The geometry of the free boundary

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- Given a fully nonlinear uniformly elliptic operator F, consider $W^{2,n}$ solutions of the following PDE

$$\begin{cases} F(D^2u) = \chi_\Omega & \text{in } B_1^+ := \mathbb{R}^n_+ \cap B_1, \\ u = 0 & \text{on } B_1' := \{x_n = 0\} \cap B_1, \end{cases}$$

where $\Omega \subset \{x_n > 0\}$ denotes an open set and $\Gamma(u) = \partial\Omega \cap B_1^+$ denotes the corresponding free boundary.

- Main problems: Investigating the dynamics and regularity of $\Gamma(u)$.
- Conjecture ($\sim$ 2004): If $\Omega = (\{u \neq 0\} \cup \{\nabla u \neq 0\}) \cap \{x_n > 0\}$, $0 \in \overline{\Gamma(u)}$, then $\Gamma(u)$ intersects $\{x_n = 0\}$ non-transversally.
- Problem (classical but explicitly stated in the 2012 book [published by AMS] "Regularity of free boundaries in obstacle-type problems"): If $\Omega = \{\nabla u \neq 0\} \cap \{x_n > 0\}$, $0 \in \overline{\Gamma(u)}$, then investigate the C^1 regularity of $\overline{\Gamma(u)}$.

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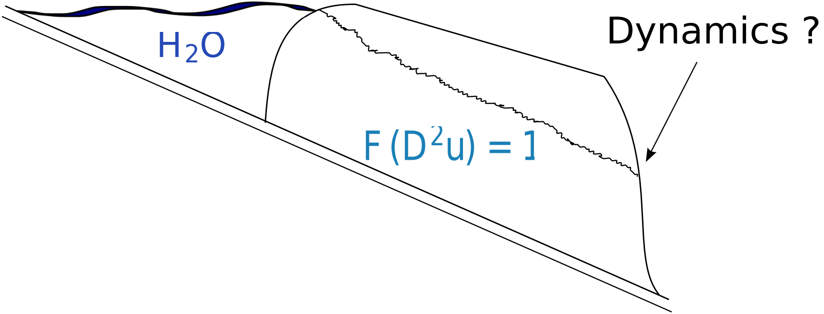
- Physical model: Dam problem, Alt-Giraldi, 1982



Dynamics if $\Omega = (\{u \neq 0\} \cup \{\nabla u \neq 0\}) \cap \{x_n > 0\}$

Denote $P_1^+(0, M, \Omega)$ as the class of bounded solutions generated by fully nonlinear uniformly elliptic convex C^1 operators, where $||u||_{L^\infty}(B_1^+) \leq M$.

Theorem (I., CPAM 2019)

There exists $r > 0$ and a modulus of continuity ω such that

$$\Gamma(u) \cap B_r^+ \subset \{x : x_n \leq \omega(|x'|)|x'|\}$$

for all $u \in P_1^+(0, M, \Omega)$, provided $0 \in \overline{\Gamma(u)}$.

Previous results

- $F(D^2u) = \Delta u$: Shahgholian & Uraltseva (Duke, 2003) (based on the Alt-Caffarelli-Friedman monotonicity formula).
- $n = 2$: (non-uniform) tangential touch was obtained in joint work with Minne (Analysis and PDE, 2016).
- Remark: monotonicity formulas are not available for fully nonlinear uniformly elliptic operators. In particular, new techniques were necessary.

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Theorem (I., CPAM 2019)

Suppose $u \in P_1^+(0, M, \Omega)$, $0 \in \overline{\{u \neq 0\}}$, *and* $\nabla u(0) = 0$. *Then the blow-up limit of* u *at the origin has the form*

$$u_0(x) = a x_1 x_n + b x_n^2$$

for $a, b \in \mathbb{R}$.

- In the interior, this was proven in the case of the Laplacian by Caffarelli-Karp-Shahgholian (Ann. of Math., 2000).
- The boundary case for the Laplacian was obtained by Shahgholian-Uraltseva (Duke, 2003).

