Entropy methods in slightly new contexts

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Entropy methods are common to study asymptotic behaviour of some evolution PDE of the type

$$\partial_t u = F(u),$$

where the solution is a function $u = u(t) \in$ (some space X), $t \geq 0$. Look for a *Lyapunov functional* or *entropy*

$$H \colon X \to [0, +\infty)$$

such that

$$\frac{\mathrm{d}}{\mathrm{d}t} H(u(t)) = -D(u(t)) \leq 0$$

and such that

$$D(u) = 0 \;\Leftrightarrow\; u \text{ is an equilibrium.}$$

Examples / difficulties

Works for:

1. Markov / sub-Markov processes ((fractional) diffusion, jump processes).
2. Equations from physics with *detailed balance*: elastic Boltzmann, coagulation-fragmentation equations.
3. Gradient flows (porous media / fast diffusion, McKean-Vlasov, aggregation equation, fractional porous media.)
4. Some equations from biology.

Does not work well (so far) for:

1. Equations from physics *without* detailed balance: coagulation equation (Smoluchowski), inelastic Boltzmann.
2. Many equations from biology.
3. Many nonlocal, nonlinear equations.

Main idea

In some equations which have diffusive behaviour one often can carry out a similar change of variables and obtain a perturbation of a known PDE, *even when there is no scale invariance*.

We will discuss some examples:

1. Linear diffusion + weak nonlinear interaction.
2. Linear diffusion in an exterior domain + Dirichlet boundary conditions.
3. Nonlinear diffusion in an exterior domain + Dirichlet boundary conditions.
4. Linear diffusion in hyperbolic space.

Self-similar behaviour of the heat equation

$$\partial_t u = \Delta u \qquad \text{on } \mathbb{R}^d \qquad \text{heat equation on } \mathbb{R}^d$$

With entropy / energy methods one can show

$$\|u\|_p \leq C t^{-\frac{d}{2}(1-\frac{1}{p})} \qquad \textbf{First-order behaviour}$$

One can change variables:

$$g(\tau, y) := e^{d\tau} u(\frac{1}{2} e^{2\tau}, e^{\tau} y)$$

and use scale invariance to get

$$\partial_\tau g = \Delta g + \mathrm{div}(yg) \qquad \text{Fokker-Planck equation}$$

With entropy methods, if G_t is the fundamental solution at 0:

$$\|u - G_t\|_p \lesssim t^{-\frac{1}{2}-\frac{d}{2}\left(1-\frac{1}{p}\right)} \qquad \textbf{Second-order behaviour}$$

The heat equation on $\mathbb{R}^d$: equivalences

$$g(t,x) = e^{dt} u\Big(\frac{1}{2}(e^{2t}-1), e^t x\Big).$$

Heat $\Leftrightarrow$ Fokker-Planck dictionary

u solves heat	$\Leftrightarrow$	g solves Fokker-Planck
$\partial_t u = \Delta u$	$\Leftrightarrow$	$\partial_t g = \Delta g + \mathrm{div}(xg)$
G_t fundamental solution	$\Leftrightarrow$	G Gaussian
$\lVert D^k u \rVert_\infty \lesssim t^{-\frac{d}{2}-\frac{k}{2}}$	$\Leftrightarrow$	$\lVert D^k g \rVert_\infty \lesssim 1$
$\lVert u - G_t \rVert_p \lesssim t^{-\frac{1}{2}-\frac{d}{2}\left(1-\frac{1}{p}\right)}$	$\Leftrightarrow$	$\lVert g - G \rVert_p \lesssim e^{-t}$

For the Fokker-Planck equation consider the entropy

$$H(g|G) := \int_{\mathbb{R}^d} g \log \frac{g}{G}$$

and use the log-Sobolev inequality to write

$$\frac{\mathrm{d}}{\mathrm{d}t} H(g|G) = -\int_{\mathbb{R}^d} g \left| \nabla \log \frac{g}{G} \right|^2 \leq -2H(g|G),$$

which shows

$$H(g|G) \leq e^{-2t} H(g_0|G).$$

By the Csiszar-Kullback inequality,

$$\|g - G\|_1 \leq \sqrt{2}\, e^{-t} \sqrt{H(g_0|G)}$$

which means precisely

$$\|u - G_t\|_1 \lesssim t^{-\frac{1}{2}}.$$

We want to adapt this to other situations.

