ is a viscosity solution of

$$-\partial_t v_{\mathbf{E}}(t,\mu) + \mathcal{H}_{\mathbf{E}}(\mu, \partial_\mu v_{\mathbf{E}}(t,\mu)) = 0, \quad \text{in } (0,T) \times \mathcal{P}_2(\mathbb{R}^d).$$

Idea for the proof:

- DPP
- Use the Superposition Principle...

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Idea for the proof:

- DPP
- Use the Superposition Principle...

Can we get a similar result for $V_{\mathbf{L}}$?

Q3: Work in progress

Theorem [Jimenez-Marigonda-Quincampoix, Jimenez]

If $v : [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ is continuous and for any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ the map $p \mapsto \mathcal{H}(\mu, p)$ is continuous in $L^2_\mu(\mathbb{R}^d)$, then the following are equivalent:

- v is viscosity subsolution of (HJ) in $\mathcal{P}_2(\mathbb{R}^d)$;
- the lift V of v is a viscosity subsolution of (HJ) in $L^2_{\mathbb{P}}(\Omega; \mathbb{R}^d)$, with Hamiltonian build by $(\star)$.

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Using the previous result, we expect to get the following.

Claim

(Under the convexity conditions) the value function $V_{\mathbf{L}}$ is a viscosity solution of

$$-\partial_t V_{\mathbf{L}}(t, X) + \mathcal{H}_{\mathbf{L}}(X, \partial_X V_{\mathbf{L}}(t, X)) = 0, \quad \text{in } (0, T) \times L^2_{\mathbb{P}}(\Omega; \mathbb{R}^d).$$

Future tasks

What next:

- Comparison principle for HJB in $\mathcal{P}_2(\mathbb{R}^d)$ and/or $L^2(\Omega;\mathbb{R}^d)/\sim$ with less regularity conditions

  Cardaliaguet, Lions, Gangbo - Tudorascu

  Marigonda - Quincampoix, Jimenez - Marigonda - Quincampoix, Jimenez

  Bertucci

  Conforti - Kraaij - Tonon

  Zidani - Jerhaoui - Aussedat

- Limiting theory for HJB: from $\mathbf{E}^N$ to $\mathbf{E}$, and from $\mathbf{L}^N$ to $\mathbf{L}$, as $N \to +\infty$.

Some references

L. Ambrosio, N. Gigli , G. Savaré

Gradient flows in metric spaces and in the space of probability measures, 2008

P. Cardaliaguet

Notes on Mean Field Games (from P.-L. Lions' lectures at Collège de France), http://www.college-de-france.fr, 2013

R. Carmona, F. Delarue

Probabilistic theory of mean field games with applications-I, 2018

G.C., S. Lisini, C. Orrieri, G. Savaré

Lagrangian, Eulerian and Kantorovich formulations of multi-agent optimal control problems: Equivalence and Gamma-convergence, 2022

C. Jimenez

Equivalence between strict viscosity solution and viscosity solution in the Wasserstein space and regular extension of the Hamiltonian in $L^2_{\mathbb{P}}$, 2024

C. Jimenez, A. Marigonda, M. Quincampoix

Dynamical systems and Hamilton-Jacobi-Bellman equations on the Wasserstein space and their L^2 *representations*, 2022

Thank you!

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