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Free boundary regularity

- In the physical case (i.e. $u \geq 0$) when the operator is of the form

$$Au = a_{ij}\partial_i\partial_j u,$$

  with $a_{ij} \in C^3$, Caffarelli proved that if at an interior free boundary point the complement of the set on which the PDE holds (i.e. the zero set for the solution and its gradient) has positive density, then locally the free boundary can be represented as the graph of a C^1 function (Acta Math., 1977).
- If the free boundary fails to satisfy the density assumption, then it may develop a cusp. An example was constructed, for instance, by Schaeffer (Ann. Scuola Norm. Sup. Pisa, 1977).

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Free boundary regularity

- It turns out, that with the assumption of zero Dirichlet boundary data, the free boundary is C^1 at contact points with the fixed boundary in the fully nonlinear setting without density assumptions. Examples show that higher regularity beyond $C^{1,\mathrm{Dini}}$ is not possible.

Theorem (I., CPAM 2019)

Suppose $u \in P_1^+(0, M, \Omega)$ is non-negative and $0 \in \overline{\Gamma(u)}$. Then there exists $r > 0$ such that $\Gamma(u)$ is the graph of a C^1 function in B_r^+.

Dynamics if $\Omega = \{\nabla u \neq 0\} \cap \{x_n > 0\}$

Theorem (I., arXiv 2023)

There exists $r > 0$ *and a modulus of continuity* ω *such that*

$$\Gamma(u) \cap B_r^+ \subset \{x : x_n \leq \omega(|x'|)|x'|\}$$

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- $F(D^2 u) = \Delta u$: non-transversal intersection was proved by Matevosyan (Comm. Partial Differential Equations, 2005).
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Outline of the proof

- Consider blow-up limits of $\{u\} \subset P_1^+(0, M, \Omega)$ at the origin:

$$\lim_{k\to\infty} \frac{u(s_k x)}{s_k^2},$$

where $s_k \to 0^+$.

- Only one of the following is true:
(1) all blow-up limits of $\{u\}$ are of the form $u_0(x) = bx_n^2$ for some $b > 0$;
(2) there exists a blow-up limit of $\{u\}$ of the form $ax_1x_n + bx_n^2$ for $a \neq 0$ and $b \in \mathbb{R}$.
- Assume the tangential touch is not true, then there exists $\epsilon > 0$ such that for all $l \in \mathbb{N}$ there exists

$$x_l \in C_\epsilon \cap B^+_{\frac{1}{l}} \cap \Gamma(u),$$

$$\mathcal{C}_\epsilon := \{x_n > \epsilon|x'|\},\ x' = (x_1, \ldots, x_{n-1}).$$

Outline of the proof

- Consider blow-up limits of $\{u\} \subset P_1^+(0, M, \Omega)$ at the origin:

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- In order to obtain a contradiction, one applies the aforementioned: either (1) or (2) is true and this is the contradiction.
  Step 1. Claim: (2) is not true. Proof: Assume (2) is true.
- Then $u_0 = ax_1x_n + bx_n^2$ is a blow-up limit of $\{u\}$ for $a \neq 0$. Mod a rotation, one obtains $a > 0$. Via the definition of a blow-up limit, there is a sequence $\{s_k\}$ such that $s_k \to 0^+$,