Diffusion + weak interaction

Example from **[C., Carrillo & Schonbek 2012]**:

$$\partial_t u = \Delta u + \mathrm{div}(u(\nabla W * u))$$

with W a smooth, compactly supported potential. The behaviour of this PDE is dominated by the diffusion term, since the interaction is not strong enough. Try the same diffusive change of variables to get:

$$\partial_\tau g = \Delta g + \mathrm{div}(yg) + e^{-d\tau}\,\mathrm{div}(g(\nabla W_\varepsilon * g)),$$

$$\text{where} \quad W_\varepsilon(y) := \varepsilon^{-d} W\left(\frac{y}{\varepsilon}\right), \qquad \varepsilon := e^{-\tau}$$

This allows us to obtain convergence of g to a Gaussian, which translates in similar “second-order” behaviour for u as before.

Recent results which work for any (reasonably smooth) W in **[Carrillo, Gómez-Castro, Yao & Zeng, 2023]**

Diffusion on an exterior domain

Dirichlet heat equation outside the unit ball U on $\mathbb{R}^d$:

$$\begin{cases} \partial_t u = \Delta u & \text{for } t > 0,\ x \in \Omega = \mathbb{R}^d \setminus \overline{U} \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Mass is lost at the boundary ∂U.

Does the solution u behave in a self-similar way, as the heat equation on the whole $\mathbb{R}^d$*?*

We can obtain results for general domains U which are not balls, but I will always use the ball as an example.

Dirichlet heat equation — Main features

In 1D **all mass is eventually lost** as $t \to +\infty$, and there is a self-similar behaviour approaching the dipole solution of the heat equation on $\mathbb{R}$.

In 2D **all mass is eventually lost** as $t \to +\infty$, and there is a self-similar behaviour approaching the fundamental solution of the heat equation on $\mathbb{R}^2$.

In $\geq$3D **only part of the mass is lost** as $t \to +\infty$, and the rest of the mass approaches the fundamental solution of the heat equation on $\mathbb{R}^d$.

We want to prove this behaviour precisely.

Herraiz 1997 Asymptotic self-similarity for $d \geq 3$.

Grigoryan & Saloff-Coste 2002 Upper and lower bounds for all d, away from the boundary.

Zhang 2003 Upper and lower bounds for all d up to the boundary.

Cortázar, Quirós & Wolanski 2012 Work on a related nonlocal equation. Asymptotic self-similarity ($d \geq 3$).

Uchiyama 2017 Asymptotic self-similarity in $d = 2$, with rates.

No rates given in previous literature except Uchiyama in $d = 2$.

Heat equation outside a ball

Conservation law. On the complement Ω of any (slightly regular) domain U we can find a function ϕ such that

$$\begin{cases} \Delta\phi = 0 & \text{on } \Omega \\ \phi = 0 & \text{on } \partial\Omega \end{cases}$$

When U is the ball of radius 1, it is explicit:

$$\begin{aligned} \phi(x) &:= 1 - |x|^{2-d}, & |x| \geq 1, \quad \text{for } d \neq 2. \\ \phi(x) &:= \log|x|, & |x| \geq 1, \quad \text{for } d = 2. \end{aligned}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u\phi = 0$$

Change of variables

$$\partial_t u = \Delta u \qquad \text{on } \mathbb{R}^d \setminus \overline{U}$$

$$v := u\phi$$

$$g(t,x) = e^{dt} v\Big(\frac{1}{2}(e^{2t}-1), e^t x\Big).$$

Heat $\Leftrightarrow$ Fokker-Planck dictionary on $\mathbb{R}^d \setminus U$

u solves the heat equation on $\mathbb{R}^d \setminus U \iff g$ solves

$$\partial_t g = \Delta g + \operatorname{div}(xg) - 2\operatorname{div}(Zg) \qquad \text{on } \Omega_t := e^{-t}\Omega$$

with

$$Z = Z(t,x) = e^t \frac{\nabla\phi(e^t x)}{\phi(e^t x)}, \qquad t > 0,\ x \in \Omega_t.$$