$$\lim_{k\to\infty} \frac{u(s_k x)}{s_k^2} = ax_1x_n + bx_n^2.$$

- Set

$$w_l = \frac{x_l}{|x_l|} \in \mathbb{S}^{n-1}$$

such that along a subsequence

$$w_l \to e \in \overline{C_\epsilon} \cap \mathbb{S}^{n-1}.$$

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- When
$$\tilde{u}_k(x) = \frac{u(s_k x)}{s_k^2},$$
one obtains
$$\sup_k ||D^2 \tilde{u}_k||_{L^\infty(B^+_{t_*})} = T = T(t_*) < \infty$$
if $t_* > 0$ via uniform estimates on $||\tilde{u}_k||_{W^{2,n}(B^+_{2t_*})}$ [I.-Minne, Analysis and PDE].
- Set $z_{l,k} = \frac{x_l}{s_k}$, $R > 1$, and $\alpha \in (0, 1/5)$. Since $\tilde{u}_k \to u_0$ in $C^{1,\alpha_1}(B^+_R)$ for any $\alpha_1 \in (0,1)$, choose such a value and observe
$$||\tilde{u}_k - u_0||_{C^{1,\alpha_1}(B^+_R)} \to 0, \qquad \inf_{C_\epsilon \setminus B^+_\alpha} |\nabla u_0(x)| \geq a_1 > 0. \quad (1)$$
- Thanks to the triangle inequality together with (1), there exists $N = N(\alpha, R, a_1, T) \in \mathbb{N}$ so that assuming $k \geq N$,
$$\inf_{(B^+_R \cap C_\epsilon)\setminus B^+_\alpha} |\nabla \tilde{u}_k(x)| \geq \frac{a_1}{3}. \quad (2)$$

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Since $z_{l,k} = \frac{x_l}{s_k}$, $s_k \to 0^+$, observe

$$|z_{1,k}| \to \infty$$

when $k \to \infty$. Hence let $k_1 \geq N$ be the smallest integer for which $|z_{1,k_1}| \geq R$; note that (2) readily implies

$$\Gamma_k \cap ((B_R^+ \cap C_\epsilon) \setminus B_\alpha^+) = \emptyset$$

when $k \geq N$ thanks to $\nabla \tilde{u}_k(x) = 0$ if $x \in \Gamma_k$. Now, since

$$\mathcal{L}_t = \{tz_{1,k_1} : t \in (0,1]\} \subset C_\epsilon$$

set $z_{k_1(1)}$ as an element on $\overline{B^+_{|x_1|}} \cap \Gamma(\tilde{u}_{k_1})$ closest to $\mathcal{L}_t$; note that from (11),

$$|x_l| \leq \frac{1}{l} \to 0,$$

therefore without loss of generality, one may assume $|x_1| < \alpha$.

In particular, if

$$A_1 = \{tz_{1,k_1} + (1-t)z_{k_1(1)} : 0 \leq t \leq 1\}$$

then there are at least two free boundary elements on A_1: z_{1,k_1}, $z_{k_1(1)}$. Note that mod a re-labeling, thanks to

$$\Gamma_{k_1} \cap ((B_R^+ \cap C_\epsilon) \setminus B_\alpha^+) = \emptyset,$$

and

$$|z_{l,k_1}| \to 0$$

when $l \to \infty$, one may assume $z_{1,k_1}, z_{k_1(1)}$ are the sole free boundary points on A_1 because if not, since there are no free boundary points in $((B_R^+ \cap C_\epsilon) \setminus B_\alpha^+)$, one can pick $z_1 \in A_1$ closest to ∂B_R, and then one can similarly choose $z_2 \in A_1$ closest to ∂B_α.

In particular,

$$t \mapsto \partial_{x_1} \tilde{u}_{k_1}(tz_{l_1,k_1} + (1-t)z_{k_1(1)}) \tag{3}$$

is absolutely continuous on $[0,1]$. Next denote $l_1 = 1$ and select the smallest integer $l_2 \geq 2$ for which $|x_{l_2}| < |x_1|$. Let $k_2 \geq k_1$ be the smallest integer for which $|z_{l_2,k_2}| > R$: $\frac{|x_{l_2}|}{s} \to \infty$ when $s \to 0^+$, therefore via $s_k \to 0$, k_2 exists. Hence one may iterate and thus obtain two elements on $\Gamma(\tilde{u}_{k_2})$:

$$z_{l_2,k_2} \in C_\epsilon \setminus B_R^+,$$

$$z_{k_2(2)} \in \overline{B^+_{|x_{l_2}|}}.$$

Observe that via iterating one can select $z_{l_m,k_m}, z_{k_m(m)} \in \Gamma(\tilde{u}_{k_m})$ s.t.

$$|z_{l_m,k_m}| \geq R,$$

$$|z_{k_m(m)}| \leq \alpha,$$

$$|z_{l_m,k_m} - z_{k_m(m)}| \geq R - \alpha > 0; \tag{4}$$

moreover, via

$$\frac{x_l}{|x_l|} \to e,$$

note

$$\frac{z_{l_m,k_m}}{|z_{l_m,k_m}|} = \frac{x_{l_m}}{|x_{l_m}|} \to e, \tag{5}$$

$$\frac{z_{k_m(m)}}{|z_{k_m(m)}|} \to e \tag{6}$$

(because $z_{l,k} = \frac{x_l}{s_k}$, one has that the free boundary points $z_{k_m(m)}$ are close to the line along e).

Now, one may show

$$\left|\frac{z_{l_m,k_m} - z_{k_m(m)}}{|z_{l_m,k_m} - z_{k_m(m)}|} - e\right| \to 0$$

as $m \to \infty$.

Note thanks to the convergence, if

$$e^m = \frac{z_{l_m,k_m} - z_{k_m(m)}}{|z_{l_m,k_m} - z_{k_m(m)}|},$$

one then has

$$e^m \to e.$$

Let

$$A_{k_m,l_m} = \{tz_{l_m,k_m} + (1-t)z_{k_m(m)} : 0 \le t \le 1\},$$

$$A_{k_m,l_m,\alpha} = \{tz_{l_m,k_m} + (1-t)z_{k_m(m)} : 0 \le t \le t(\alpha),$$
$$t(\alpha) = \sup\{t : tz_{l_m,k_m} + (1-t)z_{k_m(m)} \in B_\alpha^+\}\},$$

$$A_{k_m,l_m,R,\alpha} = \{tz_{l_m,k_m} + (1-t)z_{k_m(m)} : t(\alpha) \le t \le t(R),$$
$$t(R) = \sup\{t : tz_{l_m,k_m} + (1-t)z_{k_m(m)} \in B_R^+\}\},$$

$$A_{k_m,l_m,R} = \{tz_{l_m,k_m} + (1-t)z_{k_m(m)} : t(R) \le t \le 1\}.$$

Observe

$$
\begin{aligned}
&\frac{d}{dt}\partial_{x_1}\tilde{u}_{k_m}(tz_{l_m,k_m}+(1-t)z_{k_m(m)}) \\
&=\langle\nabla\partial_{x_1}\tilde{u}_{k_m}(tz_{l_m,k_m}+(1-t)z_{k_m(m)}),z_{l_m,k_m}-z_{k_m(m)}\rangle \\
&=\langle\nabla\partial_{x_1}\tilde{u}_{k_m}(tz_{l_m,k_m}+(1-t)z_{k_m(m)}),\frac{z_{l_m,k_m}-z_{k_m(m)}}{|z_{l_m,k_m}-z_{k_m(m)}|}\rangle|z_{l_m,k_m}-z_{k_m(m)}| \\
&=\langle\nabla\partial_{x_1}\tilde{u}_{k_m}(tz_{l_m,k_m}+(1-t)z_{k_m(m)}),e^m\rangle|z_{l_m,k_m}-z_{k_m(m)}| \\
&=\partial_{e^m x_1}\tilde{u}_{k_m}(tz_{l_m,k_m}+(1-t)z_{k_m(m)})|z_{l_m,k_m}-z_{k_m(m)}|. \qquad (7)
\end{aligned}
$$