“Transient equilibrium”

The equation $\partial_t g = \Delta g + \mathrm{div}(xg) - 2\,\mathrm{div}(Zg)$ has a “transient equilibrium” which solves

$$0 = \Delta F_t + \mathrm{div}(xF_t) - 2\,\mathrm{div}(ZF_t).$$

It is equal to (when normalised by a constant K_t)

$$F_t(x) := K_t\,\phi(e^t x)^2 e^{-\frac{|x|^2}{2}}.$$

$d = 1$	$F_t(x) = F(x) = Kx^2 G(x)$	with fixed $K > 0$
$d = 2$	$F_t(x) = K_t(t + \log\|x\|)^2 G(x)$	with $K_t \sim \frac{1}{t^2}$
$d \geq 3$	$F_t(x) = K_t\phi(e^t x)^2 G(x)$	with $K_t \to K > 0$

Evolution of “dynamic entropy”

This suggests that we consider the “dynamic entropy”

$$H(g \mid F_t) := \int_{\Omega_t} g \log \frac{g}{F_t}.$$

$$\frac{\mathrm{d}}{\mathrm{d}t} H(g \mid F_t) = -\int_{\Omega_t} g \left| \nabla \log \frac{g}{F_t} \right|^2 - \int_{\Omega_t} g \, \frac{\partial_t F_t}{F_t}$$

We need two things

1. A log-Sobolev inequality for F_t.
2. To show that the last term negligible as $t \to +\infty$.

(For details, see our preprint **[C., Gárriz & Quirós 2025]** (arxiv:2503.01492)):

Results from **[C., Gárriz & Quirós 2025]** (arxiv:2503.01492)

$$d = 1 \qquad \|u - MD_t\|_{L^1(\Omega)} \lesssim \frac{1}{t}$$

$$d = 2 \qquad \left\|u - \frac{4M}{(\log t)^2} \log|x| G_t\right\|_{L^1(\Omega)} \lesssim \frac{1}{(\log t)^2}$$

$$d \geq 3 \qquad \|u - MG_t\|_{L^1(\Omega)} \lesssim t^{-\sigma}$$

with

$$D_t(x) := K \frac{x}{t^{\frac{3}{2}}} e^{-\frac{x^2}{4t}} \qquad \text{dipole solution}$$

$$G_t(x) := (4\pi t)^{-d/2} e^{-\frac{x^2}{4t}} \qquad \text{fundamental solution}$$

$$M := \int_\Omega u_0 \phi.$$

Main outcomes (II)

For the fundamental solution p_Ω on Ω, $d \geq 3$:

$$|p_\Omega(t,x,y) - \phi(x)\phi(y)p(t,x,y)| \leq C\phi(x)\phi(y)t^{-\frac{d}{2}-\sigma}.$$

For the fundamental solution p_Ω on Ω, $d = 2$:

$$\left|p_\Omega(t,x,y) - \frac{4\phi(x)\phi(y)}{(\log t)^2}p(t,x,y)\right| \leq C\frac{\phi(x)\phi(y)}{t(\log t)^2}\left(\frac{1}{\log t} + \frac{\min\{|x|,|y|\}}{t^\sigma}\right)$$

for $|x|, |y| \lesssim \sqrt{t}$.