In particular, thanks to absolute continuity of

$$t\mapsto\partial_{x_1}\tilde{u}_{k_m}(tz_{l_m,k_m}+(1-t)z_{k_m(m)})$$

on $[0,1]$ from (3) and the fact that $z_{l_m,k_m},z_{k_m(m)}\in\Gamma(\tilde{u}_{k_m})$, it follows that $\nabla\tilde{u}_{k_m}(z_{l_m,k_m})=0$, $\nabla\tilde{u}_{k_m}(z_{k_m(m)})=0$ & via the fundamental theorem of calculus applied together with (7)

$$0 =$$

$$\partial_{x_1} \tilde{u}_{k_m}(tz_{l_m,k_m} + (1-t)z_{k_m(m)})|_{t=1} - \partial_{x_1} \tilde{u}_{k_m}(tz_{l_m,k_m} + (1-t)z_{k_m(m)})|_{t=0}$$

$$= \int_0^1 \partial_{e^m x_1} \tilde{u}_{k_m}(sz_{l_m,k_m} + (1-s)z_{k_m(m)})|z_{l_m,k_m} - z_{k_m(m)}|ds.$$

Therefore,

$$\begin{aligned}
0 &= \int_0^1 \partial_{e^m x_1} \tilde{u}_{k_m}(sz_{l_m,k_m} + (1-s)z_{k_m(m)})ds \\
&= \int_0^{t(\alpha)} \partial_{e^m x_1} \tilde{u}_{k_m}(sz_{l_m,k_m} + (1-s)z_{k_m(m)})ds \\
&+ \int_{t(\alpha)}^{t(R)} \partial_{e^m x_1} \tilde{u}_{k_m}(sz_{l_m,k_m} + (1-s)z_{k_m(m)})ds \\
&+ \int_{t(R)}^1 \partial_{e^m x_1} \tilde{u}_{k_m}(sz_{l_m,k_m} + (1-s)z_{k_m(m)})ds.
\end{aligned}$$

Set

$$J_1(m) = \int_0^{t(\alpha)} \partial_{e^m x_1} \tilde{u}_{k_m}(s z_{l_m,k_m} + (1-s) z_{k_m(m)}) ds$$

$$J_2(m) = \int_{t(\alpha)}^{t(R)} \partial_{e^m x_1} \tilde{u}_{k_m}(s z_{l_m,k_m} + (1-s) z_{k_m(m)}) ds$$

$$J_3(m) = \int_{t(R)}^{1} \partial_{e^m x_1} \tilde{u}_{k_m}(s z_{l_m,k_m} + (1-s) z_{k_m(m)}) ds.$$

Supposing $s \in [t(\alpha), t(R)]$, the $C^{2,\alpha}(B_R^+ \setminus B_\alpha^+)$ regularity yields

$$\partial_{e^m x_1} \tilde{u}_{k_m}(s z_{l_m,k_m} + (1-s) z_{k_m(m)}) \to \partial_{x_1 e} u_0 = e_n a > 0$$

uniformly; therefore

$$J_2(m) \to (t(R) - t(\alpha)) e_n a > 0. \tag{8}$$

Supposing $s \in [t(R), 1]$, then first assume

Case $\mathbf{1_{a_*}}$: $\sup_{\mathbf{m}} |\mathbf{z_{l_m,k_m}}| < \infty$.

Let $R_* > \sup_m |z_{l_m,k_m}|$. Note that thanks to absolute continuity of

$$t \mapsto \partial_{x_1} \tilde{u}_{k_m}(tz_{l_m,k_m} + (1-t)z_{k_m(m)})$$

on $[0,1]$, (observe that there aren't any free boundary points on $\{tz_{l_m,k_m} + (1-t)z_{k_m(m)} : t \in (0,1)\}$ from the construction, which via local $C^{2,\alpha}$ smoothness implies the absolute continuity) the function

$$\partial_{e^m x_1} \tilde{u}_{k_m}(sz_{l_m,k_m} + (1-s)z_{k_m(m)})$$

makes sense.