Nonlinear diffusion in $\mathbb{R}^d$

$$\partial_t u = \Delta u^m \qquad \text{on } \mathbb{R}^d \qquad \text{nonlinear diffusion on } \mathbb{R}^d$$

One can prove that, with $k = (m-1)d + 2$,

$$\|u\|_p \leq C t^{-\frac{d}{k}\left(1-\frac{1}{p}\right)} \qquad \textbf{First-order behaviour}$$

Change of variables:

$$g(\tau, y) := e^{d\tau} u(\frac{1}{k} e^{k\tau}, e^{\tau} y).$$

Use scale invariance to get

$$\partial_\tau g = \Delta g^m + \operatorname{div}(yg) \qquad \text{Nonlinear diffusion + confinement}$$

With entropy methods, if B_t is the fundamental solution at 0:

$$\|u - B_t\|_p \lesssim t^{-\gamma-\frac{d}{k}\left(1-\frac{1}{p}\right)} \qquad \textbf{Second-order behaviour}$$

1. **[Otto 1999]** (Gradient flow formulation, convexity)
2. **[Carrillo & Toscani 2000]** (Bakry-Emery method)
3. **[del Pino & Dolbeault 2002]** (GNS inequalities)
4. **[Blanchet, Bonforte, Dolbeault, Grillo & Vázquez 2009]** (Fast diffusion, $m \leq 1$)

Nonlinear diffusion outside a domain

Nonlinear diffusion outside the unit ball U on $\mathbb{R}^d$:

$$\begin{cases} \partial_t u = \Delta u^m & \text{for } t > 0,\ x \in \Omega = \mathbb{R}^d \setminus U \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Mass is lost at the boundary ∂U.

Does the solution u behave in a self-similar way, as the nonlinear diffusion equation on the whole $\mathbb{R}^d$?

Some references, all for $m \geq 1$:
[Kamin & Vázquez 1991] ($d = 1$)
[King 1991] ($d = 1, 2$)
[Brändle, Quirós & Vázquez 2006] ($d \geq 3$)

Change of variables - nonlinear diffusion in exterior domains

With the same *phi* as for the heat equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\phi = 0$$

Change of variables, with $k = (m-1)d + 2$ as before:

$$v := u\phi, \qquad g(\tau, y) = e^{d\tau} v\Big(\frac{1}{k}(e^{k\tau} - 1), e^{\tau} y\Big).$$

Then g solves, on $\Omega_\tau := e^{-\tau}\Omega$,

$$\partial_\tau g = \operatorname{div}\left(yg + \frac{\nabla g^m}{\phi(e^\tau y)^{m-1}} - (m+1) g^m \frac{\nabla[\phi(e^\tau y)]}{\phi(e^\tau y)^m} \right)$$

Dimension $d = 1$

In the case of dimension 1, $\phi(x) = x$ and the natural change of variables is a different one: take $k = 2m$. Then

$$\begin{aligned}
\partial_t g &= \partial \left(yg + \frac{\partial(g^m)}{y^{m-1}} - (m+1)\frac{g^m}{y^m} \right) \\
&= \partial \left(y^{1-\frac{1}{m}} g \, \partial \left[\frac{m}{m-1} y^{\frac{1}{m}-m} g^{m-1} + \frac{m}{m+1} y^{1+\frac{1}{m}} \right] \right) \\
&= \partial \left(y^{1-\frac{1}{m}} g \, \partial \frac{\delta H}{\delta g} \right)
\end{aligned}$$

where

$$H(g) := \int_0^\infty \frac{1}{m-1} y^{\frac{1}{m}-m} g^m + \int_0^\infty \frac{m}{m+1} y^{1+\frac{1}{m}} g$$

We have then found an entropy functional for g: we have

$$\frac{\mathrm{d}}{\mathrm{d}t} H(g) = -\int_0^\infty g\, y^{1-\frac{1}{m}} \left| \partial \left(\frac{m}{m+1} y^{1+\frac{1}{m}} + \frac{m}{m-1} y^{\frac{1}{m}-m} g^{m-1} \right) \right|^2 =: -D(g).$$

Then a natural question is: is it true that for some $\lambda > 0$,

$$\lambda (H(g) - H(g_\infty)) \leq D(g) \text{ ?}$$

We can actually prove this inequality by using the Bakry-Emery approach in the spirit of **[Carrillo & Toscani, 2000]**, with

$$\lambda := \frac{m+1}{m}.$$

This only works for $m \geq m_* \approx 0,64$
(we need $12m^3 + m^2 - 4m - 1 \geq 0$).