Thus one has that the convergence

$$||\tilde{u}_k - u_0||_{C^{1,\alpha_1}(B^+_{R_*})} \to 0 \tag{9}$$

for all $0 < \alpha_1 < 1$ is enhanced on the line away from the endpoints:

$$D^2 \tilde{u}_{k_m} \to D^2 u_0.$$

Observe

$$\partial_{e^m x_1} u_0 \to \partial_{x_1 e} u_0 = e_n a > 0.$$

This implies

$$\liminf_m \partial_{e^m x_1} \tilde{u}_{k_m}(sz_{l_m,k_m} + (1-s)z_{k_m(m)}) = e_n a > 0,$$

and Fatou yields the positivity

$$0 < (1 - t(R))e_n a \leq \liminf_m J_3(m).$$

Case $\mathbf{2_{a_*}}$: $\sup_{\mathbf{m}} |\mathbf{z_{l_m,k_m}}| = \infty$.

$$
\begin{aligned}
J_3(m) &= \int_{t(R)}^1 \partial_{e^m x_1} \tilde{u}_{k_m}(s z_{l_m,k_m} + (1-s) z_{k_m(m)}) ds \\
&= \frac{1}{|z_{l_m,k_m} - z_{k_m(m)}|} \int_{t(R)}^1 \frac{d}{ds} \partial_{x_1} \tilde{u}_{k_m}(s z_{l_m,k_m} + (1-s) z_{k_m(m)}) ds \\
&= \frac{1}{|z_{l_m,k_m} - z_{k_m(m)}|} \Big(\partial_{x_1} \tilde{u}_{k_m}(s z_{l_m,k_m} + (1-s) z_{k_m(m)})|_{s=1} \\
&- \partial_{x_1} \tilde{u}_{k_m}(s z_{l_m,k_m} + (1-s) z_{k_m(m)})|_{s=t(R)} \Big) \\
&= \frac{1}{|z_{l_m,k_m} - z_{k_m(m)}|} (0 - \partial_{x_1} \tilde{u}_{k_m}(s z_{l_m,k_m} + (1-s) z_{k_m(m)})|_{s=t(R)}),
\end{aligned}
$$

from the information that z_{l_m,k_m} is a free boundary point combined with (7). Observe via the definition,

$$
t(R) = \sup\{t : t z_{l_m,k_m} + (1-t) z_{k_m(m)} \in B_R^+\}.
$$

In particular,

$$|\partial_{x_1}\tilde{u}_{k_m}(sz_{l_m,k_m} + (1-s)z_{k_m(m)})|_{s=t(R)}| \leq T$$

for $T > 0$ independent of m via $t(R)z_{l_m,k_m} + (1-t(R))z_{k_m(m)} \in \overline{B_R^+}$ together with

$$||\tilde{u}_k - u_0||_{C^{1,\alpha_1}(B^+_{R_*})} \to 0$$

for any $R_* > 0$. Therefore since $|z_{k_m(m)}| \leq \alpha$ (mod subsequences), $|z_{l_m,k_m}| \to \infty$,

$$|J_3(m)| = \left|\frac{1}{|z_{l_m,k_m} - z_{k_m(m)}|}(-\partial_{x_1}\tilde{u}_{k_m}(sz_{l_m,k_m}+(1-s)z_{k_m(m)})|_{s=t(R)})\right| \to 0.$$

In particular, Case 1_{a_*} and Case 2_{a_*} imply

$$0 \leq \liminf_{m} J_3(m). \tag{10}$$

Also, $C^{1,1}(B_1^+)$ regularity yields

$$|J_1(m)| \leq Tt(\alpha), \tag{11}$$

$$t(\alpha) \to 0^+ \tag{12}$$

as $\alpha \to 0$. It actually follows that

$$t(\alpha) \leq \frac{2\alpha}{R - \alpha}.$$

Thus, select $R > 1$ and some small $\alpha(R, T, e, a) \in (0, 1/5)$ so that

$$\frac{t(R)e_n a}{e_n a + T} > t(\alpha).$$

Therefore (8), (10), and (11) imply

$$\begin{aligned} 0 &= \liminf_m \Big(\int_0^1 \partial_{e^m x_1} \tilde{u}_{k_m}(sz_{l_m,k_m} + (1-s)z_{k_m(m)})ds\Big) \\ &\geq (t(R) - t(\alpha))e_n a - Tt(\alpha) > 0, \end{aligned}$$

a contradiction. In particular, this completes Step 1.

Step 2. Claim: (1) is not true. If (1) is true, all blow-up limits are of the form

$$u_0 = bx_n^2.$$

Let $y_l = \frac{x_l}{r_l}$ with $r_l = |x_l|$. Observe when

$$\tilde{u}_l(x) = \frac{u(r_l x)}{r_l^2}$$

that $y_l \in \Gamma(\tilde{u}_l)$,

$$\tilde{u}_l \to bx_n^2,$$

$$y_l \to y \in \partial B_1 \cap \overline{C_\epsilon}$$

(up to a subsequence), and $y \in \Gamma(u_0)$; hence this is a contradiction because $\nabla u_0(y) \neq 0$. In particular, this proves the statement in Step 2.

Steps 1 and 2 therefore imply that the assumption that the statement of the theorem is false, is false.

Free boundary regularity

Theorem (I., arXiv 2023)

Suppose $0 \leq u \in P_1^+(0, M, \Omega)$ *and* $0 \in \overline{\Gamma(u)}$. *Then there exists* $r > 0$ *such that* $\Gamma(u)$ *is the graph of a* C^1 *function in* B_r^+.

In the physical case, the blow-up

$$u_0(x) = ax_1x_n + bx_n^2,$$

$a \neq 0$, is ruled out thanks to $u \geq 0$. Supposing $n = 2$, one may rule out the undesired blow-up in general with a maximum principle analogous to the argument in [I., Interfaces Free Bound., 2019].

Theorem (I., arXiv 2023)

Suppose $n = 2$, $u \in P_1^+(0, M, \Omega)$ and $0 \in \overline{\Gamma(u)}$. Then there exists $r > 0$ such that $\Gamma(u)$ is the graph of a C^1 function in B_r^+.

Free boundary regularity

- The idea is to show that when $R > 4$,

$$N = N(R) =$$

$$\limsup_{y_n \to 0, y_n > 0, y_n \in \{y_n : (y', y_n) \in \cup_j \Gamma(u_j)\}} \sup_{\{x : x_n > 0, |x| \leq R\}} \sup_{e \in \mathbb{S}^{n-2} \cap e_n^{\perp}}$$

$$\sup_{y \in \overline{B_{1/2}^+} \cap \{y' : (y', y_n) \in \Gamma(u_j)\}} \frac{\partial_e u_j(y_n x + y)}{x_n y_n} = 0.$$

This implies that the solution is close to a half-space solution near $\{x_n = 0\}$ and one can thereafter prove C^1 regularity via a standard technique.

The functions

$$w(x) = x^{\alpha} \sin(\frac{1}{x})$$

if $x > 0$ encode the meticulous feature of the regularity vs. tangential geometry for general curves relative to the x-axis: when $\alpha > 1$, $w(x)$ produces non-transversal intersection and this is not the case if $\alpha \in (0, 1]$; if $\alpha > 2$

$$w'(x) = \alpha x^{\alpha-1} \sin(\frac{1}{x}) - x^{\alpha-2} \cos(\frac{1}{x}) \to 0$$

as $x \to 0^+$; if $\alpha \leq 2$, $w(x)$ cannot have a $C^1([0, 1])$ extension.

Further directions

- If one varies the boundary data and equation, there exist examples in which the free boundary approaches the fixed boundary at an angle. It is of interest to study sharp conditions which yield tangential touch/non-tangential touch (e.g. work of John Andersson . . .).
- Parabolic setting (e.g. work of Lee-Park...).
- Analysis of singularities in the interior (e.g. work of Figalli-Serra and Savin-Yu).
- Monge-Ampère equations.

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Thank you for your attention :-)