Bakry-Emery—main idea

If we have that, along the flow of a solution,

$$\frac{\mathrm{d}}{\mathrm{d}t} H = -D \tag{1}$$

one can sometimes prove (surprisingly) that (along the flow)

$$\frac{\mathrm{d}}{\mathrm{d}t} D \leq -\lambda D. \tag{2}$$

(This is the second derivative of the entropy.) If we also know that $D(t) \to 0$ as $t \to +\infty$ (not too hard) then, integrating in time,

$$-D(0) \overset{(2)}{\leq} -\lambda \int_0^\infty D \overset{(1)}{=} -\lambda H(0).$$

Since the initial condition is arbitrary, this shows the functional inequality

$$\lambda H \leq D$$

for reasonable functions.

Consider now

$$\partial_t u = \Delta u \qquad \text{in } \mathbb{H}^d.$$

Studied in **[Vázquez 2022]**. Again, we would like rates.

No scale invariance—hard to find the right change of variables. Using polar coordinates ($\mathrm{d}s^2 = \mathrm{d}r^2 + (\sinh r)^2\,\mathrm{d}w^2$) we have

$$\partial_t u = \frac{1}{(\sinh r)^{d-1}} \partial_r \left((\sinh r)^{d-1} \partial_r u \right) + \frac{1}{(\sinh r)^2} \Delta_{\mathbb{S}^{d-1}} u$$

Call $\rho := (\sinh r)^{d-1} u$, the mass distribution. Then

$$\partial_t \rho = \partial_r^2 \rho + (1-d)\partial_r((\coth r)\rho) + \frac{1}{(\sinh r)^2} \Delta_{\mathbb{S}^{d-1}} \rho$$

We have

$$\partial_t \rho = \partial_r^2 \rho + (1-d)\partial_r((\coth r)\rho) + \frac{1}{(\sinh r)^2}\Delta_{\mathbb{S}^{d-1}}\rho.$$

Now call $v(t,x,w) := \rho(t, x+(d-1)t, w)$. This gives

$$\partial_t v = \partial_x^2 v + \partial_x(b(x+2t)v) + \frac{1}{(\sinh(x+2t))^2}\Delta_{\mathbb{S}^{d-1}} v$$

with

$$b(r) := \coth r - 1.$$

Finally, call

$$g(\tau, y, w) := e^{\tau} v(\frac{1}{2}e^{2\tau}, e^{\tau} y, w).$$

Then

$$\partial_\tau g = \partial_y^2 g + \partial(yg) + e^{\tau}\partial_y(b(e^{\tau}y)g) + \frac{e^{2\tau}}{(\sinh(e^{\tau}y + e^{2\tau}))^2}\Delta_{\mathbb{S}^{d-1}} g$$

If g is radially symmetric, one can prove that

$$\|g - G\|_1 \to 0 \qquad \text{as } t \to +\infty,$$

where G is standard Gaussian. This translates to

$$\|u - K_t\|_1 \to 0 \qquad \text{as } t \to +\infty,$$

where K_t is the fundamental solution starting at the center o.

Here is the idea to get rates for radially symmetric solutions: define a *transient equilibrium* by writing

$$\begin{aligned}\partial_\tau g &= \partial_y^2 g + \partial(yg) + e^\tau \partial_y (b(e^\tau y) g) \\ &= \partial_y \left(\partial_y g + yg + e^\tau b(e^\tau y) g\right)\end{aligned}$$

and setting

$$0 = \partial_y F_t(y) + y F_t(y) + e^\tau b(e^\tau y) F_t(y).$$

Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \int g(t,y) \log \frac{g(t,y)}{F_t(y)}\, \mathrm{d}y &\leq \int g \left| \partial \log \frac{g(t,y)}{F_t(y)} \right|^2 \mathrm{d}y \\ &\leq -2 \int g(t,y) \log \frac{g(t,y)}{F_t(y)}\, \mathrm{d}y.\end{aligned}$$

This gives exponential decay of the relative entropy, and implies exponential decay for other norms (such as L^1).

Thanks for listening!